Some properties of duality-rotated Maxwell fields

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Duality rotations of Maxwell fields residing in curved space-time are studied in the presence of sources, and it is shown that a general duality rotation transforms the conserved, magnetic-charge-free four-current of a Maxwell field into a new four-current which is neither conserved nor is free of magnetic charges. The necessary and sufficient condition for two Maxwell fields, in the presence of source four-currents which are both conserved and are free of magnetic charges, to go into each other under a duality rotation is obtained. As duality rotations preserve the electromagnetic energy tensor E_{ab}, this leads to conditions under which a given E_{ab}, and hence a given metric solution of the Einstein equations for a continuous system having E_{ab} as a part of it, may possess a multiple (or in particular, a dual) interpretation in terms of the electromagnetic field. In the case of non-null electromagnetic fields with vanishing Lorentz force, it is shown that a direct computation involving the given Maxwell field yields the required duality rotation provided it exists. A number of examples of duality-connected field pairs, some existing in vacuum and some others inside matter, are discussed to illustrate the theory developed.

I. INTRODUCTION

The idea of duality rotations of Maxwell fields (i.e., electromagnetic fields) was introduced into physics by Rainich, Misner, and Wheeler (RMW) in the context of the already-unified-field theory.[1,2] In view of the fact that the RMW theory is essentially a theory of source-free electromagnetic fields, duality rotations have been studied (see Witten[3] for an excellent review) under the restriction that they should preserve the source-free nature of the electromagnetic fields. In this connection, Witten[3,4] has shown that the field obtained after a nonconstant duality rotation of a source-free electromagnetic field, in general, does not satisfy the source-free Maxwell equations. This clearly shows that the source four-current of an electromagnetic field is altered by a (nonconstant) duality rotation, and in this paper we provide an extension of Witten's discussion by considering the duality rotations of electromagnetic fields in the presence of sources also. The motivation for this discussion has been provided by a paper of Tariq and Tupper[5] in which they have shown that, in some cases, an electrovac metric (see Sec. II for a definition) may be interpreted as the space-time curvature produced either by a Maxwell field without sources or by a Maxwell field with sources. In the same paper, using the spin-coefficient formalism of Newman and Penrose, Tariq and Tupper have also given prescriptions by which a source-free Maxwell field may be used to generate a Maxwell field with sources. The purpose of this paper is to bring out relevance of duality rotations to the the results of Tariq and Tupper and to develop a complete formalism to obtain all the Maxwell fields of a given electromagnetic energy tensor in electrovac as well as nonelectrovac situations.

In the following section, we begin our study of duality rotations by observing that in general the duality transform of a conserved, magnetic-charge-free source four-current is neither conserved nor is free of magnetic charges. Confining our attention to a subclass of duality rotations which produce only conserved, magnetic-charge-free four-currents, we show that such a duality rotation must obey a set of partial differential equations involving the given Maxwell field and its four-current. These equations which form a set of necessary and sufficient conditions for the existence of an "allowed" duality rotation then lead to some interesting general properties of duality-rotated Maxwell fields. Next, in Secs. III and IV, we specialize the formulas of Sec. II to Maxwell fields with zero Lorentz force and discuss some examples of duality-connected Maxwell field pairs with vanishing Lorentz force. Lastly, in Sec. V, we consider Maxwell fields with nonvanishing Lorentz force and construct an example (in flat space-time) of a duality-connected Maxwell field pair with a nonzero Lorentz force.

Since duality rotations, by definition, preserve "everything" including the electromagnetic energy tensor E_{ab} and affect only the electromagnetic field and its four-current, it follows that it is possible to replace one Maxwell field of E_{ab} by another and produce different interacting systems which are all mechanically similar to a given system of which E_{ab} is a part. Thus our discussion provides effectively a formalism for obtaining all the Maxwell fields of an E_{ab} starting from any one of its fields (provided the E_{ab} admits more than one such field) and this formalism is applicable to electrovac (Maxwell fields in vacuum) as well as nonelectrovac (Maxwell fields in matter) situations.

II. DUALITY ROTATION OF FIELDS WITH SOURCES

For the sake of completeness and future reference, we first recall some definitions and notations. We work in a space-time domain D with metric g_{ab} of signature $-+++$. As usual, a semicolon followed by an index denotes covariant differentiation with respect to g_{ab} and a comma followed by an index denotes partial differentiation. In D an electromagnetic field f_{ab} is defined. The divergence of f_{ab} gives its four-current j^a, and the divergence of the dual field $*f_{ab} \equiv \frac{1}{2}(-g)^{1/2}\epsilon_{abcd}f^{cd}$ is assumed to vanish identically, i.e., we have

$$\omega^{ab}{}_{;b} = 4\pi j^a , \tag{2.1}$$

where the complex field tensor ω_{ab} is defined to be

$$\omega_{ab} = f_{ab} + i\, *f_{ab} . \tag{2.2}$$

The equation of charge conservation

$$j^a{}_{;a} = 0 \tag{2.3}$$

is an obvious consequence of (2.1). The energy tensor E^a_b of the electromagnetic field is a real symmetric tensor defined as

$$8\pi E^a_b = \omega^{am}\,\overline{\omega}_{bm} , \tag{2.4}$$

where $\overline{\omega}_{ab}$ is the complex conjugate of ω_{ab}. The mapping

$$\omega_{ab} \rightarrow W_{ab} \equiv e^{i\theta}\omega_{ab} \tag{2.5}$$

generated by a real scalar field θ has been called a duality rotation by Misner and Wheeler.[2] Such a duality rotation affects only the electromagnetic field (and hence its four-current) and nothing else. Both the fields ω_{ab} and W_{ab} are thus defined over the same space-time domain D with metric g_{ab}. As in (2.2), we may split the duality-rotated field W_{ab} into

$$W_{ab} = F_{ab} + i\, *F_{ab} . \tag{2.6}$$

Then for the real tensors involved, (2.5) reads

$$F_{ab} = f_{ab}\cos\theta - *f_{ab}\sin\theta , \tag{2.7}$$

$$*F_{ab} = *f_{ab}\cos\theta + f_{ab}\sin\theta . \tag{2.8}$$

It is obvious that the energy tensor E_{ab} is unchanged under a duality rotation, i.e.,

$$8\pi E^a_b = \omega^{am}\,\overline{\omega}_{bm} = W^{am}\,\overline{W}_{bm} . \tag{2.9}$$

