New general relativity

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A gravitational theory is formulated on the Weitzenböck space-time, characterized by the vanishing curvature tensor (absolute parallelism) and by the torsion tensor formed of four parallel vector fields. This theory is called new general relativity, since Einstein in 1928 first gave its original form. New general relativity has three parameters c_1, c_2, and λ, besides the Einstein constant κ. In this paper we choose $c_1 = 0 = c_2$, leaving open λ. We prove, among other things, that (i) a static, spherically symmetric gravitational field is given by the Schwarzschild metric, that (ii) in the weak-field approximation an antisymmetric field of zero mass and zero spin exists, besides gravitons, and that (iii) new general relativity agrees with all the experiments so far carried out.

I. INTRODUCTION

In 1928 Einstein introduced the notion of absolute parallelism and tried to unify gravitation and electromagnetism, using tetrads with 16 degrees of freedom.[1] His attempt failed because there was no Schwarzschild solution in his simplified field equation.[2] Later, in 1961 Møller revived Einstein's idea,[3] and Pellegrini and Plebanski found a Lagrangian formulation for absolute parallelism.[4] Recently this formalism was reconsidered by Møller.[5]

In 1967, quite independently, Hayashi and Nakano started to formulate the gauge theory of the space-time translation group[6]: This theory was of no geometrical construction, but it was shown that, for a static, isotropic gravitational field, a symmetric part of their field equations is identical with the Einstein field equation in general relativity, and that, in the weak-field approximation, an antisymmetric part describes the propagation of an antisymmetric field, whose source is related to the intrinsic spin of spin-$\frac{1}{2}$ fundamental particles.[6] Miyamoto and Nakano estimated effects of exchanging this field in the microscopic system.[7] In later years Hayashi further developed the gauge theory into a more elaborate framework[8] and fixed the final form in 1973.[9] Quite recently Hayashi pointed out the connection between the gauge theory of the space-time translation group and absolute parallelism.[10]

Now we wish to unify these two developments mentioned above, following the geometry of underlying space-time structure. The Riemann-Cartan space-time U_4 is a paracompact, Hausdorff, connected C^∞ four-dimensional manifold endowed with a locally Lorentzian metric $\underline{g}$ and a linear affine connection $\underline{\Gamma}$ which is metric,

$$D_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho} = 0 \,. \tag{1.1}$$

From this equation we get

$$\Gamma^\lambda{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} + K^\lambda{}_{\cdot\mu\nu} \,, \tag{1.2}$$

where the first term denotes the Levi-Civita connection,

$$\left\{ {\lambda \atop \mu\nu} \right\} = \tfrac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \,, \tag{1.3}$$

and the second stands for the contortion tensor,

$$K^\lambda{}_{\cdot\mu\nu} = \tfrac{1}{2} (T^\lambda{}_{\cdot\mu\nu} - T_{\mu\cdot\nu}^{\cdot\lambda} - T_{\nu\cdot\mu}^{\cdot\lambda}) \tag{1.4}$$

with the torsion tensor

$$T^\lambda{}_{\cdot\mu\nu}(\Gamma) = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \,. \tag{1.5}$$

In terms of the affine connection the curvature tensor is given by

$$R^\rho{}_{\sigma\mu\nu}(\Gamma) = \partial_\mu \Gamma^\rho{}_{\sigma\nu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\lambda\mu} \Gamma^\lambda{}_{\sigma\nu} - \Gamma^\rho{}_{\lambda\nu} \Gamma^\lambda{}_{\sigma\mu} \,. \tag{1.6}$$

The Riemann-Cartan space-time has both the curvature tensor and the torsion tensor. From this space-time follow two very interesting models of space-time. One is the well-known Riemann space-time V_4, which is obtained from the U_4 by setting the torsion tensor to be identically vanishing. From (1.2) follows the Levi-Civita connection. It is well known that general relativity is the theory of gravitation on this space-time, and that it ascribes gravitation to the Riemann-Christoffel curvature tensor formed of the Levi-Civita connection.

Another interesting model is the Weitzenböck space-time A_4,[11] which is obtained from the U_4 by setting the curvature tensor to be identically vanishing,

$$R^\rho{}_{\sigma\mu\nu}(\Gamma^*) = 0 \,. \tag{1.7}$$

Or, to put it equivalently,[12] the Weitzenböck space-time is obtained by requiring the U_4 to admit absolute parallelism, i.e., to have a quadruplet (specified by $k=0,1,2,3$) of linearly independent *parallel vector fields*, $\underline{b}=\{\underline{b_k}\}=\{b_k{}^\mu\}$, which is defined by

$$D_\nu^* b_k{}^\lambda = \partial_\nu b_k{}^\lambda + \Gamma^{*\lambda}_{\mu\nu} b_k{}^\mu = 0\,. \tag{1.8}$$

Solving this equation we find the nonsymmetric affine connection,

$$\Gamma^{*\lambda}_{\mu\nu} = b_k{}^\lambda \partial_\nu b^k{}_\mu\,, \tag{1.9}$$

and the torsion tensor,

$$T^\lambda{}_{\cdot\mu\nu}(\Gamma^*) = b_k{}^\lambda(\partial_\nu b^k{}_\mu - \partial_\mu b^k{}_\nu)\,. \tag{1.10}$$

Here $\underline{b}^* = \{\underline{b^k}\} = \{b^k{}_\mu\}$ is also a quadruplet of parallel vector fields, which is inverse to $\underline{b}$. It is straightforward to see that the curvature tensor indeed vanishes identically [see (1.7)]. See Fig. 1 for reduction of the Riemann-Cartan space-time.

We will give the name, new general relativity, to the theory of gravitation on the Weitzenböck space-time, since Einstein in 1928, after inventing general relativity, considered absolute parallelism for the first time, and the main consequences of the present theory will be analogous to those of general relativity so far as macroscopic phenomena are concerned. New general relativity attributes gravitation to the torsion tensor formed of the parallel vector fields.

As is well known, general relativity is formulated by the following fundamental assumptions, which we will compare with those of new general relativity: (A) Underlying space-time is the Riemann space-time, which has the metric tensor as the basic structure. All physical laws are expressed by equations that are covariant or form invariant under the group of general coordinate transformations. (B) The equivalence principle. (C) Gravitational field equations are derivable from the action principle. (D) The field equations are partial differential equations in the field variables of not higher than the second order. (E) The gravitational field is exhaustively described by the metric tensor alone.

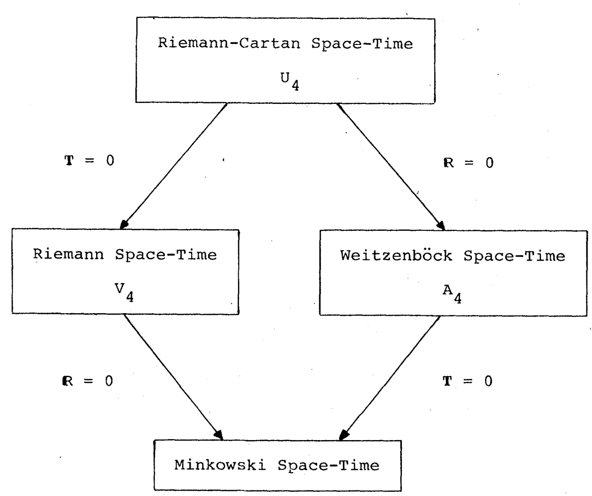

FIG. 1. The reduction of space-time is made in two particular cases: One is the Riemann space-time V_4 with a curvature tensor only ($\underline{R}$), and the other the Weitzenböck space-time A_4 with a torsion tensor alone ($\underline{T}$).

In new general relativity the fundamental assumptions are as follows: (A′) Underlying space-time is the Weitzenböck space-time, which has a quadruplet of the parallel vector fields as the fundamental structure. These parallel vector fields give rise to the metric tensor as a by-product. All physical laws are expressed by equations that are covariant or form invariant under the group of general coordinate transformations. (B′) The equivalence principle is valid only in classical physics. (C′) and (D′) are the same as (C) and (D), but at this time we start from the microscopic action principle. (E′) The gravitational field is exclusively described by a quadruplet of the parallel vector fields. As is closely related to (E′), we need to assume: (F′) All physical laws are expressed by equations that are covariant or form invariant under the group of *global* Lorentz transformations. When general relativity is extended to the domain of microscopic system, one must use tetrads and has to assume: (F) All physical laws are expressed by equations that are covariant or form invariant under the group of *local* Lorentz transformations.

We shall formulate new general relativity in the following manner: In Sec. II geometry of the Weitzenböck space-time is described in some detail, with emphasis on spinor wave functions defined in this space-time. In Sec. III microscopic matter Lagrangians are considered, such as of the electromagnetic field, of spin-$\frac{1}{2}$ fundamental particles and so forth. Their *equations of motion* are derived and then approximated by the WKB method to yield, in the classical limit, the *geodesics* of the metric $\underline{g}$,[13] along which point particles and light rays are defined to move. In Sec. IV a gravitational Lagrangian is constructed by the requirement of invariance under (1) the group of general coordinate transformations, (2) the group of global Lorentz transformations (3) the parity operation, and by the demand that (4) the Lagrangian be quadratic in the torsion tensor. Gravitational field equations are derived, with three unknown parameters, c_1, c_2, and c_3. In Sec. V a static, spherically symmetric field outside a massive neutral body is determined, with two parameters, c_1 and c_2; in this case a term proportional to c_3 is vanishing identically. In Sec. VI comparison with all the experiments so far carried out is made; firstly, we clarify how the equivalence principle is violated in microscopic systems only, and secondly, upper bounds for the parameters, c_1 and c_2, are obtained. In Sec. VII the free parameters, c_1 and c_2, are classified into two classes with c_3 arbitrary; $c_1=0=c_2$ and $c_1\neq 0\neq c_2$. (Other cases are forbidden.) The rest of the present paper concerns with the former choice of free parameters, and hence the static, isotropic field is given by the Schwarzschild solution. In this case new general relativity has only one free parameter, $\lambda=9/(4c_3)$, besides the Einstein gravitational constant κ. In Sec. VIII the group of local Lorentz transformations, which we do not assume, is seen to emerge as the dynamical symmetry group for a static, isotropic field. This new situation demands the extension of absolute parallelism. In Sec. IX, as microscopic applications, the weak-field approximation to gravitational field equations is performed. In Sec. X the coupling of an antisymmetric field to matter is discussed; it propagates in vacuum, mediating a long-range, spin-spin force among spin-$\frac{1}{2}$ fundamental particles with a coupling strength $\sqrt{\lambda}$, which is estimated by precise experimental values of quantum electrodynamics. In Sec. XI the Birkhoff theorem, that a spherically symmetric gravitational field in empty space must be static, with a metric given by the Schwarzschild solution, is proved in new general relativity. In Sec. XII we draw conclusions.

In our conventions the middle part of the Greek alphabet, $\mu,\nu,\lambda,\ldots$, refers to 0, 1, 2, and 3, while the initial part, $\alpha,\beta,\gamma,\ldots$, denotes 1, 2, and 3. In a similar way the middle part of the Latin alphabet, $i,j,k,\ldots$, means 0, 1, 2, and 3, while the initial part, $a,b,c,\ldots$, denotes 1, 2, and 3.

II. GEOMETRY OF THE WEITZENBÖCK SPACE-TIME

The space-time M is assumed to be a paracompact, Hausdorff, connected C^∞ four-dimensional manifold with a locally Lorentzian metric $\underline{g}$. Let U be a local coordinate neighborhood of $p\in M$ with local coordinates $x=\{x^\mu\}$, then we can introduce the coordinate basis $\underline{E}=\{\underline{E}_\mu\}=\{(\partial/\partial x^\mu)_p\}$ with $\mu=0$, 1, 2, and 3, and the dual basis $\underline{E}^*=\{\underline{E}^\mu\}=\{(dx^\mu)_p\}$. Every vector $\underline{V}$ at p can be written as $\underline{V}=V^\mu \underline{E}_\mu$. In particular the metric tensor $\underline{g}$ is written as

$$\underline{g}=g_{\mu\nu}\underline{E}^\mu\otimes\underline{E}^\nu\,, \tag{2.1}$$

where the metric components are simply the inner product of the coordinate basis vectors,

$$g_{\mu\nu}=g(\underline{E}_\mu,\underline{E}_\nu)=g(\underline{E}_\nu,\underline{E}_\mu)\,. \tag{2.2}$$

These components are used to raise and lower Greek indices.

By definition there exists a global system of four orthonormal vector fields $\underline{b}(p)=\{\underline{b}_i(p)\}$, such that

$$g(\underline{b}_i,\underline{b}_j)=g(\underline{b}_j,\underline{b}_i)=\eta_{ij}\,, \tag{2.3}$$

where $\eta=(\eta_{ij})=\mathrm{diag}(-1,+1,+1,+1)$. Thus the vector fields, $\underline{b}(p)=\{\underline{b}_i(p)\}$, are expressed in the old basis by

$$\underline{b}_i=b_i{}^\mu\underline{E}_\mu\,; \tag{2.4a}$$

equivalently, a global system of four orthonormal vector fields $\underline{b}^*(p)=\{\underline{b}^i(p)\}$, which are dual to $\underline{b}(p)=\{\underline{b}_i(p)\}$, is written in the old basis by

$$\underline{b}^i=b^i{}_\mu\underline{E}^\mu\,. \tag{2.4b}$$

Conversely, it follows that

$$\underline{E}_\mu=b^i{}_\mu\underline{b}_i\,, \tag{2.4c}$$

$$\underline{E}^\mu=b_i{}^\mu\underline{b}^i\,. \tag{2.4d}$$

Here the coefficients, $\{b_i{}^\mu\}$ or $\{b^i{}_\mu\}$, are 16 functions, and must staisfy

$$b_i{}^\mu b^i{}_\nu=\delta^\mu{}_\nu\,,\quad b_i{}^\mu b^j{}_\mu=\delta^j{}_i\,, \tag{2.5a}$$

$$b^i{}_\mu\eta_{ij}b^j{}_\nu=g_{\mu\nu}\,,\quad b_i{}^\mu g_{\mu\nu}b_j{}^\nu=\eta_{ij}\,. \tag{2.5b}$$

It should be remarked that Latin indices of $\underline{b}=\{\underline{b}_i\}$ and $\underline{b}^*=\{\underline{b}^i\}$ mean that they are Lorentz vectors; $\underline{b}$ is the covariant vector and $\underline{b}^*$ is the contravariant vector. From (2.4) it follows that for any vector $\underline{V}=V^\mu\underline{E}_\mu=V^i\underline{b}_i$ the components satisfy

$$V^\mu=b_i{}^\mu V^i\,,\quad V^i=b^i{}_\mu V^\mu\,. \tag{2.6}$$

This rule of converting Greek to Latin indices and vice versa is applied to any tensor of higher rank.

Now, in the Weitzenböck space-time the covariant derivative, denoted by $\underline{D}^*$, defines *absolute parallelism* with respect to the global system of the four orthonormal vector fields $\underline{b}$. By definition $\underline{D}^*$ satisfies

$$\underline{D}^*\underline{b}_i=0\,, \tag{2.7a}$$

or in the coefficient form,

$$D^*_\nu b_i{}^\lambda\equiv\partial_\nu b_i{}^\lambda+\Gamma^{*\lambda}_{\mu\nu}b_i{}^\mu=0\,. \tag{2.7b}$$

From (2.4), (2.5), and (2.7) we find

$$D^*_\nu\underline{E}_\mu=\Gamma^{*\lambda}_{\mu\nu}\underline{E}_\lambda \tag{2.8}$$

with the affine connection of absolute parallelism,

$$\Gamma^{*\lambda}_{\mu\nu}=b_i{}^\lambda\partial_\nu b^i{}_\mu=-b^i{}_\mu\partial_\nu b_i{}^\lambda\,. \tag{2.9}$$

Here the global system of the four orthonormal vector fields is called *a quadruplet of the parallel vector fields*, or simply the *parallel vector fields*. For a vector field $\underline{V}(x)=V^i(x)\underline{b}_i=V^\mu(x)\underline{E}_\mu$, the covariant derivative is given by

$$\underline{D}^*\underline{V} = (D^*_\nu V^i)\underline{E}^\nu \otimes \underline{b}_i = (D^*_\nu V^\mu)\underline{E}^\nu \otimes \underline{E}_\mu , \qquad (2.10)$$

where

$$D^*_\nu V^i = \partial_\nu V^i , \qquad (2.11)$$

$$D^*_\nu V^\mu = \partial_\nu V^\mu + \Gamma^{*\mu}_{\lambda\nu} V^\lambda . \qquad (2.12)$$

Thus, for the components V^i with respect to a quadruplet of the parallel vector fields, the covariant derivative coincides with the usual derivative.

