PHYSICAL REVIEW D VOLUME 19, NUMBER 12 15 JUNE 1979

# Static spherically symmetric solution for the field of a charged particle in a theory of gravity

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(Received 15 December 1978)

The field equations for a new theory of gravity are derived from a variational principle when electromagnetic fields are included in the Lagrangian density $\mathfrak{L}$. A static spherically symmetric solution to the field equations is obtained, which reduces to the Reissner-Nordström solution when a new gravitational coupling constant $l$ equals zero. For $l > m + (m^2 - 4\pi Q^2)^{1/2}$ black-hole event horizons are excluded from physical space-time.

## I. INTRODUCTION

A new theory of gravitation[1] has been formulated on the basis of a nonsymmetrical Hermitian $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$.

In the following, we shall study the consequences of the new theory when electromagnetic fields are present and derive the static spherically symmetric solution for the field of a charged particle. One important result that emerges from the solution is that if

$$l > m + (m^2 - 4\pi Q^2)^{1/2},$$

where $Q$ is the electric charge on the particle, the charged black-hole event horizons of the type predicted in Einstein-Maxwell theory are excluded from physical space-time. Moreover, test particles are prevented from reaching the essential singularity at $r=0$. These results generalize those obtained in Ref. 1.

## II. THE VARIATIONAL PRINCIPLE AND THE FIELD EQUATIONS

The Lagrangian density which includes contributions from the electromagnetic field $F_{\mu\nu}$ is chosen to be[1]

$$\mathfrak{L} = \sqrt{-g}\,[\,g^{\mu\nu}R_{\mu\nu}(W) + 4\pi F^{\mu\nu}F_{\mu\nu}], \tag{2.1}$$

where

$$F^{\mu\nu} = g^{\alpha\mu}g^{\beta\nu}F_{\alpha\beta} = -F^{\nu\mu},$$
$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{2.2}$$

The variational principle requires that

$$\delta\int \mathfrak{L}\, d^4x = 0. \tag{2.3}$$

Consider first the variation of the second term in (2.1) with respect to $g^{\mu\nu}$ with $\delta g^{\mu\nu}$ vanishing at the boundaries of integration and $A_\mu$ being kept constant with respect to the variation. We have

$$\delta(F^{\mu\nu}F_{\mu\nu}\sqrt{-g}) = -2E_{\beta\nu}\sqrt{-g}\,\delta g^{\beta\nu}$$
$$= 2\sqrt{-g}\,\delta g^{\beta\nu}(-\tfrac{1}{4}g_{\beta\nu}F^{\sigma\tau}F_{\sigma\tau} + g^{\alpha\mu}F_{\alpha\beta}F_{\mu\nu}), \tag{2.4}$$

where

$$E_{\mu\nu} = -g^{\alpha\beta}F_{\nu\beta}F_{\mu\alpha} + \tfrac{1}{4}g_{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} \tag{2.5}$$

is the Hermitian stress-energy tensor of the electromagnetic field.

Next consider the variations $\delta A_\mu$, the $g_{\mu\nu}$ remaining constant,

$$\delta(F^{\mu\nu}F_{\mu\nu}\sqrt{-g}) = -4(\sqrt{-g}\,F^{\mu\nu})_{,\mu}\,\delta A_\nu + 4(\sqrt{-g}\,F^{\mu\nu}\,\delta A_\nu)_{,\mu}. \tag{2.6}$$

The variations with respect to $W^\lambda_{\mu\nu}$ and $g_{\mu\nu}$ were calculated in Ref. 1. The resulting field equations are

$$g_{\mu+\nu-;\sigma} = 0, \tag{2.7}$$

$$\mathfrak{g}^{[\mu\nu]}{}_{,\nu} = 0, \tag{2.8}$$

$$R_{\mu\nu}(W) - \tfrac{1}{2}g_{\mu\nu}R(W) = 8\pi E_{\mu\nu}, \tag{2.9}$$

where $\mathfrak{X} = \sqrt{-g}\,X$ and

$$\mathfrak{F}^{\mu\nu}{}_{,\nu} = 0. \tag{2.10}$$

Equation (2.2) yields an additional relation

$$F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0. \tag{2.11}$$

The equations

$$R_{\mu\nu}(W) = R_{\mu\nu}(\Gamma) + \tfrac{2}{3}(W_{\mu,\nu} - W_{\nu,\mu}) \tag{2.12}$$

and

$$E = g^{\mu\nu}E_{\mu\nu} = 0 \tag{2.13}$$

allow (2.9) to be written as

$$R_{(\mu\nu)}(\Gamma) = 8\pi E_{(\mu\nu)}, \tag{2.14}$$

$$R_{[\mu\nu,\sigma]}(\Gamma) = 8\pi E_{[\mu\nu,\sigma]}. \tag{2.15}$$

The equation of motion of a charged test particle takes the form

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=\frac{q}{\mu}g^{(\mu\beta)}F_{\nu\beta}\frac{dx^\nu}{ds}, \qquad (2.16)$$

where $\mu$ and $q$ are the vanishing small mass and charge, respectively, of the particle.