Even though the duality rotation (2.5) does not alter the mechanical properties of the electromagnetic field such as energy, momentum, and stress, as is obvious from (2.9), it does change the field ω_{ab} and its current j^a. For example, it is well known[2] that the two fields have different invariants related by

$$F_1 = f_1\cos 2\theta - f_2\sin 2\theta , \tag{2.10}$$

$$F_2 = f_1\sin 2\theta + f_2\cos 2\theta , \tag{2.11}$$

where $2F_1 = F_{rs}F^{rs}$, $2F_2 = F_{rs}*F^{rs}$, $2f_1 = f_{rs}f^{rs}$, and $2f_2 = f_{rs}*f^{rs}$. To see how the current j^a is transformed, we take the divergence of the duality-rotated field W_{ab} to get

$$W^{ab}{}_{;b} \equiv 4\pi J^a = e^{i\theta}(4\pi j^a + i\omega^{ab}\theta_b) , \quad \theta_b \equiv \theta_{,b} \tag{2.12}$$

where we have defined, as before, the divergence of W^{ab} to be its four-current J^a. This vector current J^a, however, will be complex in general and hence involves four-currents due to magnetic charges also. Hence the duality-rotated field W^{ab} would be a physically acceptable Maxwell field[6] only if J^a is real (which ensures the elimination of magnetic charges) and divergence-free (which is the equation of continuity for J^a). The equation $J^a{}_{;a} = 0$, in view of (2.3) and (2.12), reduces to

$$\omega^{ab}\,\theta_{a;b} = \omega^{ab}\theta_{[a;b]} = 0 . \tag{2.13}$$

Thus it is *sufficient* that the scalar field θ satisfies the integrability condition

$$\theta_{a,b} - \theta_{b,a} = 0 \tag{2.14}$$

in order that J^a be divergence-free. The reality of J^a leads to the equations

$$f^{ab}\theta_b\cos\theta + (4\pi j^a - *f^{ab}\theta_b)\sin\theta = 0 , \tag{2.15}$$

$$-f^{ab}\theta_b\sin\theta + (4\pi j^a - *f^{ab}\theta_b)\cos\theta = 4\pi J^a . \tag{2.16}$$

Of these, the former is a first-order partial differential equation in θ involving only the known quantities f^{ab} and j^a. Thus, if a solution θ of (2.15), also satisfying (2.14), exists, then the new field W^{ab} and its current J^a may be obtained from (2.5) and (2.16). Hence in Eqs. (2.14) and (2.15) we have have a prescription to find a duality rotation θ which generates the Maxwell field (W_{ab}, J_a) from the Maxwell field (ω_{ab}, j_a). Observe that this prescription is quite general and is applicable to null as well as non-null electromagnetic fields. In the rest of this section we use this prescription to obtain some general properties of duality-rotated Maxwell fields. Later we specialize these results to the two cases $E^{ab}{}_{;b} = 0$ and $E^{ab}{}_{;b} \neq 0$ separately.

For the sake of completeness, we first examine (2.15) and (2.16) for constant duality rotations. When $j^a = 0$, $\theta =$ constant is a solution of (2.15). This, however, leads to a field with $J^a = 0$ as is well known.[2,3] On the other hand, when $j^a \neq 0$, arbitrary constant duality rotations are not permitted. Only $\theta = 2n\pi$, where n is zero or an integer, satisfies these equations, but it does not lead to a new field at all.

On inversion, Eqs. (2.15) and (2.16) become

$$4\pi J^a\cos\theta = 4\pi j^a - {}^*f^{ab}\theta_b\,, \quad (2.17)$$

$$4\pi J^a\sin\theta = -f^{ab}\theta_b\,. \quad (2.18)$$

These equations are convenient for a discussion of the "parallelism" of the currents J_a and j_a. Consider a situation in which the currents are both nonzero. Then, barring the exceptional events at which $\sin\theta = 0$, the vector $f^{ab}\theta_b$ is not zero, and hence it follows from (2.17) and (2.18) that J_a and j_a are "parallel," i.e., are scalar multiples of each other, if and only if

$${}^*f^{ab}\theta_b = \lambda f^{ab}\theta_b\,, \quad (2.19)$$

where λ is a real scalar. In the particular case when ${}^*f^{ab}\theta_b = 0$, λ also vanishes, otherwise it is nonzero. It is to be observed that this condition for the parallelism of the two currents is independent of (2.17) and (2.18) only when both currents are nonzero. If one of the currents is zero, (2.19) either follows from (2.17) and (2.18) or is trivially satisfied. Further, if $\theta_b \neq 0$, the scalar λ must be a solution of

$$\det({}^*f^{ab} - \lambda f^{ab}) = 0\,. \quad (2.20)$$

By going over to a locally Galilean coordinate system, it is easily seen that all four roots of this equation are real for non-null fields. For null fields this equation is satisfied for arbitrary λ. Hence situations in which both currents are nonzero and are parallel are not ruled out. This is satisfying because, as we shall show later, in the general case of electromagnetic fields inside charged matter (such as a charged perfect fluid, for example) admissible duality rotations have to satisfy (2.19) also.

In what follows, we need the reciprocal relations expressing j_a in terms of F_{ab}, ${}^*F_{ab}$, J_a, and θ. These relations are obtained by interchanging j_a and J_a, changing f_{ab} to F_{ab} and θ to $-\theta$ in (2.17) and (2.18); thus we get

$$4\pi j^a\cos\theta = 4\pi J^a + {}^*F^{ab}\theta_b\,, \quad (2.21)$$

$$4\pi j^a\sin\theta = -F^{ab}\theta_b\,. \quad (2.22)$$

From the second of these and (2.18) it follows that

$$J_a\theta^a = 0, \quad j_a\theta^a = 0\,, \quad (2.23)$$

and

$$16\pi^2 J^a j_a \sin^2\theta = 4\pi E^{ab}\theta_a\theta_b\cos\theta + \tfrac{1}{2}(f_1\cos\theta - f_2\sin\theta)\theta_a\theta^a. \quad (2.24)$$