In the Weitzenböck space-time absolute parallelism of vectors at different points of M is defined in the following way: Consider a vector $\underline{V}(p) = V^i \underline{b}_i(p)$ at p and a vector $\underline{W}(q) = W^i \underline{b}_i(q)$ at q, where the point q can be arbitrarily separated from p. The parallelism of $\underline{V}$ and $\underline{W}$ is manifest: If their components are equal with each other,

$$V^i = W^i , \qquad (2.13)$$

then the two vectors, $\underline{V}(p)$ and $\underline{W}(q)$, are parallel with each other and of equal length.

In passing we make the remark that Latin indices are used to denote components with respect to a quadruplet of the parallel vector fields, and are raised and lowered by the Minkowski metric tensor, $\{\eta_{ij}\}$ or $\{\eta^{ij}\}$.

The affine connection, $\Gamma^* = \{\Gamma^{*\lambda}_{\mu\nu}\}$, is not symmetric with respect to the exchange of lower two indices. The torsion tensor is given by

$$\Gamma^\lambda{}_{\cdot\mu\nu} \equiv \Gamma^{*\lambda}_{\mu\nu} - \Gamma^{*\lambda}_{\nu\mu} = b_i{}^\lambda (\partial_\nu b^i{}_\mu - \partial_\mu b^i{}_\nu) . \qquad (2.14)$$

The curvature tensor formed of Γ^* identically vanishes [see (1.7)], since parallel transfer of a vector is path independent owing to absolute parallelism. Thus the Weitzenböck space-time is characterized by the torsion tensor alone, and reduces to the Minkowski space-time provided the torsion tensor vanishes globally. See Fig. 1 for reduction of the Riemann-Cartan space-time. *In the Minkowski space-time the parallel vector fields, which define absolute parallelism, coincide with the coordinate basis of a Cartesian coordinate system.*

When a quadruplet of the parallel vector fields $\underline{b}$ is subject to a global, proper, orthochronous Lorentz transformation,

$$\underline{b}_i = A^j{}_i \underline{b}'_j , \qquad (2.15a)$$

$$A^j{}_i \eta_{jm} A^m{}_n = \eta_{in} , \quad \det A = 1 ,$$

$$A^0{}_0 \geq 1 , \quad \partial_\nu A^j{}_i = 0 , \qquad (2.15b)$$

new absolute parallelism defined by new parallel vector fields $\underline{b}'$ is equivalent to the original one. So geometry of the Weitzenböck space-time is invariant under the global, proper, orthochronous Lorentz group, $L^\uparrow_+ \equiv \{A = (A^j{}_i) \in \mathrm{GL}(4,R),\ A^j{}_i \eta_{jm} A^m{}_n = \eta_{in},\ \det A = 1,\ A^0{}_0 \geq 1,\ \partial_\nu A^j{}_i = 0\}$. In conformity with this invariance of underlying geometry, we demand that *physical laws should be invariant under the action of* $L^\uparrow_+$. We call this the global Lorentz invariance.

In the Weitzenböck space-time, spinors are introduced as quantities which transform like two-valued representations of the proper, orthochronous Lorentz group $L^\uparrow_+$.[14] Most elementary spinors are four kinds of two-component spinors, i.e., contravariant spinor $\{\xi^A\}$, covariant spinor $\{\chi_A\}$, dotted contravariant spinor $\{\xi^{\dot A}\}$, and dotted covariant spinor $\{\chi_{\dot A}\}$ for $A = 1$ and 2. Dotted spinors transform like the complex conjugate of undotted spinors. A spinor of higher rank is a quantity which transforms like a direct product of two-component spinors. A vector $\underline{V} = V^i \underline{b}_i$ is identified with a mixed spinor of second rank with components $V^{\dot A B}$,

$$V^{\dot A B} = \Sigma_i{}^{\dot A B} V^i , \qquad (2.16)$$

where $\{\Sigma_i{}^{\dot A B}\}$ is a set of Hermitian 2×2 matrices satisfying

$$\Sigma_i{}^{\dot A B} \epsilon_{\dot A \dot C} \Sigma_j{}^{\dot C D} + \Sigma_j{}^{\dot A B} \epsilon_{\dot A \dot C} \Sigma_i{}^{\dot C D} = -\eta_{ij} \epsilon^{BD} , \qquad (2.17)$$

where

$$(\epsilon_{\dot A \dot C}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\epsilon^{AC}) . \qquad (2.18)$$

One of the simplest choices, which we take in this paper, is

$$(\Sigma_0{}^{\dot A B}) = -I/\sqrt{2} , \quad (\Sigma_a{}^{\dot A B}) = \sigma_a/\sqrt{2} , \qquad (2.19)$$

where $\{\sigma_1, \sigma_2, \sigma_3\}$ is a set of the Pauli matrices.

The four-component Dirac spinor ψ is defined by a direct sum of a covariant spinor and a dotted contravariant spinor, and is written as a single column matrix

$$\psi = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \xi^{\dot 1} \\ \xi^{\dot 2} \end{pmatrix} \equiv \begin{pmatrix} \chi \\ \xi \end{pmatrix} . \qquad (2.20)$$

The conjugate Dirac spinor $\bar\psi$ is obtained from ψ by

$$\bar\psi = \psi^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = (\xi^1, \xi^2, \chi_{\dot 1}, \chi_{\dot 2}) . \qquad (2.21)$$

Now we extend the definition of absolute parallelism to include spinors. Consider a spinor at p, say a contravariant two-component spinor $\{\xi^A(p)\}$, and another spinor at q of the same type, say, $\{\xi^A(q)\}$. If components of these spinors are equal, i.e., $\xi^A(p) = \xi^A(q)$, then two spinors are defined to be parallel and of the same magnitude. From (2.16) it follows that absolute parallelism of spinors implies absolute parallelism of vectors and tensors: In fact, for two vectors $\underline{V}$ at point p and $\underline{W}$ at another point q, equality of spinor components, $V^{\dot{A}B}(p)=W^{\dot{A}B}(q)$, implies $V^i(p)=W^i(q)$, because $\{\Sigma_i{}^{\dot{A}B}\}$ is independent of space-time position.

When a spinor at point p is parallel transferred to another point q, its components are kept unchanged owing to absolute parallelism. Therefore, *the covariant derivative* D^*_μ *of spinors coincides with the usual derivative* ∂_μ.

Finally, we make the following important remark: The parallel vector fields $\underline{b}$ are different from the so-called tetrad fields $\underline{e}$ by an arbitrary, position-dependent Lorentz transformation, which is called a local Lorentz transformation.

III. MATTER LAGRANGIAN AND EQUATIONS OF MOTION FOR TEST PARTICLES

A. Matter Lagrangian

In new general relativity we do not identify the six extra degrees of freedom of the parallel vector fields with the electromagnetic field strength, since we now know that such an attempt failed.[2] Instead, we take the electromagnetic potential $\underline{A}=\{A_\mu\}$ as the dynamical variable independent of the parallel vector fields. The matter part of the action is then represented as a sum of the action of fundamental particles and fields, i.e., of the electromagnetic field and several kinds of spin-$\frac{1}{2}$ fundamental particles;

$$I_M=\int d^4x\sqrt{-g}\,L_M$$
$$=\int d^4x\sqrt{-g}\left(L_{\text{em}}+\sum_i L_D^{(i)}+L_{\text{int}}\right), \qquad (3.1)$$

where g is

$$g=\det(g_{\mu\nu})=-[\det(b^i{}_\mu)]^2<0\,, \qquad (3.2)$$

and L_{int} represents nongravitational interactions among fundamental particles and fields. Here the index i in the second term labels spin-$\frac{1}{2}$ fundamental particles such as quarks, electrons, muons, electron-neutrinos, muon-neutrinos, etc., all of which can be described in fairly good approximation by spinor wave functions obeying the Dirac equation. If there exist other fundamental fields besides the electromagnetic field, their action must be added to (3.1): Gauge fields for internal symmetry of fundamental particles, if they exist, can be included in (3.1) in a similar manner to the electromagnetic field.

The gauge invariance of the electromagnetic interaction shall be assumed to hold in new general relativity, because this invariance plays the fundamental role in quantum electrodynamics. The electromagnetic Lagrangian density L_{em} is then given by

$$L_{\text{em}}=-\tfrac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \qquad (3.3)$$

with

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,, \qquad (3.4)$$

which is of the same form as the electromagnetic Lagrangian density used in general relativity.

Absolute parallelism is applied to spinor wave functions of fundamental spin-$\frac{1}{2}$ particles, and the Dirac Lagrangian density L_D is given by[15]

$$L_D=\tfrac{1}{2}i\hbar b_k{}^\mu[\bar\psi\gamma^k D^*_\mu\psi-(D^*_\mu\bar\psi)\gamma^k\psi]-m\bar\psi\psi\,. \qquad (3.5a)$$

In this paper we use the unit, $\hbar=c=1$, but throughout this section we write $\hbar$ explicity for convenience of taking the semiclassical limit. For spinors the covariant derivative D^*_μ coincides with the usual derivative

$$D^*_\mu\psi=\partial_\mu\psi\,. \qquad (3.6a)$$

If we use the covariant derivative ∇_μ of general relativity,

$$\nabla_\mu\psi=(\partial_\mu+\tfrac{1}{2}i\Delta_{ij\mu}S^{ij})\psi\,, \qquad (3.6b)$$

with respect to the Ricci rotation coefficients $\{\Delta_{ij\mu}\}$,

$$\Delta_{ijk}=b_k{}^\mu\Delta_{ij\mu}=-\tfrac{1}{2}(T_{ijk}-T_{jik}-T_{kij})\,, \qquad (3.7)$$

then L_D of (3.5a) can be rewritten as

$$L_D=\tfrac{1}{2}i\hbar b_k{}^\mu[\bar\psi\gamma^k\nabla_\mu\psi-(\nabla_\mu\bar\psi)\gamma^k\psi]$$
$$-\tfrac{3}{4}\hbar a_k\bar\psi\gamma^5\gamma^k\psi-m\bar\psi\psi\,, \qquad (3.5b)$$

where $\{a^\mu\}$ is the axial-vector part of the torsion tensor,

$$a^\mu=b_k{}^\mu a^k=\tfrac{1}{6}\epsilon^{\mu\nu\rho\sigma}T_{\nu\rho\sigma}\,. \qquad (3.8)$$

This covariant derivative ∇_μ, when applied to tensors whose indices are written in Greek, becomes the usual covariant derivative with respect to the Levi-Civita connection (1.3). Here the completely antisymmetric tensors, $\epsilon=\{\epsilon^{\mu\nu\rho\sigma}\}$ and $\epsilon^*=\{\epsilon_{\mu\nu\rho\sigma}\}$, with respect to the coordinate basis are defined by[16]

$$\epsilon^{\mu\nu\rho\sigma}=(1/\sqrt{-g}\,)\delta^{\mu\nu\rho\sigma}\,,$$
$$\epsilon_{\mu\nu\rho\sigma}=\sqrt{-g}\,\delta_{\mu\nu\rho\sigma}\,, \qquad (3.9)$$

where $\delta=\{\delta^{\mu\nu\rho\sigma}\}$ and $\delta^*=\{\delta_{\mu\nu\rho\sigma}\}$ are the completely antisymmetric tensor densities of weight -1 and $+1$, respectively, with normalizations $\delta^{0123}=+1$ and $\delta_{0123}=-1$. So the completely antisymmetric tensors with respect to the parallel vector fields are defined by

$$\epsilon^{ijmn} = b^i{}_\mu b^j{}_\nu b^m{}_\rho b^n{}_\sigma \epsilon^{\mu\nu\rho\sigma} ,$$
$$\epsilon_{ijmn} = b_i{}^\mu b_j{}^\nu b_m{}^\rho b_n{}^\sigma \epsilon_{\mu\nu\rho\sigma} , \tag{3.10}$$

where $\epsilon^{(0)(1)(2)(3)} = +1$ and $\epsilon_{(0)(1)(2)(3)} = -1$ with Lorentz (Latin) indices in parentheses.

Variation of L_M with respect to A_μ gives

$$\partial_\nu(\sqrt{-g}\, F^{\mu\nu}) = \sqrt{-g}\, j^\mu , \tag{3.11}$$

where the electromagnetic current is defined by

$$j^\mu = \frac{\delta}{\delta A_\mu} L_{\text{int}} . \tag{3.12}$$

Equation (3.11) is just the Maxwell equation in general relativity, and hence the law of electromagnetism is entirely free from the influence of absolute parallelism. In space-time with a given background metric g, electromagnetic waves propagate in the same manner as in general relativity: In the short-wavelength limit, in particular, light rays propagate along the null geodesics of the metric g.

The Dirac equation is derived from L_D by taking variation with respect to $\bar{\psi}$,

$$[i\hbar b_k{}^\mu \gamma^k (D^*_\mu + \tfrac{1}{2} v_\mu) - m]\, \psi = 0 , \tag{3.13a}$$

or equivalently,

$$(i\hbar b_k{}^\mu \gamma^k \nabla_\mu - \tfrac{3}{4}\hbar a_k \gamma^5 \gamma^k - m)\psi = 0 , \tag{3.13b}$$

where $\{v_\mu\}$ is the vector part of the torsion tensor,

$$v_\mu = T^\lambda{}_{\cdot\lambda\mu} , \tag{3.14}$$

and only the gravitational interaction is included.

B. Equations of motion for massive Dirac particles

We shall derive *two equations of motion* for a freely falling Dirac particle, i.e., *the equation of orbit* and *the equation of spin precession*,[17] by applying the WKB approximation method[18] to the Dirac equation (3.13).

The particle of spin $\frac{1}{2}$ is usually represented by a four-component spinor wave function obeying the first-order Dirac equation. However, it is well known that it can equally well be described by a two-component spinor wave function obeying a second-order wave equation.[19] So there are two equivalent ways to take the classical limit for the particle of spin $\frac{1}{2}$, in accordance with which a wave equation is considered; a first-order wave equation or a second-order one.

For our present purpose of deriving the spin equation in addition to the orbit equation, it is much more convenient to start from a second-order wave equation rather than from the Dirac equation (3.13). We thus introduce a two-component spinor wave function ϕ by[20]

$$\phi = \tfrac{1}{2}(1 + \gamma^5)\psi . \tag{3.15}$$

Then, by means of the Dirac equation (3.13), this can be "solved" with respect to ψ,

$$\psi = [1 + (i/m)\hbar b_k{}^\mu \gamma^k (D^*_\mu + \tfrac{1}{2} v_\mu)]\, \phi , \tag{3.16}$$

and so we find that ϕ satisfies the second-order wave equation,

$$[i\hbar b_j{}^\mu \gamma^j (D^*_\mu + \tfrac{1}{2} v_\mu) - m] \times [i\hbar b_k{}^\nu \gamma^k (D^*_\nu + \tfrac{1}{2} v_\nu) + m]\, \phi = 0 , \tag{3.17}$$

to which we apply the WKB approximation method.