## III. THE COVARIANT FORM OF MAXWELL'S EQUATIONS

When current density sources are included, Maxwell's divergence equation (2.10) becomes

$$\mathfrak{F}^{\mu\nu}{}_{,\nu}=\mathfrak{J}^\mu, \qquad (3.1)$$

where $\mathfrak{J}^\mu$ is the electric current-density four-vector. If we define

$$F^{[\mu+\nu-]}{}_{;\sigma}=\tfrac{1}{2}(F^{\mu+\nu-}{}_{;\sigma}-F^{\nu+\mu-}{}_{;\sigma}), \qquad (3.2)$$

then

$$F^{[\mu+\sigma-]}{}_{;\sigma}=F^{\mu\sigma}{}_{,\sigma}+F^{\mu\sigma}\frac{(\sqrt{-g})_{,\sigma}}{\sqrt{-g}}, \qquad (3.3)$$

so (3.1) is equivalent to

$$F^{[\mu+\nu-]}{}_{;\nu}=J^\mu. \qquad (3.4)$$

The covariant form of the other set of Maxwell's equations (2.11) is

$$F_{\mu+\nu-;\sigma}+F_{\nu-\sigma-;\mu}+F_{\sigma+\mu+;\nu}=0. \qquad (3.5)$$

## IV. STATIC SPHERICALLY SYMMETRIC SOLUTION

In spherical polar coordinates $x^1=r$, $x^2=\theta$, $x^3=\phi$, $x^4=t$, the form of $g_{\mu\nu}$ is[2]

$$g_{\mu\nu}=\begin{bmatrix}-\alpha & 0 & 0 & w\\ 0 & -r^2 & f\sin\theta & 0\\ 0 & -f\sin\theta & -r^2\sin^2\theta & 0\\ -w & 0 & 0 & \gamma\end{bmatrix}. \qquad (4.1)$$

Since we choose the boundary conditions

$$g_{(\mu\nu)}\to\eta_{\mu\nu},\ g_{[\mu\nu]}\to 0 \text{ as } r\to\infty, \qquad (4.2)$$

where

$$\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$$

is the Minkowski metric tensor, we have $g_{[23]}\equiv 0$ everywhere.[1] The contravariant $g^{\mu\nu}$ takes the form

$$g^{\mu\nu}=\begin{bmatrix}-\frac{\gamma}{\alpha\gamma-w^2} & 0 & 0 & \frac{-w}{\alpha\gamma-w^2}\\ 0 & -\frac{1}{r^2} & 0 & 0\\ 0 & 0 & -\frac{1}{r^2\sin^2\theta} & 0\\ \frac{w}{\alpha\gamma-w^2} & 0 & 0 & \frac{\alpha}{\alpha\gamma-w^2}\end{bmatrix}. \qquad (4.3)$$

Equation (2.8) yields the solution

$$w^2=\alpha\gamma K, \qquad (4.4)$$

where

$$K=\frac{k^2}{k^2+r^4} \qquad (4.5)$$

and $k=il^2$ is a purely imaginary constant of integration. A calculation of the $R_{\mu\nu}(\Gamma)$ has been presented before.[2] Since the tensor $R_{[\mu\nu]}$ has only one nonvanishing component $R_{[14]}$ and $g_{[23]}=0$, Eq. (2.15) is identically fulfilled.

The electric field is static so that

$$A_1=A_2=A_3=0,\quad F_{14}=-F_{41}=A_4'. \qquad (4.6)$$

With $\alpha=e^\lambda$ and $\gamma=e^\nu$ (a prime denotes differentiation with respect to $r$),

$$F^{41}=(e^{\lambda+\nu}-w^2)^{-1}A_4',\quad \mathfrak{F}^{41}=\sqrt{-g}F^{41}=(e^{\lambda+\nu}-w^2)^{-1/2}r^2\sin\theta A_4'. \qquad (4.7)$$

Equation (2.10) then yields the result

$$\mathfrak{F}^{41}{}_{,1}=\sin\theta[(e^{\lambda+\nu}-w^2)^{-1/2}r^2A_4']'=0, \qquad (4.8)$$

which gives upon integration

$$A_4'=\frac{Q}{r^2}(e^{\lambda+\nu}-w^2)^{1/2}, \qquad (4.9)$$

where $Q$ is a constant of integration.