Thus the currents must necessarily be orthogonal to the gradient of θ. Further, contracting (2.22) respectively with f_{ca}, ${}^*f_{ca}$, F_{ca}, and ${}^*F_{ca}$, we obtain the following useful relations:

$$4\pi f_{ca} j^a \sin\theta = 4\pi E^b_c\theta_b\cos\theta + \tfrac{1}{2}(f_1\cos\theta - f_2\sin\theta)\theta_c\,, \quad (2.25)$$

$$4\pi\, {}^*f_{ca} j^a\sin\theta = -4\pi E^b_c\theta_b\sin\theta + \tfrac{1}{2}(f_1\sin\theta + f_2\cos\theta)\theta_c\,, \quad (2.26)$$

$$4\pi F_{ca} j^a \sin\theta = 4\pi E^b_c\theta_b + \tfrac{1}{2}F_1\theta_c\,, \quad (2.27)$$

$$4\pi\, {}^*F_{ca} j^a\sin\theta = \tfrac{1}{2}F_2\theta_c\,. \quad (2.28)$$

In deriving these, we have made use of (2.4) in its equivalent forms

$$\begin{aligned} 8\pi E^b_c &= f_{ac}f^{ab} + {}^*f_{ac}\,{}^*f^{ab} \\ &= 2(f_{ac}f^{ab} - \tfrac{1}{2}f_1\delta^b_c) \\ &= F_{ac}F^{ab} + {}^*F_{ac}\,{}^*F^{ab} \\ &= 2(F_{ac}F^{ab} - \tfrac{1}{2}F_1\delta^b_c) \end{aligned} \quad (2.29)$$

and the general identity

$$A_{ac}\,{}^*A^{ab} = {}^*A_{ac}A^{ab} = \tfrac{1}{4}(A_{rs}\,{}^*A^{rs})\delta^b_c \quad (2.30)$$

true of all antisymmetric tensors. These basic, general relations lead to a number of other interesting results under special situations. The following is one such result. In the case of a pair of non-null fields connected by a duality rotation, we may express θ entirely in terms of the energy tensor E_{ab}, as is done in the RMW theory, provided one of the fields is extremal. To see this, consider, say, f_{ab} to be extremal, i.e., let $f_2 = 0$. Then using the relation[7]

$$E^{ca}{}_{;a} = -f^{ca}j_a \quad (2.31)$$

and the algebraic property

$$E_{cd}E^{cb} = \tfrac{1}{4}(E_{rs}E^{rs})\delta^b_d \quad (2.32)$$

of the energy tensor in (2.25), we obtain

$$E_{cd}E^{ca}{}_{;a} = -[\tfrac{1}{4}E_{rs}E^{rs}\delta^b_d + \tfrac{1}{2}(E_{rs}E^{rs})^{1/2}E^b_d]\theta_b\cot\theta\,, \quad (2.33)$$

which yields the desired relation on (matrix) inversion. In obtaining this, we have used the relation

$$\begin{aligned} E_{rs}E^{rs} &= (16\pi^2)^{-1}(f_1^2 + f_2^2) \\ &= (16\pi^2)^{-1}f_1^2 \end{aligned} \quad (2.34)$$

between the invariants. Other special results following from (2.25) to (2.28) will be discussed in Sec. III.

Next we obtain a relation connecting the magnitudes of J_a, j_a, and θ_a. From (2.2), (2.4), and (2.12), it follows that

$$J_aJ^a = j_aj^a + (2\pi)^{-1}(E^{ab}\theta_a\theta_b - {}^*f^{ab}j_a\theta_b)\,. \quad (2.35)$$

Using (2.26), this may also be expressed as

$$J_a J^a = j_a j^a + (16\pi^2)^{-1}(f_1 + f_2 \cot\theta)\theta_a \theta^a\,, \tag{2.36}$$

which is a useful relation connecting the magnitudes of the currents. For a null field, this reduces to

$$J_a J^a = j_a j^a\,. \tag{2.37}$$

The invariance of the Lorentz force under arbitrary duality rotations [not necessarily those satisfying (2.14) and (2.15)] forms another important aspect of duality-rotated fields. This follows from the fact that E^{ab}, and hence $E^{ab}{}_{;b}$, is invariant under arbitrary duality rotations [see (2.31)]. This can also be seen directly as follows. Multiplying (2.12) by $\overline{W}_{ca}$ and using (2.4) and (2.5), we get

$$\overline{W}_{ca} J^a = \overline{\omega}_{ca} j^a - 2iE^b_c \theta_b\,, \tag{2.38}$$

which breaks up into the two real relations

$$F_{ca} J^a = f_{ca} j^a\,, \tag{2.39}$$

$$^*F_{ca} J^a = {}^*f_{ca} j^a + 2E^b_c \theta_b\,. \tag{2.40}$$

The first of these proves the invariance of the Lorentz force. Lastly, we note two other stray results. A double contraction of W_{ab} with J_a and j_b yields

$$W^{ab} J_a j_b = 0 = \omega^{ab} J_a j_b\,. \tag{2.41}$$

From this and (2.38) it then follows that

$$E^{ab}\theta_a J_b = 0 = E^{ab}\theta_a j_b\,. \tag{2.42}$$

So far, we have discussed some general properties of duality-rotated Maxwell fields, and it may be noted that all these results, with the exception of (2.33) and (2.37), are valid for all Maxwell fields with conserved, magnetic-charge-free four-currents. However, our discussion up to now has not taken into account that, in general, Maxwell fields form a part of interacting systems composed of matter and electromagnetic fields. The mechanical properties of such a continuous system is described by an energy-momentum tensor τ_{ab} which is the sum of the Maxwell energy tensor E_{ab} and the material energy tensor T_{ab}. The total energy tensor τ_{ab} obeys the conservation law

$$\tau^{ab}{}_{;b} = T^{ab}{}_{;b} + E^{ab}{}_{;b} = 0\,, \tag{2.43}$$

and the equations of motion of the interacting system are to be obtained from (2.43) and the Maxwell equations. In the special theory of relativity (STR), where the space-time metric is given *a priori*, one does not have to go beyond (2.43) whereas in the general theory of relativity (GTR), (2.43) follows from the Einstein equations