We seek a semiclassical solution of (3.17) with the following form:

$$\phi = \exp\!\left(\frac{i}{\hbar} S\right) \phi_0 , \tag{3.18}$$

by assuming that $\hbar$ is very small compared to S. Using (3.18) in (3.17) and then putting each order of $(\hbar/i)$ to zero, we find (up to the first order)

$$(\hbar/i)^0 :\ g^{\mu\nu}(\partial_\mu S)(\partial_\nu S) + m^2 = 0 , \tag{3.19}$$

$$(\hbar/i)^1 :\ \{2g^{\mu\nu}(\partial_\mu S)(D^*_\nu + \tfrac{1}{2} v_\nu) - b_j{}^\mu b_k{}^\nu [D^*_\mu(\partial_\nu S)] \gamma^j \gamma^k\}\, \phi_0 = 0 . \tag{3.20a}$$

The last equation is rewritten as

$$\{2g^{\mu\nu}(\partial_\mu S)\nabla_\nu + g^{\mu\nu}[\nabla_\mu(\partial_\nu S)] + \tfrac{3}{2} i \epsilon_{ijmn} b^{m\mu}(\partial_\mu S) a^n S^{ij}\}\phi_0 = 0 \tag{3.20b}$$

with help of the relation between D^*_ν and ∇_ν,

$$D^*_\nu \phi_0 = (\nabla_\nu + \tfrac{1}{2} i K_{ij\nu} S^{ij})\phi_0 ,$$
$$D^*_\nu(\partial_\mu S) = \nabla_\nu(\partial_\mu S) - K^\lambda{}_{\cdot\mu\nu} \partial_\lambda S , \tag{3.21}$$

with $\{K_{ij\nu}\}$ and $\{K_{\lambda\mu\nu}\}$ being the contortion tensor defined by

$$\begin{aligned} K_{ij\nu} &= b_i{}^\lambda b_j{}^\mu K_{\lambda\mu\nu} \\ &= \tfrac{1}{2} b_i{}^\lambda b_j{}^\mu (T_{\lambda\mu\nu} - T_{\mu\lambda\nu} - T_{\nu\lambda\mu}) \\ &= -\Delta_{ij\nu} . \end{aligned} \tag{3.22}$$

The applicability condition of the semiclassical solution (3.18) is that when it is used in (3.17) the terms of order $(\hbar/i)^0$ are much larger than those of order $(\hbar/i)^1$. Estimating $|\partial_\mu S|/\hbar \sim 1/(\text{wavelength}) \equiv 1/\lambda$, $D^*_\nu \phi_0 \sim \phi_0/w$ with w the width of the wave packet, and $|D^*_\nu(\partial_\mu S)| \sim \hbar/\lambda L$ with L being the distance over which the parallel vector fields $\{b^k{}_\mu\}$ vary considerably, we obtain the following inequality:

$$L \gg \lambda, \quad w \gg \lambda . \tag{3.23}$$

Equation (3.19) is the Hamilton-Jacobi equation which describes the motion of freely falling particles in general relativity.[21] The complete solution $S(x; \alpha_1, \alpha_2, \alpha_3)$ with three free parameters, α_1, α_2, and α_3, determines the classical orbit by

$$\frac{\partial S}{\partial \alpha_a} = \beta_a (= \text{const}), \quad (a = 1, 2, 3) . \tag{3.24}$$

When the trajectory $x^\mu(\tau)$ defined by (3.24) is parametrized by the proper time τ, it satisfies the geodesic equation,

$$\frac{dx^\mu}{d\tau}=\frac{1}{m}g^{\mu\nu}\partial_\nu S\equiv U^\mu \text{ (four-velocity)}, \tag{3.25}$$

$$\frac{d^2x^\mu}{d\tau^2}+\left\{{\mu \atop \lambda\nu}\right\}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau}=0. \tag{3.26}$$

Given the solution $S(x;\alpha_1,\alpha_2,\alpha_3)$ of (3.19), Eq. (3.20) can be solved to define the spinor wave function ϕ_0 in terms of S. By virtue of (3.16), the semiclassical expression for the Dirac spinor wave function in terms of S and ϕ_0 is given by

$$\psi=\exp\left(\frac{i}{\hbar}S\right)\psi_0, \tag{3.27}$$

$$\psi_0=\left(1-\frac{\partial_\mu S}{m}b_k{}^\mu\gamma^k\right)\phi_0$$
$$=(1-b_k{}^\mu U_\mu\gamma^k)\phi_0. \tag{3.28}$$

The probability current, $j^\mu=b_k{}^\mu\bar\psi\gamma^k\psi$, then takes the following form in the semiclassical approximation:

$$j^\mu=\rho U^\mu, \tag{3.29}$$

where ρ is defined by

$$\rho=-2b_k{}^\mu U_\mu\bar\phi_0\gamma^k\phi_0. \tag{3.30}$$

Equation (3.20b) of ϕ_0 ensures that j^μ satisfies the continuity equation,

$$\nabla_\mu j^\mu=\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,j^\mu)=0. \tag{3.31}$$

The expression (3.29) for the probability current shows that, in the semiclassical approximation, the probability may be regarded as following along the classical trajectory.

We can form a wave packet by superposing the solutions of (3.27) with different values of parameters, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$:

$$\psi(x)=\int\rho(\alpha)d^3\alpha\exp\left[\frac{i}{\hbar}S(x;\alpha)\right]\psi_0(x;\alpha), \tag{3.32a}$$

where $\rho(\alpha)$ is a weight function with a "sharp peak" of width $\Delta\alpha$ at $\alpha=\bar\alpha\equiv(\bar\alpha_1,\bar\alpha_2,\bar\alpha_3)$. Here a sharp peak means that the following conditions are satisfied: (1) The ratio $\hbar/\Delta\alpha$ is negligibly small compared to the macroscopic scale, and (2) the ratio $\hbar/(\Delta\alpha)^2$ satisfies the inequality

$$\frac{\hbar}{(\Delta\alpha)^2}\gg\frac{\partial^2 S}{\partial\alpha_a\partial\alpha_b}. \tag{3.32b}$$

Since $\psi_0(x;\alpha)$ is not highly oscillating with respect to α, it follows from the inequality (3.32b) that Eq. (3.32a) can be rewritten as

$$\psi(x)=\exp\left[\frac{i}{\hbar}S(x;\bar\alpha)\right]\psi_0(x;\bar\alpha)\times\int\rho(\alpha)d^3\alpha\exp\left[\frac{i}{\hbar}\sum_a(\alpha_a-\bar\alpha_a)\frac{\partial S}{\partial\bar\alpha_a}\right]. \tag{3.32c}$$

In the integral of (3.32c) compensation takes place almost everywhere except in the space-time region satisfying

$$\frac{\partial S}{\partial\bar\alpha_a}\lesssim\frac{\hbar}{\Delta\alpha}. \tag{3.33a}$$

According to the condition (1) stated above, the right-hand side of (3.33a) is negligibly small compared to the macroscopic scale. Therefore, in the macroscopic scale, the wave packet (3.32c) has nonvanishing amplitude only along a world line $x^\mu(\tau)$ defined by

$$\frac{\partial S}{\partial\bar\alpha_a}=0. \tag{3.33b}$$

Here τ is the proper time along the world line. *The wave packet thus propagates along a classical trajectory $x^\mu(\tau)$ satisfying the geodesic equation (3.26).*

Now we turn our consideration to the motion of spin for a spin-$\frac{1}{2}$ particle described by the spinor wave function (3.32c). The spin polarization is described by the spinor wave function $\psi_0(x(\tau);\bar\alpha)$, since other two factors of (3.32c) are scalar functions which have nothing to do with intrinsic spin polarization. We introduce a new spinor wave function $\psi_0'(x(\tau);\bar\alpha)$ by

$$\psi_0'=\frac{1}{\sqrt\rho}\psi_0. \tag{3.34a}$$

Then it is normalized like

$$b_k{}^\mu\bar\psi_0'\gamma^k\psi_0'=U^\mu \text{ (four-velocity)}, \tag{3.34b}$$

and accordingly it can be taken as the normalized spinor wave function which describes spin polarization. From (3.20a)–(3.20b) and (3.28)–(3.31), it follows after a little algebra that the normalized spinor wave function ψ_0' satisfies

$$\left(\frac{D^*}{d\tau}+iU^\mu K_{\mu ij}S^{ij}\right)\psi_0'=0, \tag{3.35a}$$

or equivalently,

$$\left(\frac{\nabla}{d\tau}+\frac{3i}{4}\epsilon_{ijmn}b^m{}_\mu U^\mu a^n S^{ij}\right)\psi_0'=0, \tag{3.35b}$$

where $D^*/d\tau$ and $\nabla/d\tau$ mean covariant differentiation along a classical trajectory $x^\mu(\tau)$, $D^*/d\tau=U^\mu D^*_\mu$ and $\nabla/d\tau=U^\mu\nabla_\mu$, respectively. Equations (3.35) describe the temporal change of the spin state as a spin-$\frac{1}{2}$ particle moves along a classical trajectory $x^\mu(\tau)$. The second term of (3.35b) represents the effect of absolute parallelism on the spin precession in new general relativity.

We define the spin vector $\{S^\mu\}$ by

$$S^\mu = -\tfrac{1}{2} b_k{}^\mu \bar{\psi}_0' \gamma^5 \gamma^k \psi_0', \tag{3.36a}$$

which has only three independent components, because (3.28) and (3.34a) give

$$U^\mu S_\mu = 0. \tag{3.36b}$$

It follows from (3.35) that $\{S^\mu\}$ satisfies

$$\frac{D^*}{d\tau} S^\mu = -(K^{\lambda\mu\nu} - K^{\lambda\nu\mu}) U_\lambda S_\nu, \tag{3.37a}$$

or equivalently,

$$\frac{\nabla}{d\tau} S^\mu = -\tfrac{3}{2} \epsilon^{\mu\nu\rho\sigma} U_\nu a_\rho S_\sigma. \tag{3.37b}$$

These are the classical equations of spin precession in new general relativity. The right-hand side of (3.37b) represents the effect of absolute parallelism. When the axial-vector part of the torsion tensor vanishes, (3.37b) reduces to the equation of spin precession in general relativity. For a nonrelativistic particle in a weak gravitational field,[22] (3.37b) becomes

$$\frac{d\vec{S}}{dt} = -\tfrac{3}{2} \vec{a} \times \vec{S}, \tag{3.38}$$

where $\vec{S}$ and $\vec{a}$ are the space components of $\{S^\mu\}$ and $\{a^\mu\}$, respectively.

C. Equations of motion for neutrinos and antineutrinos

Neutrinos and antineutrinos are described by two-component spinor wave functions which are obtained from (four-component) Dirac spinor wave functions of (2.20) by putting $\chi = 0$ for neutrinos and $\xi = 0$ for antineutrinos, respectively. We shall consider only antineutrinos, since neutrinos can be treated in a similar manner. For antineutrinos, the Dirac equation (3.13a) becomes the Weyl equation for a right-handed massless particle,[23]

$$i\hbar b_k{}^\mu \sigma^k (D_\mu^* + \tfrac{1}{2} v_\mu)\chi = 0. \tag{3.39}$$

The semiclassical solution, $\chi = \exp(iS/\hbar)\chi_0$, must satisfy

$$b_k{}^\mu (\partial_\mu S) \sigma^k \chi_0 = 0, \tag{3.40}$$

and hence we get

$$\det(b_k{}^\mu \partial_\mu S \sigma^k) = -g^{\mu\nu}(\partial_\mu S)(\partial_\nu S) = 0, \tag{3.41}$$

which is just the Hamilton-Jacobi equation for a massless particle in general relativity. The classical trajectory $x^\mu(\sigma)$ defined by (3.24) satisfies the following equations:

$$\frac{dx^\mu}{d\sigma} = g^{\mu\nu}\partial_\nu S \equiv p^\mu \text{ (four-momentum)}, \tag{3.42}$$

$$\frac{d^2x^\mu}{d\sigma^2} + \left\{ {\mu \atop \lambda\nu} \right\} \frac{dx^\lambda}{d\sigma} \frac{dx^\nu}{d\sigma} = 0, \tag{3.43}$$

where σ is the affine parameter along the trajectory: The normalization of σ is fixed by (3.42).

The four momentum $\{p^\mu\}$ is null, due to (3.41). The current $\{j^\mu = b_k{}^\mu \chi^\dagger \sigma^k \chi\}$ is also null, because χ is a two-component spinor. Since these two null vectors, $\{p^\mu\}$ and $\{j^\mu\}$, are orthogonal with each other by virtue of (3.40), they must be proportional to one another,

$$j^\mu = \rho p^\mu, \tag{3.44}$$

where ρ is a positive-definite scalar function. We can apply the same argument as for the massive case, to show that *antineutrinos move along the classical trajectory satisfying (3.43) in the short-wavelength limit.*

We take $\chi_0/\sqrt{\rho}$ as the normalized spinor wave function for antineutrinos. The spin vector (3.36a) then becomes

$$S^\mu = \frac{1}{2\rho} b_k{}^\mu \chi_0^\dagger \sigma^k \chi_0 = \frac{1}{2\rho} j^\mu = \tfrac{1}{2} p^\mu, \tag{3.45}$$

showing that antineutrinos are of helicity $+\frac{1}{2}$ as they should. Therefore, *the classical equations of motion for neutrinos and antineutrinos in new general relativity are the same as those in general relativity.*

IV. GRAVITATIONAL LAGRANGIAN IN VACUUM

We shall construct a gravitational Lagrangian density in vacuum,

$$I_G = \int d^4x \sqrt{-g} L_G. \tag{4.1}$$

For this purpose we enumerate the basic postulates which the above action must obey:

(1) Invariance under the group of general coordinate transformations; for arbitrary change of coordinates the parallel vector fields transform like

$$b'_k{}^\mu(x') = (\partial x'^\mu / \partial x^\nu) b_k{}^\nu(x). \tag{4.2}$$

(2) Invariance under the group of global, proper, orthochronous Lorentz transformations $L_+^\uparrow$; for its element $A = (A^j{}_k)$ with $A^t \eta A = \eta$, $\det A = 1$, $A^0{}_0 \geq 1$, and $\partial_\mu A = 0$, the parallel vector fields change like

$$b_k{}^\mu(x) = A^j{}_k b'_j{}^\mu(x). \tag{4.3}$$

(3) L_G be invariant under the parity operation; by parity operation we mean the Lorentz transformation, $\underline{b}_{(0)} \to \underline{b}_{(0)}$ and $\underline{b}_{(a)} \to -\underline{b}_{(a)}$, where Lorentz

indices are enclosed by parentheses.

(4) L_G be quadratic terms in the torsion tensor, besides a cosmological term.

Now, the torsion tensor is given by

$$T^{\lambda}_{\cdot\mu\nu}=b_k{}^{\lambda}(\partial_\nu b^k{}_\mu-\partial_\mu b^k{}_\nu)\,, \tag{4.4}$$

which is *reducible* with respect to the group of global Lorentz transformations. It is convenient to perform an *irreducible* decomposition, from which we can construct a gravitational Lagrangian density. The torsion tensor is decomposed into three irreducible parts under this group[24a]:

$$t_{\lambda\mu\nu}=\tfrac{1}{2}(T_{\lambda\mu\nu}+T_{\mu\lambda\nu})+\tfrac{1}{6}(g_{\nu\lambda}v_\mu+g_{\nu\mu}v_\lambda)-\tfrac{1}{3}g_{\lambda\mu}v_\nu\,, \tag{4.5}$$

$$v_\mu=T^{\lambda}_{\cdot\lambda\mu}\,, \tag{4.6}$$

$$a_\mu=\tfrac{1}{6}\epsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma}\,, \tag{4.7}$$

where $\epsilon^*=\{\epsilon_{\mu\nu\rho\sigma}\}$ is the completely antisymmetric tensor, introduced in (3.9). In fact, let $\rho(m,n)$ be an irreducible representation of the proper orthochronous Lorentz group, where $2m$ and $2n$ take non-negative integer numbers.[24b] Then the tensor $\{t_{\lambda\mu\nu}\}$ transforms according to $\rho(\frac{3}{2},\frac{1}{2})\oplus\rho(\frac{1}{2},\frac{3}{2})$ of 16 dimensions (the Young table [21] minus traces), the vector $\{v_\mu\}$ according to $\rho(\frac{1}{2},\frac{1}{2})$ of 4 dimensions, and finally the axial-vector $\{a_\mu\}$ according to $\rho(\frac{1}{2},\frac{1}{2})$ of 4 dimensions (the Young table [111]). The torsion tensor is conversely written in terms of the three irreducibles,

$$T_{\lambda\mu\nu}=\tfrac{2}{3}(t_{\lambda\mu\nu}-t_{\lambda\nu\mu})+\tfrac{1}{3}(g_{\lambda\mu}v_\nu-g_{\lambda\nu}v_\mu)+\epsilon_{\lambda\mu\nu\rho}a^\rho\,. \tag{4.8}$$

The tensor $\{t_{\lambda\mu\nu}\}$ has the following properties derived from the defining equation (4.5):

$$t_{\lambda\mu\nu}=t_{\mu\lambda\nu}\,, \tag{4.9}$$

$$g^{\mu\nu}t_{\lambda\mu\nu}=0=g^{\lambda\mu}t_{\lambda\mu\nu}\,, \tag{4.10}$$

$$t_{\lambda\mu\nu}+t_{\mu\nu\lambda}+t_{\nu\lambda\mu}=0\,. \tag{4.11}$$

The above postulates of (1) to (4) require that the most general Lagrangian density be of the form

$$L_G=a_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu})+a_2(v^\mu v_\mu)+a_3(a^\mu a_\mu)+a_0\,, \tag{4.12}$$

where a_1, a_2, and a_3 are free parameters, while a_0 is a cosmological term.