We must now solve the field equation (2.14). Since

$$E_{11}=-\tfrac{1}{2}e^\lambda\frac{Q^2}{r^4},\quad E_{22}=\tfrac{1}{2}r^2\frac{Q^2}{r^4},\quad E_{33}=\tfrac{1}{2}r^2\sin^2\theta\frac{Q^2}{r^4},\quad E_{44}=\tfrac{1}{2}e^\nu\frac{Q^2}{r^4}, \qquad (4.10)$$

we get

$$e^{-\lambda}R_{11}(\Gamma)+e^{-\nu}R_{44}(\Gamma)=\frac{\lambda'+\nu'}{r}+\frac{4K}{r^2}=0, \qquad (4.11)$$

where $K$ is given by (4.5). The $R_{22}(\Gamma)$ equation is

$$-\frac{\nu'}{r}e^{-\lambda}+\frac{\lambda'}{r}e^{-\lambda}-\frac{2}{r^2}(e^{-\lambda}-1)-\frac{4K}{r^2}e^{-\lambda}=8\pi\frac{Q^2}{r^4}. \qquad (4.12)$$

Multiplying (4.11) by $e^{-\lambda}$ and substituting the resulting expression for $-(\nu'/r)e^{-\lambda}$ into (4.12) we find

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=4\pi\frac{Q^2}{r^4}. \qquad (4.13)$$

This can be integrated to give

$$\alpha=\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)^{-1}. \qquad (4.14)$$

By virtue of (4.5), Eq. (4.11) can be written in the form

$$\left[\lambda+\nu+\ln\left(\frac{r^4}{r^4+k^2}\right)\right]'=0. \tag{4.15}$$

Hence

$$\lambda+\nu+\ln\left(\frac{r^4}{r^4+k^2}\right)=\text{const} \tag{4.16}$$

This gives, since $k^2=-l^4$, the result

$$\alpha\gamma\frac{r^4}{r^4-l^4}=C^2, \tag{4.17}$$

where $C^2$ is a constant of integration. Using Eq. (4.4) we get

$$w^2=\frac{C^2k^2}{r^4}. \tag{4.18}$$

For $r\to\infty$ we have $w\to 0$ and $g_{(\mu\nu)}\to\eta_{\mu\nu}$. Thus from (4.17) we must have $C^2=1$ and (4.17) and (4.18) give

$$\gamma=\left(1-\frac{l^4}{r^4}\right)\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right), \qquad w=\pm\frac{il^2}{r^2}. \tag{4.19}$$

## V. PHYSICAL INTERPRETATION OF THE SOLUTION

From Eqs. (4.14) and (4.19) we obtain the metric exterior to a charged particle

$$ds^2=\left(1-\frac{l^4}{r^4}\right)\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)dt^2-\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)^{-1}dr^2-r^2d\Omega^2. \tag{5.1}$$

Moreover, from (4.9), (4.14), and (4.19) we find that

$$F_{14}=A_4'=\frac{Q}{r^2}. \tag{5.2}$$

This justifies our identifying $Q$ with the electric charge on the particle.

The metric (5.1) has three null surfaces

$$r_1=l,\quad r_2=m+(m^2-4\pi Q^2)^{1/2},$$
$$r_3=m-(m^2-4\pi Q^2)^{1/2},$$

where the latter two null surfaces occur only when $m^2>4\pi Q^2$. When $r<l$, we see that $ds^2<0$ and the radius $r_1=l$ defines a sphere inside which space is Euclidean four-dimensional. This sphere excludes the essential singularity at $r=0$ from physical space-time, as in the case of the vacuum solution for an uncharged particle with $Q=0$.[1] When $l<r_2$, the line element (5.1) can form black-hole event horizons at $r=r_2$ and $r=r_3$, similar to those that occur in the Reissner-Nordström solution,[3,4] when $m^2>4\pi Q^2$. However, when $l>r_2$, the standard black-hole event horizons are *excluded* from physical space-time, that space-time for which the metric is locally Minkowskian and the proper time along the path of a test particle remains real. Test particles moving along paths determined by the equation of motion (2.16) will be deflected for $r>l$, so no particle can reach $r=0$, as in the uncharged case.[1]

In the present work there has been no attempt made to unify the gravitational and electromagnetic fields. In contrast to previous work,[5] the total gravitational field is described by the nonsymmetric metric $g_{\mu\nu}$, while the electromagnetic field $F_{\mu\nu}$ is treated as a separate field in the same way as in the Einstein-Maxwell theory.


## ACKNOWLEDGMENT

This work was supported by the Natural Sciences and Engineering Research Council of Canada.



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