$$G_{ab} = -8\pi\tau_{ab}\,, \tag{2.44}$$

which lie at a deeper level and serve to determine the space-time metric. We now examine the consequence of duality rotation of Maxwell fields in light of the Eqs. (2.43) and (2.44). We have seen that whenever a solution θ (not necessarily unique) exists for Eqs. (2.14) and (2.15) for a given Maxwell field (f_{ab}, j_a), the corresponding Maxwell energy tensor E_{ab} possesses more than one physically admissible Maxwell field. Such an E_{ab} may be said to have a *multiple interpretation* (if there are just two such Maxwell fields, then the corresponding E_{ab} has a *dual interpretation*). With each Maxwell field of an E_{ab} we may associate the same material energy tensor T_{ab} [we assume that $T_{ab} + E_{ab}$ obeys (2.43)] and obtain a distinct interacting system consisting of matter and electromagnetic fields. It is evident that all such interacting systems have the same mechanical properties and equations of motion, and belong to the same *class* with respect to duality rotations. Here we recall again that duality rotations, by definition, affect only the Maxwell fields and their currents, and nothing else, with the consequence that the material energy tensor T_{ab}, all geometrical quantities such as the space-time metric g_{ab} and the Einstein tensor G_{ab}, are left unaffected. Such classes of interacting systems may be considered both in STR and in GTR. In GTR, unlike STR, the existence of such classes of systems is more than a mere curiosity; all systems belonging to the same class with respect to duality rotations produce one and the same space-time curvature through (2.44). In the special case of systems with $T^{ab} = 0$, i.e., *electrovac systems*, conditions under which a metric solution of (2.44) has a dual interpretation in the sense that it may be interpreted as the space-time curvature produced either by an electromagnetic field without sources, or by an electromagnetic field with sources, have been discussed by Tariq and Tupper[5] and also by McIntosh.[8] Tariq and Tupper do not base their discussion on duality rotations. McIntosh uses duality rotations in his discussion, but only in the restricted sense that it carries a non-null Maxwell field into its extremal field. He shows that a second Maxwell field with a real source j_a is associated with every non-null source-free Maxwell field $(F_{ab}, J_a = 0)$ of an electrovac solution of (2.44), provided the extremal field f_{ab} of F_{ab} has $f^{ab}{}_{;b} \neq 0$ and $^*f^{ab}{}_{;b} = 0$. Some of our results, especially (2.17), (2.18), and (2.23), in the special case $J_a = 0$, are contained in his discussion. However, the purpose of his discussion being entirely different from ours, he does not consider the problem of finding the duality-rotated counterpart of a given Maxwell field under the most general condition. Our discussion presented above includes and generalizes the results of Tariq and Tupper and some of the results of McIntosh and what is more, yields the corresponding results for *nonelectrovac systems* (i.e., systems with $T_{ab} \neq 0$).

To summarize, we note that the necessary and sufficient condition for a given Einstein-Maxwell metric [i.e., a solution of (2.44)] to have a dual, or in general a multiple, interpretation is that there exist nontrivial solutions of the system of Eqs. (2.14) and (2.15). In some cases (2.19) must also be included in the prescription for determining θ. For example, in the case of interacting systems such as charged perfect fluids, the electromagnetic four-current j^a and the four-velocity U^a of matter are two timelike vectors related by

$$j^a = \sigma U^a, \tag{2.45}$$

where σ is the invariant charge density of the system. Since U^a, being a purely geometrical quantity determined by the space-time geometry, is unaltered by duality rotations, the duality-rotated current J^a has to obey a relation similar to (2.45), i.e.,

$$J^a = \sigma' U^a \tag{2.46}$$

where σ' is the new invariant charge density, and we get into a contradiction unless the duality-rotated current J^a is parallel to j^a. Thus we have to demand that θ has to satisfy (2.19) also, thereby eliminating those duality rotations which may satisfy (2.14) and (2.15), but fail to produce parallel four-currents. But we must note that it is not always necessary to include (2.19) in the prescription for determining θ. In the case of electrovac systems with $T_{ab} = 0$, (2.19) is evidently irrelevant. Even in the case of nonelectrovac systems with $T_{ab} \neq 0$, (2.19) is not to be imposed provided one of the four-currents is either null or spacelike, as then there cannot be a relation of the type (2.45) between the timelike four-velocity field of matter and the four-current.

Lastly, before passing, we wish to make a remark on the relation of the results obtained here with those of the RMW theory[2,3] of non-null electromagnetic fields in which the extremal Maxwell root of a non-null E_{ab} is mapped onto a Maxwell field with zero current. It must be observed that the theory developed here does not include the RMW theory because the extremal Maxwell root of a non-null E_{ab} has, in general, a complex four-divergence and we have considered only real currents.

III. SYSTEMS WITH VANISHING LORENTZ FORCE

Since the Lorentz force is invariant under duality rotations, Maxwell fields may be classified with respect to duality rotations into two groups depending on whether or not the Lorentz force vanishes. In this section, we consider only the special case of fields with vanishing Lorentz force.

For a system consisting of matter and electromagnetic fields, the vanishing of the Lorentz force implies [see (2.31) and (2.43)]

$$E^{ab}{}_{;b} = T^{ab}{}_{;b} = 0, \tag{3.1}$$

so that the electromagnetic and matter fields do not interact. Moreover, when the Lorentz force is zero, the four-current is not totally arbitrary as is evident from the following results:

R1: For Maxwell fields f_{ab} with $f_2 \neq 0$

$$f_{ab}j^b = 0 \Leftrightarrow j^b = 0.$$

This is evident in a local Galilean coordinate system in which, say, f_{01} and f_{23} are the only nonzero components of f_{ab} at the event.

R2: For Maxwell fields with $f_2 = 0$ and $f_1 > 0$, the vanishing of $f_{ab}j^b$ does not place any condition on j^b.

This follows because the field is purely magnetic in this case and hence there exists a local Galilean coordinate system in which, say, f_{23} is the only nonzero component of f_{ab} at the event.

R3: For Maxwell fields with $f_2 = 0$ and $f_1 < 0$, $f_{ab}j^b = 0 \Rightarrow j_b j^b \geqslant 0$, where the equality holds only when $j_b = 0$.
This follows because the field is purely electric in this case and in a suitable local Galilean coordinate system f_{ab} has only $f_{01} \neq 0$.

R4: For Maxwell fields with $f_2 = 0$ and $f_1 = 0$, i.e., for null fields, $f_{ab}j^b = 0 \Rightarrow j_b j^b \geqslant 0$.

This is easily proved by choosing the spatial coordinates such that $f^{01} = f^{13}$ are the only surviving components of f_{ab} at the event.

Thus we see that when the Lorentz force vanishes, the four-current of the field can be nonzero only when the invariant f_2 also vanishes.