In Appendix A we will treat a case of lifting up the postulate of (3), by adding to (4.12) parity-violating terms like $(v^\mu a_\mu)$ and $(\epsilon_{\mu\nu\rho\sigma}t^{\lambda\mu\nu}t_{\lambda}^{\cdot\rho\sigma})$. In the rest of this paper we shall neglect the cosmological term, so we have the gravitational action of

$$I_G=\int d^4x\sqrt{-g}\,L_G=\int d^4x\sqrt{-g}\,[a_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu})+a_2(v^\mu v_\mu)+a_3(a^\mu a_\mu)]\,. \tag{4.13}$$

Next we observe the identity

$$\int d^4x\sqrt{-g}\,R(\{\ \})=\int d^4x\sqrt{-g}\,[-\tfrac{2}{3}(t^{\lambda\mu\nu}t_{\lambda\mu\nu})+\tfrac{2}{3}(v^\mu v_\mu)-\tfrac{3}{2}(a^\mu a_\mu)]\,, \tag{4.14}$$

where $R(\{\ \})$ denotes the Riemann scalar curvature. It is given by the contraction of the Ricci tensor, which is again the contraction of the Riemann-Christoffel curvature tensor:

$$R^{\rho}_{\cdot\sigma\mu\nu}(\{\ \})=\partial_\mu\{^{\rho}_{\sigma\nu}\}-\partial_\nu\{^{\rho}_{\sigma\mu}\}+\{^{\rho}_{\lambda\mu}\}\{^{\lambda}_{\sigma\nu}\}-\{^{\rho}_{\lambda\nu}\}\{^{\lambda}_{\sigma\mu}\}\,, \tag{4.15}$$

$$R_{\mu\nu}(\{\ \})=R^{\lambda}_{\cdot\mu\lambda\nu}(\{\ \})\,, \tag{4.16}$$

$$R(\{\ \})=g^{\mu\nu}R_{\mu\nu}(\{\ \})\,. \tag{4.17}$$

Here the symbol $\{^{\rho}_{\sigma\nu}\}$ denotes the Levi-Civita connection (1.3) of the Riemann space-time. Since the Weitzenböck space-time has the vanishing of the curvature tensor [see (1.7)], the identity of (4.14) should be taken as purely mathematical. Using $\kappa=8\pi G/c^4=8\pi G$ with G the Newton gravitational constant, we finally rewrite the gravitational action in the following form:

$$I_G=\int d^4x\sqrt{-g}\left(\frac{1}{2\kappa}R(\{\ \})+c_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu})+c_2(v^\mu v_\mu)+c_3(a^\mu a_\mu)\right). \tag{4.18}$$

Comparing (4.13) with (4.18), we find that the parameters are effectively (under integration symbol) related to each other by

$$c_1=a_1+\frac{1}{3\kappa}\,,\quad c_2=a_2-\frac{1}{3\kappa}\,,\quad c_3=a_3+\frac{3}{4\kappa}\,. \tag{4.19}$$

It should be mentioned that one of the free parameters, c_1, c_2, and c_3, must be nonzero; otherwise, the left-hand side of a gravitational field equation would become symmetric, while the right-hand side, the energy-momentum tensor of spin-$\frac{1}{2}$ fundamental particles, would become nonsymmetric. This is a contradiction.

It is easy to derive a gravitational field equation. For the sake of completeness we write down a gravitational field equation when matter is present, by adding to the vacuum gravitational action a matter action I_M, which satisfies the postulates of (1) to (4),

$$I = I_G + I_M\,, \tag{4.20}$$

$$I_M = \int d^4x\sqrt{-g}\,L_M\,. \tag{4.21}$$

From this action follows the following field equation, by taking variation with respect to the parallel vector fields $b^k{}_\nu$ and then multiplying with $\eta^{kj}b_j{}^\mu$:

$$G^{\mu\nu}(\{\ \}) + 2\kappa D^*_\lambda F^{\mu\nu\lambda} + 2\kappa v_\lambda F^{\mu\nu\lambda} + 2\kappa H^{\mu\nu} - \kappa g^{\mu\nu}L' = \kappa T^{\mu\nu}\,. \tag{4.22}$$

Here the first term denotes the Einstein tensor of general relativity,

$$G^{\mu\nu}(\{\ \}) \equiv R^{\mu\nu}(\{\ \}) - \tfrac{1}{2}g^{\mu\nu}R(\{\ \})\,, \tag{4.23}$$

and the tensor $\{F^{\mu\nu\lambda}\}$ stands for

$$\begin{aligned} F^{\mu\nu\lambda} &= c_1(t^{\mu\nu\lambda} - t^{\mu\lambda\nu}) + c_2(g^{\mu\nu}v^\lambda - g^{\mu\lambda}v^\nu) \\ &\quad -\tfrac{1}{3}c_3\epsilon^{\mu\nu\lambda\rho}a_\rho \\ &= -F^{\mu\lambda\nu}\,. \end{aligned} \tag{4.24}$$

The fourth term $\{H^{\mu\nu}\}$ is defined by

$$H^{\mu\nu} = T^{\rho\sigma\mu}F_{\rho\sigma}{}^{\cdot\cdot\nu} - \tfrac{1}{2}T^{\nu\rho\sigma}F^{\mu}{}_{\cdot\rho\sigma} = H^{\nu\mu}\,, \tag{4.25}$$

which is shown to be symmetric upon inserting the irreducible decomposition (4.8) of the torsion tensor. Finally, L' is given by

$$L' = c_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu}) + c_2(v^\mu v_\mu) + c_3(a^\mu a_\mu)\,. \tag{4.26}$$

A source term is, as usual, defined by

$$\begin{aligned} \delta\int d^4x\sqrt{-g}\,L_M &= \int d^4x\sqrt{-g}\,T_k^{\cdot\mu}\delta b^k{}_\mu \\ &= -\int d^4x\sqrt{-g}\,T_\mu^{\cdot k}\delta b_k{}^\mu\,. \end{aligned} \tag{4.27}$$

Therefore, an energy-momentum tensor is given by

$$\sqrt{-g}\,T_{\mu\nu} = -\eta_{kj}b^j{}_\nu\delta\sqrt{-g}\,L_M/\delta b_k{}^\mu\,, \tag{4.28a}$$

or equivalently,

$$\sqrt{-g}\,T^{\mu\nu} = \eta^{kj}b_j{}^\mu\delta\sqrt{-g}\,L_M/\delta b^k{}_\nu\,. \tag{4.28b}$$

For instance, the energy-momentum tensor of the electromagnetic field and the Dirac field is calculated from the above formulas with (3.3) and (3.5) to be

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu\sigma}g^{\rho\sigma} + g_{\mu\nu}L_{\rm em}\,, \tag{4.29}$$

$$\begin{aligned} T_{\mu\nu} &= -\tfrac{1}{2}ib^k{}_\nu[\bar\psi\gamma_k\partial_\mu\psi - (\partial_\mu\bar\psi)\gamma_k\psi] + g_{\mu\nu}L_D \\ &= -\tfrac{1}{2}ib^k{}_\nu[\bar\psi\gamma_k\nabla_\mu\psi - (\nabla_\mu\bar\psi)\gamma_k\psi] \\ &\quad + \tfrac{1}{4}\epsilon_{\lambda\rho\sigma\nu}b_k{}^\lambda K^{\rho\sigma}_{\cdot\cdot\mu}\bar\psi\gamma^5\gamma^k\psi \\ &\quad + g_{\mu\nu}(L_D^{\rm GR} - \tfrac{3}{4}a_k\bar\psi\gamma^5\gamma^k\psi)\,, \end{aligned} \tag{4.30a}$$

with the Dirac Lagrangian $L_D^{\rm GR}$ used in general relativity

$$L_D^{\rm GR} = \tfrac{1}{2}ib_k{}^\mu[\bar\psi\gamma^k\nabla_\mu\psi - (\nabla_\mu\bar\psi)\gamma^k\psi] - m\bar\psi\psi\,. \tag{4.30b}$$

It is useful to split the gravitational field equation into the symmetric and antisymmetric parts:

$$G^{\mu\nu}(\{\ \}) + 2\kappa D^*_\lambda F^{(\mu\nu)\lambda} + 2\kappa v_\lambda F^{(\mu\nu)\lambda} + 2\kappa H^{\mu\nu} - \kappa g^{\mu\nu}L' = \kappa T^{(\mu\nu)}\,, \tag{4.31}$$

$$2D^*_\lambda F^{[\mu\nu]\lambda} + 2v_\lambda F^{[\mu\nu]\lambda} = T^{[\mu\nu]}\,, \tag{4.32}$$

where

$$\begin{aligned} F^{(\mu\nu)\lambda} &= \tfrac{1}{2}(F^{\mu\nu\lambda} + F^{\nu\mu\lambda})\,, \\ F^{[\mu\nu]\lambda} &= \tfrac{1}{2}(F^{\mu\nu\lambda} - F^{\nu\mu\lambda})\,, \\ T^{(\mu\nu)} &= \tfrac{1}{2}(T^{\mu\nu} + T^{\nu\mu})\,, \\ T^{[\mu\nu]} &= \tfrac{1}{2}(T^{\mu\nu} - T^{\nu\mu})\,. \end{aligned} \tag{4.33}$$

Furthermore, it is often useful to rewrite the field equation in Latin indices:

$$G^{ij}(\{\ \}) + \frac{2\kappa}{\sqrt{-g}}\,\partial_\lambda(\sqrt{-g}\,F^{ij\lambda}) + 2\kappa H^{ij} - \kappa\eta^{ij}L' = \kappa T^{ij}\,, \tag{4.34}$$

where

$$\begin{aligned} G^{ij}(\{\ \}) &= b^i{}_\mu b^j{}_\nu G^{\mu\nu}(\{\ \})\,, \\ F^{ij\lambda} &= b^i{}_\mu b^j{}_\nu F^{\mu\nu\lambda}\,, \\ H^{ij} &= b^i{}_\mu b^j{}_\nu H^{\mu\nu}\,, \quad T^{ij} = b^i{}_\mu b^j{}_\nu T^{\mu\nu}\,. \end{aligned} \tag{4.35}$$

Finally the symmetric and antisymmetric parts of the gravitational field equation are derived as follows:

$$G^{ij}(\{\ \}) + \frac{2\kappa}{\sqrt{-g}}\,\partial_\lambda(\sqrt{-g}\,F^{(ij)\lambda}) + 2\kappa H^{ij} - \kappa\eta^{ij}L' = \kappa T^{(ij)}\,, \tag{4.36}$$

$$2\partial_\lambda(\sqrt{-g}\,F^{[ij]\lambda}) = \sqrt{-g}\,T^{[ij]}\,. \tag{4.37}$$

V. THE STATIC, ISOTROPIC GRAVITATIONAL FIELD IN VACUUM

Let us consider a static, isotropic gravitational field produced by a static, spherical body, assuming that the spin of constituent particles of a body, if it exists, can be completely neglected. The state of a central body then does not change under space inversion, besides time reversal and space rotation. Therefore, it is possible to find a set of coordinates, $x^0 = t$, x^1, x^2, and x^3, such that the parallel vector fields $\underline{b} = \{\underline{b_k}\} = \{b_k{}^\mu\}$ are *form invariant* under time reversal, space inversion, and space rotation,

$$t \to -t\,,\quad \underline{b}_{(0)} \to -\underline{b}_{(0)}$$
(time reversal) , (5.1a)

$$x^\alpha \to -x^\alpha, \quad \underline{b}_{(a)} \to -\underline{b}_{(a)} \quad \text{(space inversion)}, \tag{5.1b}$$

$$x^\alpha \to R_{\alpha\beta}x^\beta, \quad \underline{b}_{(a)} \to R_{ac}\underline{b}_{(c)} \quad \text{(space rotation)}, \tag{5.1c}$$

where, to avoid confusion, Latin indices in b_k are enclosed in parentheses, and $R = (R_{ac}) = (R_{\alpha\beta})$ is a position-independent 3×3 orthogonal matrix

$$RR^t = R^tR = I, \quad \det R = 1. \tag{5.1d}$$

Then, as is shown in Appendix B, by using the freedom to redefine the radius

$$x'^\alpha = \Phi(r)x^\alpha, \quad r \equiv (x^\alpha x^\alpha)^{1/2},$$

we can assume, without loss of generality, that the parallel vector fields $\underline{b} = \{\underline{b}_k\} = \{b_k{}^\mu(x)\}$ have a diagonal form,

$$b_{(0)}{}^0 = C(r), \quad b_{(0)}{}^\alpha = 0 = b_{(a)}{}^0, \quad b_{(a)}{}^\alpha = D(r)\delta_a^\alpha, \tag{5.2}$$

with two unknown functions of r, C, and D. The invariant distance ds^2 is then expressed in the isotropic form,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ij}b^i{}_\mu b^j{}_\nu dx^\mu dx^\nu = -A(r)dt^2 + B(r)dx^\alpha dx^\alpha, \tag{5.3a}$$

where the metric coefficients, A and B, are related to C and D by

$$A = 1/C^2, \quad B = 1/D^2. \tag{5.3b}$$

The set of coordinates $\{x^\mu\}$ is therefore the isotropic coordinate system.