Now we consider the duality rotation of fields with vanishing Lorentz force. Setting $f_{ab}j^b = 0$ in (2.25) we get

$$E^b_c\theta_b = (8\pi)^{-1}(f_2\tan\theta - f_1)\theta_c, \tag{3.2}$$

which shows that θ_b is an eigenvector of E^b_c belonging to the eigenvalue $(8\pi)^{-1}(f_2\tan\theta - f_1)$. But the eigenvalues of E^b_c are given by[9] $\epsilon, \epsilon, -\epsilon, -\epsilon$ where $\epsilon = (8\pi)^{-1}(f_1^2 + f_2^2)^{1/2}$ and to be consistent we must have

$$(f_2\tan\theta - f_1)^2 = (f_1^2 + f_2^2)\,. \tag{3.3}$$

This is an identity for null fields as well as for non-null fields with $f_2 = 0$. However, for fields with $f_2 \neq 0$, this leads to

$$\tan 2\theta = -f_2/f_1\,. \tag{3.4}$$

We shall consider this again a little later. Substituting (3.2) in (2.24), we see that

$$J_a j^a = 0\,. \tag{3.5}$$

Thus, the currents are orthogonal and hence can never be simultaneously timelike. As a consequence, in such cases where it is necessary to have the two currents timelike and parallel, it is impossible to satisfy (3.5) unless one of the currents vanishes.

Further, using (3.2) in (2.26), we obtain

$$4\pi\, {}^*f_{ca} j^a = (f_1 + f_2\cot 2\theta)\theta_c\,. \tag{3.6}$$

This equation can be solved for θ_c in the case of non-null fields. In obtaining this solution it is convenient to classify duality-connected Maxwell field pairs into three types as follows:

Type I: $j^a \neq 0,\ J^a \neq 0$.
Type II: $j^a \neq 0,\ J^a = 0$, or $j^a = 0,\ J^a \neq 0$.
Type III: $j^a = 0,\ J^a = 0$.

With this classification, we show below that every non-null Maxwell field pair with vanishing Lorentz force must necessarily be of type II.

In the case of type I non-null Maxwell field pairs, both invariants f_2 and F_2 must vanish (see R1 to R4) which, however, is possible only when $\sin(2\theta) = 0$ [see (2.11)]. Similarly for type III non-null Maxwell field pairs, as $j^a = J^a = 0$, Eqs. (2.25) to (2.28) clearly show that $\theta =$ constant is the only permitted solution. Thus, whenever the Lorentz force vanishes, non-null Maxwell field pairs of the types I and III can differ at best by trivial constant duality rotations.

For type II non-null Maxwell field pairs with $j^a \neq 0$ and $J^a = 0$, R1 to R4 show that $f_2 = 0$ which when substituted in (3.6) yields

$$\theta_c = 4\pi f_1^{-1}({}^*f_{ca} j^a)\,. \tag{3.7}$$

If this vector field satisfies (2.14) and (2.15), then its integral

$$\theta = 4\pi \int f_1^{-1}({}^*f_{ca} j^a)dx^c + \text{constant} \tag{3.8}$$

is the required duality rotation which maps the given field $(f_{ab}, j_a \neq 0)$ onto the source-free field $(F_{ab}, J_a = 0)$.

On the other hand, for type II non-null Maxwell field pairs with $j^a = 0$ and $J^a \neq 0$, i.e., when the given Maxwell field f_{ab} is source-free and it is required to find a duality mapping θ which sends f_{ab} onto a field F_{ab} with sources, (3.6) leads to (3.4). As f_{ab} is not null, f_1 and f_2 cannot vanish simultaneously and this equation (3.4) is determinate for θ. The θ obtained by solving (3.4) would be the required duality rotation provided its gradient

$$\theta_c = \tfrac{1}{2}(f_1^2 + f_2^2)^{-1}(f_2 f_{1c} - f_1 f_{2c})\,, \tag{3.9}$$

where f_{1c} and f_{2c} are respectively the partial derivatives of f_1 and f_2 with respect to x^c, is nonzero and satisfies (2.15).

Thus, in the preceding two paragraphs, we have a very simple prescription for obtaining the duality-rotated counterpart of a non-null Maxwell field f_{ab} with vanishing Lorentz force. This prescription involves a straightforward computation and checking and does not require a solution of simultaneous partial differential equations as is required by the general prescription (2.14) and (2.15).

For null Maxwell fields with vanishing Lorentz force, Eqs. (3.2) and (3.6) reduce to

$${}^*f_{ca} j^a = 0 \tag{3.10}$$

and

$$E^b_c \theta_b = 0\,. \tag{3.11}$$

These equations, unlike the corresponding ones for non-null fields, do not lead to an explicit solution for θ_b, and one is thus forced to use the general prescription (2.14) and (2.15) to find θ. However, in such cases where it is convenient, (3.11) may first be used to determine a general solution θ_b which may then be checked for consistency through (2.14) and (2.15). The other equation (3.10) is a necessary condition for the existence of a nontrivial duality rotation of f_{ab} and provides a useful preliminary check in the case of fields with $j_a \neq 0$.

Regarding the currents of a null Maxwell field pair (with zero Lorentz force), we have the relations [see (2.37), (3.5), and R4]

$$j_a J^a = 0,\quad j_a j^a = J_a J^a \geq 0\,, \tag{3.12}$$

which show that for type-I pairs, the two (nonzero) currents are either null vectors with the relation $J_a = \alpha j_a$ where α is a scalar, or mutually orthogonal spacelike vectors of the same magnitude, and for type-II pairs the nonzero current must necessarily be null. Here it must be observed that, unlike in the case of non-null Maxwell field pairs, all three types I, II, and III of null Maxwell field pairs are possible.

IV. EXAMPLES OF SYSTEMS WITH VANISHING LORENTZ FORCE

We now illustrate the discussion of the preceding section with a few examples. Of these, examples 1, 2, and 5 deal with electrovac fields ($T_{ab}=0$) while the other two deal with fields inside matter ($T_{ab}\neq 0$).

Example 1. As an example of a non-null electrovac field, we consider the Tariq-Tupper solution[5] of (2.44) with $T_{ab}=0$ and $E_{ab}\neq 0$. In this interesting solution, the electromagnetic field resides in a curved space-time domain with the metric

$$ds^2=-(dt-2z\,d\varphi)^2+r^2d\varphi^2+(2a^2r^2)^{-1}(dr^2+dz^2)\,, \tag{4.1}$$

where a is an arbitrary real constant and $x^1=t$, $x^2=r$, $x^3=z$, $x^4=\varphi$. The electromagnetic field together with its invariants and the four-current is given by

$$\begin{aligned}&(\omega_{12},\omega_{13},\omega_{14})=(\mp 2i/r,0,0)\,,\\&(\omega_{34},\omega_{42},\omega_{23})=\mp 2(1,-2iz/r,0)\,,\\&(\omega^{12},\omega^{13},\omega^{14})=\pm 4a^2(ir,2z,0)\,,\\&(\omega^{34},\omega^{42}\ \ \omega^{23})=(\mp 4a^2,0,0)\,,\end{aligned} \tag{4.2}$$

$$\omega_{ab}\omega^{ab}=4(f_1+if_2)=32a^2,\quad (-g)^{1/2}=(2a^2r)^{-1}\,, \tag{4.3}$$

$$\begin{aligned}&j^a=(\pm 2a^2/\pi,0,0,0)\,,\\&f_{ab}j^b=0,\quad [(-g)^{1/2}\omega^{ab}]_{,b}=4\pi(-g)^{1/2}j^a\,.\end{aligned} \tag{4.4}$$

Using these in prescription (3.7) we obtain

$$\theta_2=2/r,\quad \theta_1=\theta_3=\theta_4=0\,. \tag{4.5}$$

It is easy to check that this vector field satisfies (2.14) and (2.15) so that its integral

$$\theta=2\ln r+\text{constant} \tag{4.6}$$

is the required duality rotation. This leads to a new field (W_{ab}, J_a) with

$$W_{ab}=\pm\exp(2\ln 2r-2b)\omega_{ab},\ \ J_a=0 \tag{4.7}$$

where we have expressed the arbitrary constant in (4.6) as $(-2b\pm\pi/2)$ in terms of a new constant b so that (4.7) is precisely the source-free electromagnetic field quoted by Tariq and Tupper. [Note that the field quoted by Tariq and Tupper contains misprints; the correct values are given by (4.7).]

Example 2. Here we consider a null electrovac solution of (2.44) in a conformally flat space-time. There exist a number of convenient forms in which conformally flat null electrovac solutions are displayed, but the following[10] appears to be simple. "If a_i is any constant null vector in the Minkowski space-time, i.e., if

$$-a_0{}^2+a_1{}^2+a_2{}^2+a_3{}^2=0\,, \tag{4.8}$$

and if $U=U(a_ix^i)$ is any function of the invariant parameter a_ix^i satisfying

$$(2\dot{U}^2-U\ddot{U})>0\,, \tag{4.9}$$

where $\dot{U}$ denotes the derivative of U with respect to a_ix^i, then the conformally flat metric

$$ds^2=U^2(-dt^2+dx^2+dy^2+dz^2) \tag{4.10}$$

with $x^0=t$, $x^1=x$, $x^2=y$, and $x^3=z$ and the null electromagnetic field f_{ij} with components

$$\begin{aligned}&(f_{01},f_{02},f_{03})=V(a_0{}^2-a_1{}^2,-a_1a_2,-a_1a_3)\,,\\&(f_{23},f_{31},f_{12})=V(0,a_0a_3,-a_0a_2)\,,\end{aligned} \tag{4.11}$$

where $V=(a_0{}^2-a_1{}^2)^{-1/2}U$, together form a null electrovac solution of (2.44)." For purposes of illustration, we shall now find a duality rotation of the special solution with

$$a_1=a_3=0,\quad a_0=a_2=1\,, \tag{4.12}$$

for which only

$$f^{12}=f^{01}={}^*f^{23}=-{}^*f^{03}=-U^{-3} \tag{4.13}$$

survive. For this source-free ($j^a=0$) field, (2.15) simplifies to

$$\theta_0=\theta_2\,, \tag{4.14}$$

$$\theta_1+\theta_3\tan\theta=0\,. \tag{4.15}$$

Using these in (2.16), we find the duality-rotated four-current to be

$$4\pi J^a=U^{-3}(\theta_1\sin\theta-\theta_3\cos\theta)(1,0,-1,0)\,. \tag{4.16}$$

This is evidently a null current [see (4.10)] and its form suggests that it is sufficient to consider the solutions of (4.14) and (4.15) in the two special situations in which (i) $\theta_1=\theta_3=0$ and (ii) $\theta_0=\theta_2=0$. When θ_1 and θ_3 vanish, the general solution of (4.14) and (4.15) is given by

$$\theta=\theta(t+y)+\text{constant}\,, \tag{4.17}$$

where $\theta(t+y)$ is an arbitrary differentiable function of its argument $(t+y)$. Such a function evidently satisfies (2.14) also and thus (4.17) generates admissible duality rotations. Regarding the duality-rotated field F^{ab}, obtainable by using (4.13) and (4.17) in (2.7), we note that it is also source-free [see (4.16)], and forms with f_{ab} a type-III field pair. Moreover, each choice of a functional form for θ in (4.17) generates its own F_{ab}, and thus we have a whole family of source-free null electromagnetic fields F_{ab} corresponding to the same null energy tensor E^b_c. In other words, E^b_c has a multiple interpretation. It is well known[3] that a null electromagnetic energy tensor does not determine its electromagnetic field up to a constant duality rotation. Peres[11] has given one such example and the above is another example of this type.

On the other hand, when $\theta_0=\theta_2=0$, (4.14) follows trivially and the complete integral of (4.15) is given by[12]

$$\theta+z\cot\theta=x-b, \tag{4.18}$$

where b is an arbitrary constant. Using this, and $\theta_0=\theta_2=0$, it is easy to see that the integrability condition (2.14) is satisfied. Thus (4.18) generates an admissible duality rotation which when operating on the f^{ab} of (4.13) produces a new field F^{ab}. It may be observed that f^{ab} and F^{ab} together form a type-II field pair. The null (but nonzero) four-current J_a of the field F^{ab} is already given in (4.16). However, as θ cannot be determined as an explicit function of (x,z), it is not possible to display the field F^{ab} and its current J^a as explicit functions of the coordinates.