We write the gravitational field equation (4.22) as

$$I^{\mu\nu} = \kappa T^{\mu\nu}, \tag{5.4a}$$

where $\{I^{\mu\nu}\}$ is defined by

$$I^{\mu\nu} \equiv G^{\mu\nu}(\{\,\}) + 2\kappa D^*_\lambda F^{\mu\nu\lambda} + 2\kappa v_\lambda F^{\mu\nu\lambda} + 2\kappa H^{\mu\nu} - \kappa g^{\mu\nu}L'. \tag{5.4b}$$

For a static, isotropic gravitational field with (5.2), $\{I^{\mu\nu}\}$ is given by

$$I^{00} = -\frac{1+\kappa(c_1+4c_2)}{AB}\left[\epsilon\left(\frac{A'}{A}\right)' + (1-2\epsilon)\left(\frac{B'}{B}\right)' + \frac{2}{r}\left(\epsilon\frac{A'}{A} + (1-2\epsilon)\frac{B'}{B}\right) + \frac{\epsilon}{4}\left(\frac{A'}{A}\right)^2 + \frac{\epsilon}{2}\left(\frac{A'}{A}\right)\left(\frac{B'}{B}\right) + \tfrac{1}{4}(1-4\epsilon)\left(\frac{B'}{B}\right)^2\right], \tag{5.5a}$$

$$I^{0\alpha} = 0 = I^{\alpha 0}, \tag{5.5b}$$

$$\begin{aligned} I^{\alpha\beta} = {} & \frac{\delta^{\alpha\beta}}{2B^2}[1+\kappa(c_1+4c_2)]\left[(1-2\epsilon)\left(\frac{A'}{A}\right)' + \left(\frac{B'}{B}\right)' + \frac{1}{r}\left((1-2\epsilon)\frac{A'}{A} + \frac{B'}{B}\right) + \tfrac{1}{2}(1-3\epsilon)\left(\frac{A'}{A}\right)^2 + \epsilon\left(\frac{A'}{A}\right)\left(\frac{B'}{B}\right)\right] \\ & - \frac{1}{2B^2}\frac{x^\alpha x^\beta}{r^2}[1+\kappa(c_1+4c_2)]\left[(1-2\epsilon)\left(\frac{A'}{A}\right)' + \left(\frac{B'}{B}\right)' - \frac{1}{r}\left((1-2\epsilon)\frac{A'}{A} + \frac{B'}{B}\right) + \tfrac{1}{2}(1-4\epsilon)\left(\frac{A'}{A}\right)^2 \right. \\ & \left. - (1-3\epsilon)\left(\frac{A'}{A}\right)\left(\frac{B'}{B}\right) - \frac{1}{2}\left(\frac{B'}{B}\right)^2\right], \end{aligned} \tag{5.5c}$$

where the parameter ϵ is a constant defined by

$$\epsilon \equiv \frac{\kappa(c_1+c_2)}{1+\kappa(c_1+4c_2)}, \tag{5.6}$$

and a prime means differentiation with respect to r. It is shown in Appendix C that the constant $[1+\kappa(c_1+4c_2)]$ is a nonzero number.

There is no appearance of the parameter c_3, but only the parameters, c_1 and c_2, owing to a static, isotropic gravitational field. In other words, we can say nothing about the parameter c_3 in this case. Now we proceed to study a solution of the field equation (5.4) with (5.5) for the following three cases: (A) the Newtonian limit, (B) the post-Newtonian approximation, and (C) an exact solution in vacuum.

A. The Newtonian limit

We assume that a central gravitating body is a nonrelativistic system with all the components of $T^{\alpha\beta}$ being negligibly small compared to T^{00}; $T^{00} \gg |T^{\alpha\beta}| \simeq 0$. Then the gravitational field is weak; the metric coefficients, A and B, are nearly unity, $A \simeq 1 \simeq B$, and terms quadratic in A' and B' can be ignored in the field equation (5.4) with (5.5). We then find that the gravitational field equation in the Newtonian limit is given by

$$-[1+\kappa(c_1+4c_2)]\left\{\epsilon A'' + (1-2\epsilon)B'' + \frac{2}{r}[\epsilon A' + (1-2\epsilon)B']\right\} = \kappa T^{00}, \tag{5.7a}$$

$$(1-2\epsilon)A'+B'=0. \tag{5.7b}$$

The external solution satisfying the boundary condition,

$$\lim_{r\to\infty} A(r)=\lim_{r\to\infty} B(r)=1, \tag{5.8}$$

is

$$A(r)=1-\frac{2}{(1-\epsilon)(1-4\epsilon)[1+\kappa(c_1+4c_2)]}\frac{Gm}{r}, \tag{5.9a}$$

$$B(r)=1+\frac{2(1-2\epsilon)}{(1-\epsilon)(1-4\epsilon)[1+\kappa(c_1+4c_2)]}\frac{Gm}{r}, \tag{5.9b}$$

with m the total mass of the source,

$$m=\int T^{00}(x)d^3x=4\pi\int r^2T^{00}(r)dr. \tag{5.10}$$

It was found in Sec. III that the trajectory of a test particle is determined by the geodesic equation (3.26), which reduces for a nonrelativistic particle to

$$\frac{d^2x^\alpha}{dt^2}=\frac{1}{2}\frac{\partial}{\partial x^\alpha}g_{00}(x)$$
$$=-\frac{1}{(1-\epsilon)(1-4\epsilon)[1+\kappa(c_1+4c_2)]}\frac{\partial}{\partial x^\alpha}\left(-\frac{Gm}{r}\right). \tag{5.11a}$$

Here the solution (5.9a) is used in the final step. We demand that the trajectory of a nonrelativistic test particle, specified by $x^\alpha(t)$, obeys the Newton equation of motion

$$\frac{d^2x^\alpha}{dt^2}=-\frac{\partial}{\partial x^\alpha}\phi, \tag{5.11b}$$

where ϕ is a gravitational potential, which takes the form

$$\phi=-Gm/r \tag{5.11c}$$

for a gravitational field around a spherical body with mass m. Accordingly, the parameters, c_1 and c_2, must satisfy the condition

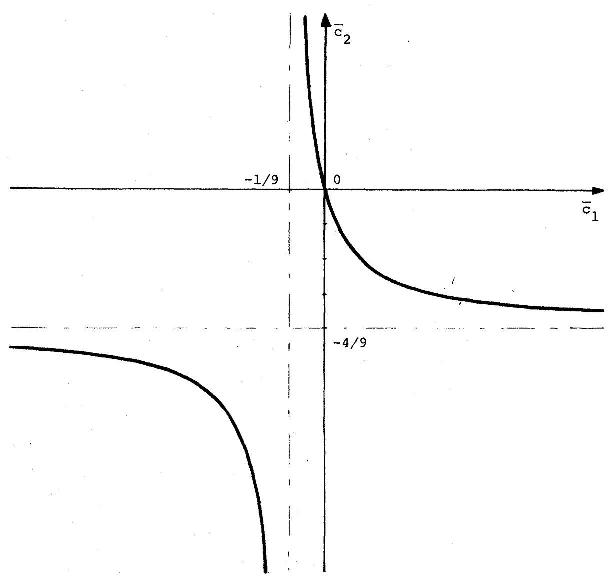

FIG. 2. The curve of (5.12b): $\bar{c}_1=\kappa c_1$ and $\bar{c}_2=\kappa c_2$. $\bar{c}_2=-4\bar{c}_1/(1+9\bar{c}_1)$.

$$(1-\epsilon)(1-4\epsilon)[1+\kappa(c_1+4c_2)]=1, \tag{5.12a}$$

which we shall assume hereafter. This condition is called the Newton approximation condition. In terms of $\bar{c}_1=\kappa c_1$ and $\bar{c}_2=\kappa c_2$, the Newton approximation condition reads as

$$4\bar{c}_1+\bar{c}_2+9\bar{c}_1\bar{c}_2=0. \tag{5.12b}$$

From this follow the two cases, $c_1=0=c_2$ and $c_1\neq 0\neq c_2$. See Fig. 2 for the curve specified by (5.12b). Now, combining (5.6) and (5.12b), we find

$$\bar{c}_1=-\frac{\epsilon}{3(1-\epsilon)},\quad \bar{c}_2=\frac{4\epsilon}{3(1-4\epsilon)}. \tag{5.12c}$$

Since ϵ is observable in solar-system experiments, as will be shown in Sec. VI, we draw the curves of (5.12c) versus ϵ in Fig. 3.

B. Vacuum solution in the post-Newtonian approximation

The field equation (5.4) with (5.5) can be rewritten in vacuum as follows,

$$\epsilon\left(\frac{A'}{A}\right)'+(1-2\epsilon)\left(\frac{B'}{B}\right)'+\frac{2}{r}\left(\epsilon\frac{A'}{A}+(1-2\epsilon)\frac{B'}{B}\right)+\frac{\epsilon}{4}\left(\frac{A'}{A}\right)^2+\frac{\epsilon}{2}\left(\frac{A'}{A}\right)\left(\frac{B'}{B}\right)+\tfrac{1}{4}(1-4\epsilon)\left(\frac{B'}{B}\right)^2=0, \tag{5.13a}$$

$$(1-2\epsilon)\left(\frac{A'}{A}\right)'+\left(\frac{B'}{B}\right)'+\frac{1}{r}\left((1-2\epsilon)\frac{A'}{A}+\frac{B'}{B}\right)+\tfrac{1}{2}(1-3\epsilon)\left(\frac{A'}{A}\right)^2+\epsilon\left(\frac{A'}{A}\right)\left(\frac{B'}{B}\right)=0, \tag{5.13b}$$

$$(1-2\epsilon)\left(\frac{A'}{A}\right)'+\left(\frac{B'}{B}\right)'-\frac{1}{r}\left((1-2\epsilon)\frac{A'}{A}+\frac{B'}{B}\right)+\tfrac{1}{2}(1-4\epsilon)\left(\frac{A'}{A}\right)^2-(1-3\epsilon)\left(\frac{A'}{A}\right)\left(\frac{B'}{B}\right)-\frac{1}{2}\left(\frac{B'}{B}\right)^2=0. \tag{5.13c}$$

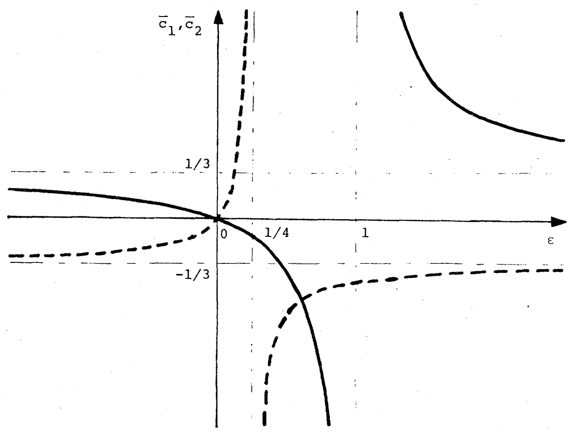

FIG. 3. The curves of (5.12c): The solid curve is for $\bar{c}_1$, and the dashed curve is for $\bar{c}_2$.

In the spatial region far outside the Schwarzschild radius, i.e., $r \gg GM$, the metric coefficients, $A(r)$ and $B(r)$, can be expanded in a small parameter (GM/r),

$$A(r) = 1 - 2\frac{GM}{r} + 2\beta\left(\frac{GM}{r}\right)^2 + \cdots, \tag{5.14a}$$

$$B(r) = 1 + 2\gamma\frac{GM}{r} + 2\delta\left(\frac{GM}{r}\right)^2 + \cdots, \tag{5.14b}$$

where M is the gravitational mass of a central gravitating body, and β, γ, and δ are expansion parameters to be determined by the field equation. Using (5.14) in (5.13), and putting each order of (GM/r) equal to zero, we find that the parameters, β, γ, and δ, are given by

$$\begin{aligned} &\beta = 1 - \epsilon/2, \quad \gamma = 1 - 2\epsilon, \\ &\delta = \tfrac{3}{4}(1 - 3\epsilon + \tfrac{8}{3}\epsilon^2). \end{aligned} \tag{5.15}$$

It is to be noticed that the Newton approximation condition (5.12a) is not used to derive (5.15), although the above results are consistent with (5.12a).

C. Exact vacuum solution

The gravitational field studied in the previous two subsections is weak. Now we derive an *exact* solution of the vacuum field equation (5.13), which allows us to study a strong gravitational field in new general relativity.

After slight modification of $[2\times(5.13\text{b}) - (5.13\text{c})]$ we obtain

$$\frac{d}{dr}\left[r^3\left((1-2\epsilon)\frac{A'}{A} + \frac{B'}{B}\right)\right] + \frac{r^3}{2}\left(\frac{A'}{A} + \frac{B'}{B}\right)\left((1-2\epsilon)\frac{A'}{A} + \frac{B'}{B}\right) = 0. \tag{5.16}$$

This equation can be integrated to give

$$(1-2\epsilon)\frac{A'}{A} + \frac{B'}{B} = (AB)^{-1/2}\frac{f_1}{r^3}, \tag{5.17a}$$

with f_1 an integration constant, which can be fixed by using (5.14) with (5.15) in (5.17a):

$$f_1 = (1-\epsilon)(1-4\epsilon)(GM)^2. \tag{5.17b}$$

In the same way we get from $[3\times(5.13\text{b}) - (5.13\text{c}) - 2\times(5.13\text{a})]$

$$(1-3\epsilon)\frac{A'}{A} + 2\epsilon\frac{B'}{B} = (AB)^{-1/2}\frac{f_2}{r^2}, \tag{5.18a}$$

with f_2 an integration constant given by

$$f_2 = 2(1-\epsilon)(1-4\epsilon)(GM). \tag{5.18b}$$

From the combination $[(1-5\epsilon)\times(5.17\text{a}) + 2\epsilon\times(5.18\text{a})]$ it follows that

$$\frac{d}{dr}(AB)^{1/2} = 2\epsilon\frac{GM}{r^2} + \frac{1-5\epsilon}{2}\frac{(GM)^2}{r^3}, \tag{5.19}$$

and, therefore, remembering that the boundary condition for A and B, denoted by (5.8), is expressed by

$$\lim_{r\to\infty} AB = 1, \tag{5.20}$$

we obtain

$$\begin{aligned} (AB)^{1/2} &= 1 - 2\epsilon\frac{GM}{r} - \frac{1-5\epsilon}{4}\left(\frac{GM}{r}\right)^2 \\ &= \left(1 - \frac{GM}{pr}\right)\left(1 + \frac{GM}{qr}\right), \end{aligned} \tag{5.21}$$

where two constants, p and q, are defined by

$$\begin{aligned} p &\equiv \frac{2}{1-5\epsilon}\{[(1-\epsilon)(1-4\epsilon)]^{1/2} - 2\epsilon\} \\ &= 2 + \epsilon + O(\epsilon^2), \\ q &\equiv \frac{2}{1-5\epsilon}\{[(1-\epsilon)(1-4\epsilon)]^{1/2} + 2\epsilon\} \\ &= 2 + 9\epsilon + O(\epsilon^2). \end{aligned} \tag{5.22a}$$

Here ϵ is assumed to be

$$\epsilon \leq \tfrac{1}{4}, \tag{5.22b}$$

which covers the important case of $\epsilon = 0$; for $\frac{1}{4} < \epsilon < 1$, p and q become complex values. Substitution of (5.21) into (5.17a) and (5.18a) finally gives

$$\begin{aligned} A(r) &= \left(1 - \frac{GM}{pr}\right)^{p}\left(1 + \frac{GM}{qr}\right)^{-q}, \\ B(r) &= \left(1 - \frac{GM}{pr}\right)^{2-p}\left(1 + \frac{GM}{qr}\right)^{2+q}. \end{aligned} \tag{5.23}$$

It can be shown by direct calculations that this solution indeed satisfies the field equation (5.13).

The parallel vector fields of (5.2) are thus given by

$$b_{(0)}{}^{0}=\frac{1}{\sqrt{A}}=\left(1-\frac{GM}{pr}\right)^{-p/2}\left(1+\frac{GM}{qr}\right)^{q/2},$$
$$b_{(0)}{}^{\alpha}=0=b_{(a)}{}^{0},\tag{5.24}$$
$$b_{(a)}{}^{\alpha}=\frac{\delta_a^\alpha}{\sqrt{B}}=\left(1-\frac{GM}{pr}\right)^{-1+p/2}\left(1+\frac{GM}{qr}\right)^{-1-q/2}\delta_a^\alpha,$$

in a static, isotropic gravitational field. The invariant distance ds^2 of (5.3a) becomes

$$\begin{aligned}ds^2&=-\left(1-\frac{GM}{pr}\right)^{p}\left(1+\frac{GM}{qr}\right)^{-q}dt^2+\left(1-\frac{GM}{pr}\right)^{2-p}\left(1+\frac{GM}{qr}\right)^{2+q}dx^\alpha dx^\alpha\\&=-\left(1-\frac{GM}{pr}\right)^{p}\left(1+\frac{GM}{qr}\right)^{-q}dt^2+\left(1-\frac{GM}{pr}\right)^{2-p}\left(1+\frac{GM}{qr}\right)^{2+q}[dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)],\end{aligned}\tag{5.25}$$

where we have introduced the spherical polar coordinates by

$$x^1=r\sin\theta\cos\phi,\quad x^2=r\sin\theta\sin\phi,\quad x^3=r\cos\theta.\tag{5.26}$$

If the parameter ϵ of (5.6) is *exactly zero*, then two constants, p and q, are exactly equal to 2, and hence this metric coincides with the Schwarzschild metric written in the isotropic coordinates[25]:

$$\begin{aligned}ds^2&=-\frac{(1-GM/2r)^2}{(1+GM/2r)^2}dt^2+\left(1+\frac{GM}{2r}\right)^4dx^\alpha dx^\alpha\\&=-\frac{(1-GM/2r)^2}{(1+GM/2r)^2}dt^2\\&\quad+\left(1+\frac{GM}{2r}\right)^4[dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)].\end{aligned}\tag{5.27}$$

VI. COMPARISON WITH EXPERIMENTS

A. The equivalence principle

It has been verified experimentally to very high accuracy[26] that the world line of a freely falling test body is independent of its composition and structure. The equivalence principle implies that the unique world line of a test body coincides with the geodesics of the metric g. It was shown in Sec. III that by taking the short-wavelength limit of the Maxwell and Dirac equations the photon and Dirac particles in the classical limit are to travel along the geodesics of the metric g. Thus, new general relativity is compatible with the equivalence principle in this limit.