Example 3. This is an example of a non-null field with $T_{ab}\neq 0$. The solution considered is that of Misra and Pandey[13] and it describes an axially symmetric stationary solution of (2.44) with charged incoherent matter and a zero-mass scalar meson field. This solution is a generalization of the earlier solution of Som and Raychaudhury[14] for a charged dust and includes the effects of an interacting meson field. In signature +2, the metric and the electromagnetic field are given by

$$ds^2=-dt^2-(\alpha^2r^2-1)r^2d\varphi^2+e^{2k}(dr^2+dz^2)$$
$$-2\alpha r^2dtd\varphi, \tag{4.19}$$

$$x^0=t,\quad x^1=r,\quad x^2=z,\quad x^3=\varphi,\quad (-g)^{1/2}=re^{2k}, \tag{4.20}$$

$$k=\tfrac{1}{2}(\lambda^2-\alpha^2)r^2+4\pi\mu^2\ln r,\quad \alpha,\lambda,\mu \text{ constants} \tag{4.21}$$

$$f_{31}=\lambda r,\quad f^{01}=-\lambda\alpha re^{-2k},\quad f^{31}=\lambda r^{-1}e^{-2k},$$
$${}^*f_{02}=\lambda,\quad {}^*f_{23}=-\lambda\alpha r^2,\quad {}^*f^{02}=-\lambda e^{-2k}, \tag{4.22}$$

$$f_1=\lambda^2e^{-2k},\quad f_2=0,\quad 2\pi j^0=-\lambda\alpha e^{-2k}, \tag{4.23}$$

$$f_{ab}j^b=0,\quad [(-g)^{1/2}\omega^{ab}]_{,b}=4\pi(-g)^{1/2}j^a, \tag{4.24}$$

where we have shown only the nonzero terms. As before θ_c may be computed from (3.7) and we get

$$\theta_2=2\alpha,\quad \theta_0=\theta_1=\theta_3=0. \tag{4.25}$$

This is evidently integrable and yields

$$\theta=\beta+2\alpha z, \tag{4.26}$$

where β is a constant of integration. The duality-rotated field is given by

$$F_{31}=\lambda r\cos(\beta+2\alpha z),\quad F_{02}=-\lambda\sin(\beta+2\alpha z),$$
$$F_{23}=\alpha\lambda r^2\sin(\beta+2\alpha z),\quad J_a=0. \tag{4.27}$$

We see that the non-null field pair (f_{ab},F_{ab}) is of type II as it should be. In their paper,[13] Misra and Pandey have also given a discussion of the equilibrium of the electromagnetic field (f_{ab},j_a) with the dust and meson fields basing their arguments essentially on the constant nonzero ratio (ρ/σ) of the dust and charge densities. It is interesting to see how the conclusions about equilibria change if one considers instead of (f_{ab},j_a) its duality-rotated counterpart $(F_{ab},J_a=0)$ which has zero charge density. However, we do not wish to discuss this aspect here.

Example 4. As an example of a null electromagnetic field with $T_{ab}\neq 0$, we consider the Ozsvath[15] solution describing a null field in equilibrium with incoherent matter, and the formulas relevant for our discussion are collected below in signature +2:

$$ds^2=-dt^2+dx^2+a^2\sin^2x dy^2+(\cos x dy+dz)^2, \tag{4.28}$$

$$x^0=t,\quad x^1=x,\quad x^2=y,\quad x^3=z,\quad (-g)^{1/2}=a\sin x,$$
$$a\equiv[(3p+1)/4p]^{1/2},\quad p\geq 1 \text{ is a constant parameter}, \tag{4.29}$$

$$(\omega_{01},\omega_{02},\omega_{03})=e^{-i\varphi}(-ib,ab\sin x,0),$$
$$(\omega_{31},\omega_{23},\omega_{12})=e^{-i\varphi}(-ib,-ab\sin x,ib\cos x),$$
$$(\omega^{01},\omega^{02},\omega^{03})=e^{-i\varphi}(ib,-ba^{-1}\csc x,ba^{-1}\cot x),$$
$$(\omega^{23},\omega^{31},\omega^{12})=e^{-i\varphi}(-ba^{-1}\csc x,-ib,0), \tag{4.30}$$

$$\varphi\equiv a(z+t),\quad 2b\equiv-[(p+1)(p-1)/p(3p+1)]^{1/2}, \tag{4.31}$$

$$\omega_{ab}\omega^{ab}=0,\quad j^a=0,\quad [(-g)^{1/2}\omega^{ab}]_{,b}=0. \tag{4.32}$$

A glance at (4.30) suggests that we try a duality rotation of ω_{ab} through $\theta=\varphi$. A direct check shows that $\theta=\varphi-\pi/2$ is an allowed duality rotation (of course, there are other possible duality rotations) and this leads to the new field $W_{ab}=\omega_{ab}e^{i(\varphi-\pi/2)}$ with the null four-current

$$4\pi J^a=b\cot x(1,0,0,-1). \tag{4.33}$$

Thus we have another example of a null field pair of type II. However, as we have already remarked, the new current J^a, being null, is not related to the timelike four-velocity of the dust.

A generalization of the Ozsvath solution discussed above has been given by Misra and Narain,[16] and even in this case one can find a duality rotation of the source-free null field leading to a null field with a null current. In passing we wish to make a small observation on the solution of Misra and Narain.[16] The arbitrary constant C in their solution must be set equal to zero, otherwise the null field quoted would have a complex four-current, and would not be source-free as is claimed therein.

Example 5. Unlike the previous examples, this is a null electromagnetic field in special relativity and we have constructed this example specifically to obtain a null field pair of type I.

In an otherwise empty domain of the Minkowski world with the metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad (4.34)$$

we consider a null electromagnetic field with the nonzero components

$$f^{01} = f^{31} = -{}^*f^{02} = {}^*f^{23} = a, \quad (4.35)$$

where a is an arbitrary differentiable function of x only. The four-current of this field is given by

$$4\pi j^a = (a_1, 0, 0, a_1), \quad a_1 = da/dx \quad (4.36)$$

and it satisfies

$$\omega^{ab}{}_{,b} = 4\pi j^a, \quad j_a j^a = 0, \quad f_{ab} j^b = 0. \quad (4.37)$$

For this field, Eq. (2.15) reduces to

$$\theta_0 + \theta_3 = 0, \quad (4.38)$$

$$a\theta_1 \cos\theta + a\theta_2 \sin\theta + a_1 \sin\theta = 0, \quad (4.39)$$

which when used in (2.16) yields the new current

$$4\pi J^a = a\theta_1 \csc\theta (1, 0, 0, 1). \quad (4.40)$$

Since our interest lies only in building model Maxwell field pairs, we consider only the following special solutions of (4.38) and (4.39) corresponding to some specific choices of the function $a(x)$:

$$\begin{aligned}&\theta_0 = \theta_2 = \theta_3 = 0, \quad a = e^{-x} = -a_1\\ &\theta = \sin^{-1}(e^{x+\alpha}), \quad \alpha = \text{constant}\end{aligned} \quad (4.41)$$

$$\begin{aligned}&\theta_0 = \theta_1 = \theta_3 = 0, \quad a = e^{-x} = -a_1\\ &\theta = y + \beta, \quad \beta = \text{constant},\end{aligned} \quad (4.42)$$

$$\begin{aligned}&\theta_1 = \theta_2 = 0, \quad a = \text{constant}, \quad a_1 = 0\\ &\theta = \theta(z - t) + \gamma, \quad \gamma = \text{constant}.\end{aligned} \quad (4.43)$$

It is seen from (4.36) and (4.40) that these special duality rotations lead respectively to null Maxwell field pairs of types I, II, and III. In (4.43) θ is any arbitrary differentiable function of $(z - t)$ and it leads to a whole family of electromagnetic fields all having the same electromagnetic energy tensor, as in example 2.

V. SYSTEMS WITH NONVANISHING LORENTZ FORCE

For a pair of Maxwell fields connected by a duality rotation, both currents must be nonzero when the (duality-invariant) Lorentz force is nonzero and as such only type-I pairs are possible. As another consequence of a nonzero Lorentz force, only systems with a nonzero T_{ab} are permitted [see (2.43)]. Also, no simplification is possible in the determination of θ as the Lorentz force is nonzero and one has to use the general prescription (2.14), (2.15), and (2.19). [Of course, (2.19) is not always necessary.] Only in the case of null fields, as $j^a = 0$, do Eqs. (2.25) and (2.26) appear to put a strong restriction on θ, namely

$$\det({}^*f_{ca} + \tan\theta f_{ca}) = 0. \quad (5.1)$$

However, this equation does not determine $\tan\theta$ at all [see the remarks following (2.20)].

To show that Maxwell field pairs with nonzero Lorentz force are nevertheless possible, we give an example. In the absence of examples in general relativity (we have not succeeded in identifying such examples in GTR, but they would be interesting) we construct a simple non-null Maxwell field pair in flat space-time.

Consider a purely electric field of magnitude a acting along the x axis of an inertial coordinate system. If we assume a to be an arbitrary differentiable function of x and t, then the nonzero components of the field, together with its invariants and four-current, are given by

$$f^{01} = {}^*f^{23} = a, \quad (5.2)$$

$$f_1 = -a^2, \quad f_2 = 0, \quad (5.3)$$

$$4\pi j^a = (a_1, -a_0, 0, 0), \quad a_i = a_{,i}, \quad (5.4)$$

$$16\pi^2 j_a j^a = (a_0{}^2 - a_1{}^2), \quad \omega^{ab}{}_{,b} = 4\pi j^a, \quad (5.5)$$

$$4\pi f_{ab} j^b = (aa_0, aa_1, 0, 0). \quad (5.6)$$

Note that the Lorentz force given by (5.6) is nonzero. Substituting from these in (2.15) and simplifying, we get

$$\sin\theta = k/a, \quad (5.7)$$

where k is a constant of integration. This satisfies (2.14) and is therefore admissible. The duality-rotated field and its current are given by

$$F^{01} = (1 - k^2/a^2)^{1/2} a, \quad F^{23} = -k, \quad (5.8)$$

$$J^b = a(a^2 - k^2)^{-1/2} j^b. \quad (5.9)$$

Thus the two currents (5.4) and (5.9) are both nonzero and are parallel, and we have an example of a (type-I) non-null Maxwell field pair with nonvanishing Lorentz force. Further, the two currents are both timelike, null, or spacelike according to $a_0{}^2 \gtreqless a_1{}^2$.

As already remarked, a nonzero T_{ab} is needed to keep these electromagnetic fields in equilibrium and make them satisfy the conservation law (2.43). It is not difficult to find such a T_{ab}. For simplicity we consider a to be a function of x only so that $a_0 = 0$. Then the two four-currents are timelike and the energy tensor E^{ab} has the nonzero components

$$E^{00} = -E^{11} = E^{22} = E^{33} = a^2/8\pi . \qquad (5.10)$$

It is easy to check that this E^{ab} and the perfect fluid distribution with

$$T^{ab} = (\mu + p)U^a U^b + p\eta^{ab} , \qquad (5.11)$$

where η^{ab} is the Minkowskian metric (4.34),

$$U^a = \delta^a_0, \quad p = a^2/8\pi , \qquad (5.12)$$

and μ is any arbitrary function of (x, y, z) independent of t, satisfy the conservation law (2.43).

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[1]G. Y. Rainich, Trans. Am. Math. Soc. 27, 106 (1925).

[2]C. W. Misner and J. A. Wheeler, Ann. Phys. (N. Y.) 2, 525 (1957).

[3]L. Witten, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), Chap. 9.

[4]L. Witten, Phys. Rev. 115, 206 (1959).

[5]N. Tariq and B. O. J. Tupper, J. Math. Phys. 17, 292 (1976).

[6]If need be, one can also retain the magnetic charges in J_a by dropping the reality conditions (2.15) and (2.16). Then (2.14) would be the only condition on θ.

[7]J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960), p. 358.

[8]C. B. G. McIntosh, Gen. Relativ. Gravit. 9, 277 (1978).

[9]J. L. Synge, *Relativity: The Special Theory*, second edition (North-Holland, Amsterdam, 1972), p. 325.

[10]A. V. Gopala Rao, Ph.D. thesis, 1978 (unpublished).

[11]A. Peres, Phys. Rev. 118, 1105 (1960).

[12]G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw-Hill, New York, 1961), p. 280.

[13]R. M. Misra and D. B. Pandey, Ann. Phys. (N. Y.) 71, 293 (1972).

[14]M. M. Som and A. K. Raychaudhuri, Proc. R. Soc. London A304, 81 (1968).

[15]J. Ozsvath, J. Math. Phys. 6, 1265 (1965).

[16]R. M. Misra and Udit Narain, Proc. Natl. Inst. Sci. India 35A, 771 (1969).