In general relativity implications of the equivalence principle are concisely expressed by the conservation law,

$$\nabla_\nu T^{\mu\nu}_{\rm GR}=0,\tag{6.1}$$

where $\{T^{\mu\nu}_{\rm GR}\}$ is the matter energy-momentum tensor appearing on the right-hand side of the Einstein field equation. It follows from the conservation law that the world line of a freely falling test body is the geodesics of the metric g. The characteristic feature of general relativity is that the conservation law of (6.1) is a consequence of the Einstein gravitational field equation, and hence that mechanical equations of motion for matter are consequences of the same gravitational field equation.

Now we shall show that almost the same property holds also in new general relativity based on the Weitzenböck space-time. From the invariance of the gravitational action under the group of general coordinate transformations follows the identity[27]

$$\sqrt{-g}\,B^{k\nu}\partial_\mu b_{k\nu}-\partial_\nu(\sqrt{-g}\,B_\mu{}^{\nu})\equiv0,\tag{6.2}$$

where

$$\sqrt{-g}\,B^{k\nu}=\sqrt{-g}\,b^k{}_\mu B^{\mu\nu}\equiv-\delta\sqrt{-g}\,L_G/\delta b_{k\nu}.\tag{6.3}$$

After slight modification, this identity can be rewritten as

$$\nabla_\nu B^{\mu\nu}-K^{\nu\lambda\mu}B_{\nu\lambda}\equiv0,\tag{6.4}$$

where $\{K^{\nu\lambda\mu}\}$ is the contortion tensor given by (3.22). From the definition of (6.3) it follows that the gravitational field equation takes the form

$$B^{\mu\nu}=T^{\mu\nu},\tag{6.5}$$

with the matter energy-momentum tensor $\{T^{\mu\nu}\}$ defined by (4.28). Using (6.5) in the identity (6.4), we get the response equation to gravitation,

$$\nabla_\nu T^{\mu\nu}-K^{\nu\lambda\mu}T_{\nu\lambda}=0.\tag{6.6}$$

This is the conservation law of new general relativity, corresponding to the conservation law (6.1) of general relativity. The energy-momentum tensor $\{T^{\mu\nu}\}$ is not symmetric in new general relativity. However, an antisymmetric part $\{T^{[\mu\nu]}\}$ is due to the contribution from the intrinsic spin of spin-$\frac{1}{2}$ fundamental particles. For macroscopic bodies such as a test body employed in terrestrial experiments and astrophysical objects such as planets and stars, effects due to the intrinsic spin of spin-$\frac{1}{2}$ fundamental particles can be ignored, and hence their energy-momentum tensor can be supposed to be symmetric and of the same form as that of general relativity. Therefore, an energy-momentum tensor of macroscopic bodies satisfies the conservation law

$$\nabla_\nu T^{\mu\nu} = 0 \tag{6.7}$$

owing to the antisymmetric property of the contortion tensor $\{K^{\nu\lambda\mu}\}$ with respect to ν and λ. The only exception seems to be compact stellar objects such as neutron stars and black holes: The spin direction of neutrons may happen to be aligned over the macroscopic scale inside neutron stars. If this is indeed the case, the gravitational response of neutron matter should be described by Eq. (6.6) instead of by the conservation law (6.1)

The equivalence principle is thus satisfied for macroscopic bodies in new general relativity, and the world line of a test body coincides with the geodesics of the metric $\underline{g}$, although the metric $\underline{g}$ itself may be different from that of general relativity.

In the microscopic scale new general relativity violates the equivalence principle, since effects due to the intrinsic spin of spin-$\frac{1}{2}$ fundamental particles cannot be ignored there, and an antisymmetric part of the energy-momentum tensor should be seriously taken into account. The motion of the intrinsic spin of a freely falling spin-$\frac{1}{2}$ fundamental particle, for example, does not satisfy the equivalence principle. As was shown in (3.37b), the spin vector $\{S^\mu\}$ obeys the equation of motion

$$\nabla S^\mu/d\tau = -\tfrac{3}{2}\epsilon^{\mu\nu\rho\sigma}\,U_\nu a_\rho S_\sigma \tag{6.8a}$$

with

$$U^\mu \equiv \frac{dx^\mu}{d\tau}, \quad \frac{\nabla S^\mu}{d\tau} \equiv \frac{dS^\mu}{d\tau} + \left\{{\mu \atop \lambda\nu}\right\} U^\lambda S^\nu , \tag{6.8b}$$

where τ is the proper time. In order for this equation of motion for the spin vector to meet with the equivalence principle, the right-hand side should vanish. Therefore, unless the axial-vector part $\{a^\mu\}$ of the torsion tensor happens to vanish identically, the equation of motion for the spin vector violates the equivalence principle.[28] Another important implication of new general relativity for microscopic phenomena is the prediction of universal spin-spin interaction, caused by an antisymmetric part of the energy-momentum tensor. This interaction, if it exists, contributes to the hyperfine splitting of the atomic energy levels, and it shall be discussed in Sec. X.

B. Comparison with solar-system experiments

Since the invariant distance ds^2 of (5.3a) is written in the isotropic coordinates, the post-Newtonian parameters of the expansion (5.14a)–(5.14b), β and γ, are the Eddington-Robertson parameters.[29] Thus, by virtue of (5.15), the Eddington-Robertson parameters of new general relativity are given by

$$\beta = 1 - \epsilon/2, \quad \gamma = 1 - 2\epsilon . \tag{6.9}$$

The values of β and γ have been measured by the solar-system experiments:

$$\gamma = \begin{cases} 1.00 \pm 0.06 \text{ (retardation of radio waves}^{30}\text{)}, & (6.10\text{a}) \\ 1.014 \pm 0.018 \text{ (solar deflection}^{31}\text{)}, & (6.10\text{b}) \end{cases}$$

$$\tfrac{1}{3}(2 + 2\gamma - \beta) = 1.003 \pm 0.005 \quad \text{(perihelion advances}^{32}\text{)}, \tag{6.11}$$

$$\eta \equiv 4\beta - \gamma - 3 = -0.001 \pm 0.015 \quad \text{(lunar laser ranging}^{33}\text{)}. \tag{6.12}$$

From (6.9) it follows that the Nordtvedt parameter η is vanishing in new general relativity;

$$\eta = 0 . \tag{6.13}$$

For the sake of safety, we here adopt the value (6.10b) for γ. Using (6.9) in (6.10b) and (6.11), we get

$$\epsilon = \begin{cases} -0.007 \pm 0.009 & \text{from (6.10b)}, \\ -0.003 \pm 0.004 & \text{from (6.11)}. \end{cases} \tag{6.14a}$$

Combining these two values for ϵ as if they were independent, we are led to

$$\epsilon = -0.004 \pm 0.004 . \tag{6.14b}$$

This value of ϵ satisfies our assumption of $\epsilon \leq \frac{1}{4}$; see (5.22b).

By virtue of (5.12c), two dimensionless constants, κc_1 and κc_2, can be expressed as

$$\begin{aligned} \kappa c_1 &= -\frac{\epsilon}{3(1-\epsilon)} = -\frac{\epsilon}{3} + O(\epsilon^2), \\ \kappa c_2 &= \frac{4\epsilon}{3(1-4\epsilon)} = \frac{4\epsilon}{3} + O(\epsilon^2). \end{aligned} \tag{6.15}$$

Use of (6.14b) then gives

$$\kappa c_1 = 0.001 \pm 0.001, \quad \kappa c_2 = -0.005 \pm 0.005 . \tag{6.16}$$

Rewriting the gravitational Lagrangian density L_G of (4.18) as

$$L_G = \frac{1}{2\kappa}[R(\{\ \}) + 2\kappa c_1 (t^{\lambda\mu\nu} t_{\lambda\mu\nu}) + 2\kappa c_2 (v^\mu v_\mu) + 2\kappa c_3 (a^\mu a_\mu)]\,, \tag{6.17}$$

we find that the strength of the c_1 and c_2 terms are severely restricted by the solar-system experiments.

VII. THE CASE OF $c_1=0=c_2$

We have seen in the last section that the c_1 and c_2 terms of L_G are, if they exist, very severely restricted by the solar-system experiments. Therefore, we shall henceforce ignore these two terms and assume that $c_1 = 0 = c_2$. The case of $c_1 \neq 0 \neq c_2$ shall be discussed in a separate paper.

The gravitational Lagrangian density L_G then becomes

$$L_G = \frac{1}{2\kappa} R(\{\ \}) + c_3 (a^\mu a_\mu)\,, \tag{7.1}$$

and the gravitational field equation, (4.31) and (4.32), can be expressed as

$$G^{\mu\nu}(\{\ \}) + K^{\mu\nu} = \kappa T^{(\mu\nu)}\,, \tag{7.2}$$

$$b_i{}^\mu b_j{}^\nu \partial_\rho(\sqrt{-g}\, J^{ij\rho}) = \lambda\sqrt{-g}\, T^{[\mu\nu]}\,, \tag{7.3}$$

where we have introduced a new parameter λ by

$$\lambda = \frac{9}{4c_3}\,, \tag{7.4}$$

and $\{K^{\mu\nu}\}$ and $\{J^{\mu\nu\rho}\}$ are defined by

$$K^{\mu\nu} = \frac{\kappa}{\lambda}\{\tfrac{1}{2}[\epsilon^{\mu\rho\sigma\lambda}(T^{\nu}_{\cdot\rho\sigma} - T_{\rho\sigma}^{\cdot\cdot\nu}) + \epsilon^{\nu\rho\sigma\lambda}(T^{\mu}_{\cdot\rho\sigma} - T_{\rho\sigma}^{\cdot\cdot\mu})]a_\lambda - \tfrac{3}{2}a^\mu a^\nu - \tfrac{3}{4}g^{\mu\nu}a^\rho a_\rho\}\,, \tag{7.5}$$

$$J^{\mu\nu\rho} = b_i{}^\mu b_j{}^\nu J^{ij\rho} = -\tfrac{3}{2}\epsilon^{\mu\nu\rho\sigma}a_\sigma\,. \tag{7.6}$$

Taking the combination of $[(7.2) + (\kappa/\lambda)\times(7.3)]$, the gravitational field equation is rewritten as

$$G^{\mu\nu}(\{\ \}) + L^{\mu\nu} = \kappa T^{\mu\nu}\,, \tag{7.7}$$

with $\{L^{\mu\nu}\}$ defined by

$$L^{\mu\nu} = \frac{\kappa}{2\lambda}\{a_\lambda[\epsilon^{\mu\rho\sigma\lambda}(T^{\nu}_{\cdot\rho\sigma} - T_{\rho\sigma}^{\cdot\cdot\nu}) + \epsilon^{\nu\rho\sigma\lambda}(T^{\mu}_{\cdot\rho\sigma} - T_{\rho\sigma}^{\cdot\cdot\mu})] - 3a^\mu a^\nu - \tfrac{3}{2}g^{\mu\nu}a^\rho a_\rho + 3\epsilon^{\mu\nu\rho\sigma}(b^i{}_\rho\,\partial_\sigma a_i + a_\rho v_\sigma)\}\,, \tag{7.8}$$

where $\{a_i = b_i{}^\mu a_\mu\}$ is a scalar with respect to general coordinate transformations.

As is evident by the definition of the torsion tensor (4.4) and its irreducible components of (4.5)–(4.7), the second term $\{L^{\mu\nu}\}$ of (7.7) does not transform like a tensor under a *local* Lorentz transformation

$$\begin{aligned} &\underline{b}_k(x) = A^j{}_k(x)\underline{b}'_j(x)\,, \\ &A^j{}_k(x)\eta_{jm}A^m{}_n(x) = \eta_{kn}\,. \end{aligned} \tag{7.9}$$

The energy-momentum tensor of the electromagnetic field depends on the parallel vector fields $\underline{b}$ only through the metric tensor, $g_{\mu\nu} = b^i{}_\mu \eta_{ij} b^j{}_\nu$, and hence it is locally Lorentz invariant. The energy-momentum tensor of spin-$\frac{1}{2}$ fundamental particles, however, is not locally Lorentz invariant, due to the second term of the second line of (4.30a), i.e., $\frac{1}{4}\epsilon_{\lambda\rho\sigma\nu} b_k{}^\lambda K^{\rho\sigma}_{\cdot\cdot\mu}\bar{\psi}\gamma^5\gamma^k\psi$. Thus, the energy-momentum tensor of matter is not locally Lorentz invariant, unless effects due to the intrinsic spin of spin-$\frac{1}{2}$ fundamental particles can be neglected. Therefore, *the gravitational field equation of* (7.7) *is not invariant under a local Lorentz transformation.*

The gravitational field equation is considerably simplified in the particular case which satisfies the following two conditions: (1) The axial-vector part of the torsion tensor vanishes identically,

$$a^\mu = \tfrac{1}{6}\,\epsilon^{\mu\nu\rho\sigma} b^k{}_\nu(\partial_\sigma b_{k\rho} - \partial_\rho b_{k\sigma}) = 0\,, \tag{7.10}$$

and (2) effects due to the intrinsic spin of spin-$\frac{1}{2}$ fundamental particles can be neglected. The first condition implies that the left-hand side of (7.7) becomes the Einstein tensor $G^{\mu\nu}(\{\ \})$. The second condition, on the other hand, allows us to treat spin-$\frac{1}{2}$ fundamental particles as if they were spinless; the energy-momentum tensor $\{T^{\mu\nu}\}$ on the right-hand side of (7.7) can then be identified with the energy-momentum tensor $\{T^{\mu\nu}_{\rm GR}\}$ used in general relativity. Thus, in this particular case the gravitational field equation (7.7) is identical with the Einstein field equation,

$$G^{\mu\nu}(\{\ \}) = \kappa T^{\mu\nu}_{\rm GR}\,. \tag{7.11}$$

For example, suppose that the metric in the invariant distance,

$$ds^2 = -A(x)(dx^0)^2 + B(x)(dx^1)^2 + C(x)(dx^2)^2 + D(x)(dx^3)^2\,, \tag{7.12}$$

is an exact solution of the Einstein field equation (7.11), where $A(x)$, $B(x)$, $C(x)$, and $D(x)$ are functions of x. Define the parallel vector fields $\underline{b} = \{\underline{b}_k\}$ by

$$\underline{b}_{(0)}=\frac{1}{\sqrt{A}}\underline{E}_0,\quad \underline{b}_{(1)}=\frac{1}{\sqrt{B}}\underline{E}_1,$$
$$\underline{b}_{(2)}=\frac{1}{\sqrt{C}}\underline{E}_2,\quad \underline{b}_{(3)}=\frac{1}{\sqrt{D}}\underline{E}_3, \tag{7.13}$$

with $\underline{E}_\mu=\partial/\partial x^\mu$ and Latin indices in $\underline{b}_k$ enclosed in parentheses. Then they form a system of four orthonormal vectors with their contravariant components given by

$$b_{(0)}{}^0=1/\sqrt{A}\,,\quad b_{(1)}{}^1=1/\sqrt{B}\,,$$
$$b_{(2)}{}^2=1/\sqrt{C}\,,\quad b_{(3)}{}^3=1/\sqrt{D}\,, \tag{7.14a}$$
$$b_k{}^\mu=0 \text{ otherwise},$$

and their covariant components given by

$$b^{(0)}{}_0=\sqrt{A}\,,\quad b^{(1)}{}_1=\sqrt{B}\,,$$
$$b^{(2)}{}_2=\sqrt{C}\,,\quad b^{(3)}{}_3=\sqrt{D}\,, \tag{7.14b}$$
$$b^k{}_\mu=0 \text{ otherwise}.$$

In this case the axial-vector part of the torsion tensor, formed of $\underline{b}$, vanishes identically;

$$a^\mu=0\,. \tag{7.15}$$

Therefore, the parallel vector fields (7.13) are an exact solution of the gravitational field equation (7.7) with the source term, $T^{\mu\nu}=T^{\mu\nu}_{\rm GR}$. The metric of the form (7.12) covers, among others, a number of static vacuum solutions with high symmetry of the Einstein field equation,[34] such as the Schwarzschild solution, the Reissner-Nordström solution and the Weyl solution, and the Friedmann model[35] in cosmology.

VIII. GEOMETRY OF THE EXTENDED WEITZENBÖCK SPACE-TIME

In this section we shall consider the particular case discussed in the last section: Namely, we shall assume that the parallel vector fields, $\underline{b}=\{\underline{b}_k\}$, satisfy both the condition (7.10) and the condition that spin-$\frac{1}{2}$ fundamental particles can be treated as if they were spinless. The gravitational field equation (7.7) is then apparently of the same form as the Einstein field equation, but the geometrical background of these two equations are quite different: In general relativity the Einstein equation defines the Riemann space-time, while in new general relativity the gravitational field equation is to define the parallel vector fields of the Weitzenböck space-time.

Let the parallel vector fields, $\underline{b}=\{\underline{b}_k\}$, be a solution of the gravitational field equation (7.7): Namely, we suppose that $\underline{b}=\{\underline{b}_k\}$ simultaneously satisfies both the condition (7.10) and the Einstein field equation (7.11). New parallel vector fields, $\underline{b}'=\{\underline{b}'_k\}$, obtained from $\underline{b}$ by a *local* Lorentz transformation (7.9), also satisfy the Einstein field equation by virtue of the local Lorentz invariance of the Einstein field equation. The condition (7.10), on the other hand, is not fulfilled by $\underline{b}'$ in general, because the axial-vector part of the torsion tensor, $\{a^\mu\}$, transforms like

$$a'^\mu(x)=a^\mu(x)-\tfrac{1}{3}\epsilon^{\mu\nu\rho\sigma}b^j{}_\nu b^k{}_\rho A^m{}_j(x)A_{mk}(x)_{,\sigma} \tag{8.1}$$

under a local Lorentz transformation (7.9). Here A_{mk} is defined by $A_{mk}=\eta_{mj}A^j{}_k$.

The new parallel vector fields $\underline{b}'$ thus satisfy the gravitational field equation (7.7), if and only if the transformation matrix $[A^j{}_k(x)]$ obeys the condition,

$$\epsilon^{\mu\nu\rho\sigma}b^j{}_\nu b^k{}_\rho A^m{}_j(x)A_{mk,\sigma}=0\,, \tag{8.2}$$

which ensures the condition (7.10) for $\underline{b}'$. In the present particular case, therefore, *the gravitational field equation* (7.7) *is invariant under those local Lorentz transformations which satisfy the condition* (8.2), *and the parallel vector fields are defined by the gravitational field equation with ambiguity of making those local Lorentz transformations*. This ambiguity does not lead to any observable effects, because the Maxwell and Dirac equations, (3.11) and (3.13), respectively, are also invariant under those transformations.

In the Weitzenböck space-time the parallel vector fields should be defined only with arbitrariness of making a *global* Lorentz transformation, and there is no room for making any local Lorentz transformations. In the present particular case, however, the new parallel vector fields $\underline{b}'$ connected with $\underline{b}$ by a local Lorentz transformation satisfying (8.2) should be regarded as equivalent to b', because the Maxwell, Dirac, and gravitational field equations are all invariant under the transformation from $\underline{b}$ to $\underline{b}'$. We are thus forced to generalize the concept of absolute parallelism in the following manner: *Absolute parallelism defined by* $\underline{b}'$ *shall be regarded as equivalent to that defined by* $\underline{b}$, *provided that* $\underline{b}$ *and* $\underline{b}'$ *are connected with each other by a local Lorentz transformation subjecting to* (8.2). We shall refer to this new parallelism as *extended absolute parallelism*, and space-time endowed with extended absolute parallelism shall be called the *extended Weitzenböck space-time*. The geometry of the extended Weitzenböck space-time then is invariant under those local Lorentz transformations which fulfil the condition (8.2).

For given parallel vector fields $\underline{b}$, we denote by $\Lambda(\underline{b})$ the set of those local Lorentz transformations which fulfil the condition (8.2). The set $\Lambda(\underline{b})$ does not form a Lie group: Namely, for two elements of $\Lambda(\underline{b})$, A and A', the inverse A^{-1} and the product $A'A$ do not belong to $\Lambda(\underline{b})$ in general. However, for an infinitesimal local Lorentz transformation,

$$A^j{}_k(x) = \delta^j{}_k + \omega^j{}_k(x), \quad \omega_{jk} + \omega_{kj} = 0, \quad |\omega_{jk}| \ll 1, \tag{8.3}$$

the condition (8.2) becomes

$$\epsilon^{\mu\nu\rho\sigma} b^j{}_\nu b^k{}_\rho \omega_{jk}(x)_{,\sigma} = 0, \tag{8.4}$$

by neglecting the second- and higher-order terms of ω_{jk}. Since the condition (8.4) is linear in ω_{jk}, the infinitesimal neighborhood of the unit element in $\Lambda(\underline{b})$ has some of a Lie-algebra property: The inverse, $(A^{-1})^j{}_k = \delta^j{}_k - \omega^j{}_k$, and the product, $(A'A)^j{}_k = \delta^j{}_k + \omega'{}_{jk} + \omega_{jk}$, satisfy (8.4) for any two infinitesimal local Lorentz transformations, A and A', belonging to $\Lambda(\underline{b})$.

As an example of the extended Weitzenböck space-time, consider the static isotropic space-time, which has the Schwarzschild metric for the present case of $c_1 = 0 = c_2$. Written in the isotropic coordinates used in Sec. V, the Schwarzschild metric is expressed in the isotropic form,

$$ds^2 = -A(r)dt^2 + B(r)dx^\alpha dx^\alpha, \tag{8.5}$$

with

$$A(r) = \frac{(1 - GM/2r)^2}{(1 + GM/2r)^2}, \quad B(r) = \left(1 + \frac{GM}{2r}\right)^4, \tag{8.6}$$

and the parallel vector fields $\underline{b}$ defined by (5.24) are

$$\underline{b}_{(0)} = \frac{1}{\sqrt{A}} \underline{E}_0, \quad \underline{b}_{(a)} = \frac{1}{\sqrt{B}} \delta^\alpha_a \underline{E}_\alpha, \tag{8.7}$$

with $\{\underline{E}_\mu\}$ the coordinate basis; $\underline{E}_\mu = \partial/\partial x^\mu$. The axial-vector part of the torsion tensor, formed of $\underline{b}$, thus vanishes identically, and so the static, isotropic space-time is the extended Weitzenböck space-time. Besides the parallel vector fields $\underline{b}$ of (8.7), there exists an infinitely large number of parallel vector fields, which are all equivalent to $\underline{b}$ and with each other: All parallel vector fields are related to $\underline{b}$ of (8.7) by local Lorentz transformations satisfying the condition (8.2). The condition (8.4) for an infinitesimal local Lorentz transformation specified by (8.3) can easily be solved in this case: The solution which leaves $\underline{b}$ static is

$$\omega_{(a)(b)} = \delta^\alpha_a \delta^\beta_b (H_{\alpha,\beta} - H_{\beta,\alpha}), \quad \omega_{(0)(a)} = -\omega_{(a)(0)} = \delta^\alpha_a H_{0,\alpha}, \tag{8.8}$$

with H_μ arbitrary small functions independent of t, where Latin (Lorentz) indices in ω_{jk} are enclosed by parentheses.

For a finite local Lorentz transformation it is not easy to solve the Eq. (8.2). However, we can find a special kind of the parallel vector fields in the static, isotropic space-time by looking for such a set of coordinates $\{x^\mu\}$ that the Schwarzschild metric is expressed in the diagonal form of (7.12). The parallel vector fields, defined by (7.13), in such coordinates are equivalent to the parallel vector fields of (8.7) defined in the isotropic coordinates. An example is given by the spherical polar coordinates (t, r, θ, ϕ) introduced by (5.26): The Schwarzschild metric reads as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\theta^2 + D(r,\theta)d\phi^2 \tag{8.9}$$

with A and B still given by (8.6), and

$$C(r) = r^2\left(1 + \frac{GM}{2r}\right)^4, \quad D(r,\theta) = r^2\left(1 + \frac{GM}{2r}\right)^4 \sin^2\theta. \tag{8.10}$$

Thus, the system of four orthonormal vectors, $\underline{b}'$,

$$\underline{b}'_{(0)} = \frac{1}{\sqrt{A}}\underline{E}_t, \quad \underline{b}'_{(1)} = \frac{1}{\sqrt{B}}\underline{E}_r, \quad \underline{b}'_{(2)} = \frac{1}{\sqrt{C}}\underline{E}_\theta, \quad \underline{b}'_{(3)} = \frac{1}{\sqrt{D}}\underline{E}_\phi, \tag{8.11}$$

with

$$\underline{E}_t = \partial/\partial t, \quad \underline{E}_r = \partial/\partial r, \quad \underline{E}_\theta = \partial/\partial\theta, \quad \underline{E}_\phi = \partial/\partial\phi, \tag{8.12}$$

is also a solution of the gravitational field equation in vacuum, and can be taken as the parallel vector fields, which are related to $\underline{b}$ of (8.7) by a local space rotation

$$\underline{b}'_{(0)} = \underline{b}_{(0)}, \quad \underline{b}'_{(a)} = R_{ac}\underline{b}_{(c)}, \tag{8.13}$$

with

$$(R_{ac}) = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}. \tag{8.14}$$

The parallel vector fields $\underline{b}'$ are usually used as tetrad fields in quantum field theory for the Schwarzschild space-time.[36] The Schwarzschild metric is of the form (7.12) also in the Kruskal-Szekeres coordinates,[37] and so we can use this coordinate system to form parallel vector fields $\underline{b}''$ by (7.13).

IX. THE WEAK-FIELD APPROXIMATION

Further insights into new general relativity can be gained by applying the set of the gravitational field equation, (7.2) and (7.3), to weak-field situations

$$b^k{}_\mu(x) = \delta^k{}_\mu + a^k{}_\mu(x), \quad |a^k{}_\mu| \ll 1, \quad b_k{}^\mu(x) = \delta_k{}^\mu + c_k{}^\mu(x), \quad |c_k{}^\mu| \ll 1, \tag{9.1a}$$

since in this case the particle spectrum of the new general relativity can be clarified by the use of the unitary, irreducible representations of the Poincaré group. In this situation we can expand the field equation in $a^k{}_\mu$ and $c_k{}^\mu$ and can keep only lowest terms. Thus we need not distinguish Latin indices from Greek indices, which are now raised and lowered by the Minkowski metric tensor, $\{\eta^{\mu\nu}\}$ or $\{\eta_{\mu\nu}\}$: We shall use Greek indices throughout this section. From (2.5a) follows

$$a^{\nu}_{\cdot\mu}+c^{\cdot\nu}_{\mu}=0\,, \tag{9.1b}$$

and hence we take $\{a_{\mu\nu}\}$ as the basic field variable. We shall decompose the weak field $\{a_{\mu\nu}\}$ into its symmetric and antisymmetric parts,

$$a_{\mu\nu}=\tfrac{1}{2}h_{\mu\nu}+A_{\mu\nu}\,, \tag{9.2}$$

with $h_{\mu\nu}=h_{\nu\mu}$ and $A_{\mu\nu}=-A_{\nu\mu}$. The components of the metric tensor are then written as

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\,. \tag{9.3}$$

The antisymmetric field makes no contribution to the space-time metric in this approximation, implying that it is associated with the intrinsic spin of spin-$\frac{1}{2}$ fundamental particles.

The Einstein tensor becomes to lowest order in $h_{\mu\nu}$

$$G^{(1)}_{\mu\nu}(\{\,\})=-\tfrac{1}{2}[\Box\bar{h}_{\mu\nu}-\partial^{\lambda}(\partial_\mu\bar{h}_{\nu\lambda}+\partial_\nu\bar{h}_{\mu\lambda}) +\eta_{\mu\nu}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma}]\,, \tag{9.4}$$

where $\Box=\partial^\mu\partial_\mu$ and $\bar{h}_{\mu\nu}$ are defined as usual by

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}h,\quad h=\eta^{\mu\nu}h_{\mu\nu}\,. \tag{9.5}$$

The second term of (7.2), i.e., $\{K^{\mu\nu}\}$, is of second order in $h_{\mu\nu}$ and $A_{\mu\nu}$, and hence can be ignored. The left-hand side of (7.3), when indices are lowered, becomes in the weak-field approximation

$$b_{i\mu}b_{j\nu}\partial_\rho(\sqrt{-g}\;J^{ij\rho})=-[\Box A_{\mu\nu}-\partial^{\lambda}(\partial_\mu A_{\nu\lambda}-\partial_\nu A_{\mu\lambda})]\,, \tag{9.6a}$$

because the axial-vector part $\{a^\mu\}$ of the torsion tensor is given by

$$a^\mu=\tfrac{1}{3}\epsilon^{\mu\nu\rho\sigma}\partial_\nu A_{\rho\sigma}\,. \tag{9.6b}$$

Thus in the weak-field approximation the symmetric and antisymmetric parts of the gravitational field equation are given by

$$\Box\bar{h}_{\mu\nu}-\partial^{\lambda}(\partial_\mu\bar{h}_{\nu\lambda}+\partial_\nu\bar{h}_{\mu\lambda})+\eta_{\mu\nu}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma}=-2\kappa T_{(\mu\nu)}\,, \tag{9.7}$$

$$\Box A_{\mu\nu}-\partial^{\lambda}(\partial_\mu A_{\nu\lambda}-\partial_\nu A_{\mu\lambda})=-\lambda T_{[\mu\nu]}\,. \tag{9.8}$$

It follows from these equations that the symmetric field $\{h_{\mu\nu}\}$ and the antisymmetric field $\{A_{\mu\nu}\}$ are completely decoupled from each other. The $\{h_{\mu\nu}\}$ obeys the linearized Einstein field equation. The nonsymmetric energy-momentum tensor $\{T_{\mu\nu}\}$ is taken to lowest order in the weak fields; namely, it is independent of $\{h_{\mu\nu}\}$ and $\{A_{\mu\nu}\}$, and satisfies the ordinary conservation law in special relativity,

$$\partial_\nu T^{\mu\nu}=0\,, \tag{9.9}$$

by virtue of the response equation (6.6).

Multiplying ∂^ν on both sides of (9.7) and (9.8), we find that both the symmetric and antisymmetric parts of $\{T_{\mu\nu}\}$ satisfy the conservation law,

$$\partial_\nu T^{(\mu\nu)}=0\,, \tag{9.10a}$$

$$\partial_\nu T^{[\mu\nu]}=0\,. \tag{9.10b}$$

By virtue of (9.9), these two equations are not independent of each other. The conservation law (9.10b) imposes a severe restriction on the form of spin tensor of matter. In fact, due to the Tetrode formula in special relativity,[39]

$$2T^{[\mu\nu]}=\partial_\rho S^{\mu\nu\rho}\,, \tag{9.11}$$

Eq. (9.10b) is automatically satisfied if and only if a spin tensor, $\{S^{\mu\nu\rho}\}$, is totally antisymmetric with respect to its three indices. Thus, the gravitational field equation demands that a spin tensor be expressed as

$$S^{\mu\nu\rho}=\epsilon^{\mu\nu\rho\sigma}J_{5\sigma} \tag{9.12}$$

by an axial-vector current, $\{J_{5\sigma}\}$. For Dirac particles, $\{J_{5\sigma}\}$ is given by

$$J_{5\sigma}=-\tfrac{1}{2}\bar{\psi}\gamma^5\gamma_\sigma\psi\,. \tag{9.13}$$

As can be checked by direct calculation, the linearized field equations of (9.7)–(9.8) are invariant under gauge transformations,

$$h'_{\mu\nu}=h_{\mu\nu}-\partial_\mu J_\nu-\partial_\nu J_\mu\,, \tag{9.14a}$$

$$A'_{\mu\nu}=A_{\mu\nu}+\partial_\mu H_\nu-\partial_\nu H_\mu\,, \tag{9.14b}$$

with J_μ and H_μ arbitrary small functions which leave the fields weak. In a particular case, $J_\mu=2H_\mu=\Lambda_\mu$, these gauge transformations give rise to an infinitesimal coordinate transformation, $x^\mu\to x'^\mu=x^\mu+\Lambda^\mu(x)$. By means of these gauge freedoms, we can put the gauge conditions,

$$\partial_\nu\bar{h}^{\mu\nu}=0\,, \tag{9.15}$$

$$\partial_\nu A^{\mu\nu}=0\,, \tag{9.16}$$

which we shall assume henceforth. Then the field equations of (9.7)–(9.8) become

$$\Box\bar{h}_{\mu\nu}=-2\kappa T_{(\mu\nu)}\,, \tag{9.17}$$

$$\Box A_{\mu\nu}=-\lambda T_{[\mu\nu]}\,. \tag{9.18}$$

We shall restrict our discussions to the antisymmetric field $\{A_{\mu\nu}\}$, because the physics of the $\{h_{\mu\nu}\}$ field is well known.[39] The retarded solution of (9.18) is given by

$$A_{\mu\nu}(\vec{x},t)=\frac{\lambda}{4\pi}\int d^3x'\,\frac{T_{[\mu\nu]}(\vec{x}',t-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}. \tag{9.19}$$

Suppose that we observe the gravitational field in the space region far outisde a source; therefore, we can calculate the solution of (9.19) to lowest order in $1/r \equiv 1/|\vec{x}|$, using the expansion

$$\begin{aligned}|\vec{x}-\vec{x}'| &= r-\frac{\vec{x}\cdot\vec{x}'}{r}+O\left(\frac{1}{r}\right),\\ |\vec{x}-\vec{x}'|^{-1} &= \frac{1}{r}+\frac{\vec{x}\cdot\vec{x}'}{r^3}+O\left(\frac{1}{r^3}\right).\end{aligned} \tag{9.20}$$

We assume that the energy-momentum tensor can be expressed as Fourier integral or as a sum of Fourier components; suppose we calculate for a single Fourier component,

$$\begin{aligned}T_{[\mu\nu]}(\vec{x},t) &= T_{[\mu\nu]}(\vec{x},\omega)e^{-i\omega t}\\ &\quad+\bar{T}_{[\mu\nu]}(\vec{x},\omega)e^{+i\omega t}.\end{aligned} \tag{9.21}$$

where a bar means complex conjugation. In the wave zone the solution (9.19) then becomes just like a plane wave

$$A_{\mu\nu}(\vec{x},t)=d_{\mu\nu}(\vec{x},\omega)e^{ikx}+\bar{d}_{\mu\nu}(\vec{x},\omega)e^{-ikx}, \tag{9.22}$$

with the wave vector,

$$\vec{k}\equiv\omega\hat{x},\quad k^0\equiv\omega,\quad(\hat{x}=\vec{x}/r), \tag{9.23}$$

and the polarization tensor

$$d_{\mu\nu}(\vec{x},\omega)=\frac{\lambda}{4\pi r}\int d^3x'\,T_{[\mu\nu]}(\vec{x}',\omega)e^{-i\vec{k}\cdot\vec{x}'}. \tag{9.24}$$

The wave vector $\{k^\mu\}$ is a null vector, and the polarization tensor satisfies the conditions,

$$k^\nu d_{\mu\nu}(\vec{x},\omega)=0,\quad k^\nu\bar{d}_{\mu\nu}(\vec{x},\omega)=0 \tag{9.25}$$

by virtue of the conservation law (9.10b). Since r is very large, the $\vec{x}$ dependence of $d_{\mu\nu}(\vec{x},\omega)$ can be neglected, and so the plane wave (9.22) satisfies the d'Alembert equation, $\Box A_{\mu\nu}(\vec{x},t)=0$, and the gauge condition (9.16).

The energy-momentum tensor $\{t^{\mu\nu}\}$ of the $\{A_{\mu\nu}\}$ field is given by

$$\begin{aligned}t^{\mu\nu} &= -[\partial L_A/\partial(\partial_\nu A_{\rho\sigma})]\partial^\mu A_{\rho\sigma}+\eta^{\mu\nu}L_A\\ &= -\left(\frac{3}{2\lambda}\right)a_\lambda\epsilon^{\lambda\rho\sigma\nu}\partial^\mu A_{\rho\sigma}+\left(\frac{9}{4\lambda}\right)\eta^{\mu\nu}(a_\lambda a^\lambda),\end{aligned} \tag{9.26}$$

where L_A is the linearized Lagrangian density of the $\{A_{\mu\nu}\}$ field,

$$L_A=c_3(a_\mu a^\mu)=\left(\frac{9}{4\lambda}\right)(a_\mu a^\mu). \tag{9.27}$$

We use the plane wave solution (9.22) in (9.26), and average $t^{\mu\nu}$ over a space-time region much larger than $|\vec{k}|^{-1}$. The average kills all terms proportional to $\exp(\pm 2ikx)$, and we are left with only the $\vec{x}$-independent terms,

$$\begin{aligned}\langle t^{\mu\nu}\rangle &= (2/\lambda)\,\mathrm{Re}(k^\mu k^\nu d^{\rho\sigma}\bar{d}_{\rho\sigma}-2k^\mu k^\rho d^{\nu\sigma}\bar{d}_{\rho\sigma})\\ &\quad+(2/\lambda)\eta^{\mu\nu}k_\rho k^\sigma d^{\rho\lambda}\bar{d}_{\sigma\lambda}.\end{aligned} \tag{9.28}$$

Using the condition (9.25), we then find

$$\langle t^{\mu\nu}\rangle=(4/\lambda)|d_{12}|^2k^\mu k^\nu, \tag{9.29}$$

where we have chosen the direction of $\vec{k}$ as the third axis. Therefore, *only the* (12) *component is physically significant, and the energy density* t^{00} *is positive definite if the constant* λ *is positive.*

The (12) component, d_{12}, does not change at all under a rotation around the third axis; in fact, for such a rotation the rotation matrix (R_{ab}) satisfies $R_{13}=0=R_{23}$, and hence the d_{12} transforms like a scalar,

$$\begin{aligned}d'_{12} &= R_{1a}R_{2b}d_{ab}=(R_{11}R_{12}-R_{12}R_{21})d_{12}\\ &= \det(R_{ab})d_{12}=d_{12}.\end{aligned} \tag{9.30}$$

The physically significant component d_{12} is thus of helicity zero. In the terminology of elementary particle physics *the* $\{A_{\mu\nu}\}$ *field is a massless field of spin* 0.

In the above discussion the $\{A_{\mu\nu}\}$ field is assumed to be a classical field. The quantization of the $\{A_{\mu\nu}\}$ field can be performed consistently, and the resulting quantized theory does not involve ghost states.[40]

The space components of the solution (9.22), which decrease as $1/r$, contribute to the energy-momentum tensor, but the (0α) components do not, and hence they are of no physical significance. This fact suggests that the next terms of $A_{0\alpha}(\vec{x},t)$, which are proportional to $1/r^2$, are important. In order to eliminate the $(1/r)$ term from $A_{0\alpha}(\vec{x},t)$, we rewrite the (0α) components of (9.19) as follows:

$$\begin{aligned}A_{0\alpha}(\vec{x},t) &= \frac{\lambda}{4\pi}\int d^3x'\,T_{[0\alpha]}(\vec{x}',t-r)\\ &\quad\times\left[\frac{1}{r}+\frac{\vec{x}\cdot\vec{x}'}{r^3}+O\left(\frac{1}{r^3}\right)\right]+\frac{\partial}{\partial t}\xi_\alpha,\end{aligned} \tag{9.31}$$

with ξ_α given by

$$\begin{aligned}\xi_\alpha &= \frac{\lambda}{4\pi}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{\partial^n}{\partial t^n}\int d^3x'\,T_{[0\alpha]}(\vec{x}',t-r)\\ &\quad\times\frac{(r-|\vec{x}-\vec{x}'|)^{n+1}}{|\vec{x}-\vec{x}'|}.\end{aligned} \tag{9.32}$$

The integral in (9.31) can be rewritten by using the relations, (9.11) and (9.12):

$$\int d^3x'\,T_{[0\alpha]}(\vec{x}',t-r)=\epsilon_{0\alpha\beta\gamma}\int d^3x'\,J_{5\gamma'\beta}(\vec{x}',t-r)=0,$$

$$\int d^3x' T_{[0\alpha]}(\vec{x}', t-r)(\vec{x}\cdot\vec{x}')$$
$$= \tfrac{1}{2}\epsilon_{0\alpha\beta\gamma}\int d^3x' J_{5\gamma',\beta}(\vec{x}', t-r)(\vec{x}\cdot\vec{x}')$$
$$= \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int d^3x' J_{5\gamma}(\vec{x}', t-r)x^{\beta}$$
$$= \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}x^{\beta}S^{\gamma}(t-r),$$

where $\vec{S}=\{S^{\alpha}\}$ is a *total intrinsic spin* of the source,

$$S_{\alpha}(t) = \int d^3x J_{5\alpha}(\vec{x}, t) = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int d^3x S^{\beta\gamma 0}(\vec{x}, t). \tag{9.33}$$

We now perform a gauge transformation (9.14b) with H_{μ} given by

$$H_{\alpha} = \xi_{\alpha}, \quad H_0 = 0. \tag{9.34}$$

Then, dropping a prime on $A_{0\alpha}$, we finally get

$$A_{0\alpha}(\vec{x}, t) = \frac{\lambda}{8\pi}\epsilon_{\alpha\beta\gamma}\frac{x^{\beta}S^{\gamma}(t-r)}{r^3} + O\left(\frac{1}{r^3}\right). \tag{9.35}$$

Since ξ_{α} decrease as $1/r$, the change of the space components, $\delta A_{\alpha\beta} = \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha}$, decreases as $1/r^2$, and hence the $(1/r)$ terms of $A_{\alpha\beta}$ do not change under this gauge transformation. The expression (9.35) is to be compared with the asymptotic expression for $h_{0\alpha}(\vec{x}, t)$,[41]

$$h_{0\alpha}(\vec{x}, t) = \frac{\kappa}{4\pi}\epsilon_{\alpha\beta\gamma}\frac{x^{\beta}M^{\gamma}(t-r)}{r^3}, \tag{9.36}$$

where $\vec{M}=\{M^{\alpha}\}$ is a *total angular momentum* of the source,

$$M_{\alpha}(t) = \epsilon_{\alpha\beta\gamma}\int d^3x\, x^{\beta}T^{(\gamma 0)}(\vec{x}, t). \tag{9.37}$$

See Table I for an illustration of $\{h_{\mu\nu}\}$ and $\{A_{\mu\nu}\}$ in the asymptotic region.

TABLE I. Asymptotic expressions for $h_{\mu\nu}$ and $A_{\mu\nu}$ far from a weakly gravitating system. The result for $h_{\mu\nu}$ is well known (Ref. 41), but we list it here for comparison's sake.

$\mu\nu$	$h_{\mu\nu}$	$A_{\mu\nu}$
00	$-\left[1-\frac{2GM}{r}+O\left(\frac{1}{r^2}\right)\right]$	0
0α	$\frac{\kappa}{4\pi}\epsilon_{\alpha\beta\gamma}\frac{x^{\beta}M^{\gamma}}{r^3}+O\left(\frac{1}{r^3}\right)$	$\frac{\lambda}{8\pi}\epsilon_{\alpha\beta\gamma}\frac{x^{\beta}S^{\gamma}}{r^3}+O\left(\frac{1}{r^3}\right)$
$\alpha\beta$	$\left(1+\frac{2GM}{r}\right)\delta_{\alpha\beta}$ + [gravitational radiation terms that die out as $O(1/r)$]	[gravitational radiation terms that die out as $O(1/r)$]

X. COUPLING OF AN ANTISYMMETRIC FIELD

As is well known, the symmetric field $\{h_{\mu\nu}\}$ can be neglected in atomic phenomena. So we shall study the coupling of an antisymmetric field, assuming that the metric tensor is the Minkowski metric tensor,

$$g_{\mu\nu} = \eta_{\mu\nu}. \tag{10.1}$$

It is then convenient to employ a Cartesian coordinate system $\{x^{\mu}\}$: The tetrad fields associated with it, which we denote by $\underline{e}=\{\underline{e}_k\}=\{e_k{}^{\mu}=\delta_k{}^{\mu}\}$, are related to the parallel vector fields $\underline{b}$ by a local Lorentz transformation,

$$\underline{e}_k = \Lambda^j{}_k(x)\underline{b}_j,$$
$$\Lambda^j{}_k(x) = \delta^j{}_k - \delta_k{}^{\mu}\delta^{j\nu}A_{\mu\nu}(x). \tag{10.2}$$

Here we assume that an antisymmetric field $\{A_{\mu\nu}\}$ is so weak that we can neglect the second- and higher-order terms of $\{A_{\mu\nu}\}$.

In Sec. II, the Dirac spinor wave function was introduced by referring to the parallel vector fields $\underline{b}$; we denote it here by ψ_b. The Dirac spinor wave function ψ_e, which is defined by referring to the tetrad fields $\underline{e}$, is related to ψ_b by the local Lorentz transformation (10.2);

$$\psi_e = U(\Lambda)\psi_b, \quad U(\Lambda) = 1 - \tfrac{1}{2}i A_{\mu\nu}S^{\mu\nu}. \tag{10.3}$$

It should be remarked here that the spinor wave function ψ_e is usually used in atomic physics to describe the electron.

Suppose that ψ_b satisfies the Dirac equation (3.13b), then Eq. (10.3) implies that ψ_e satisfies

$$(i\gamma^{\mu}\partial_{\mu} - \tfrac{3}{4}a_{\mu}\gamma^5\gamma^{\mu} - m)\psi_e = 0 \tag{10.4}$$

by virtue of the following property of the covariant derivative ∇_{μ}:

$$U(\Lambda)\nabla_{\mu}^{(b)}\psi_b = \nabla_{\mu}^{(e)}\psi_e = \partial_{\mu}\psi_e. \tag{10.5}$$

Here $\nabla_{\mu}^{(b)}$ and $\nabla_{\mu}^{(e)}$ mean the covariant derivative defined by the Ricci rotation coefficients formed of $\underline{b}$ and $\underline{e}$, respectively: $\nabla_{\mu}^{(e)}$ coincides with the usual derivative ∂_{μ}, since $e_k{}^{\mu} = \delta_k{}^{\mu}$.

Now we apply the Dirac equation (10.4) to the electron in the hydrogen atom, including the electromagnetic interaction between the electron and the proton by the minimal principle

$$\partial_{\mu} \to \partial_{\mu} + ieA_{\mu}, \tag{10.6}$$

where $(-e)$ is the electric charge of the electron, and the electromagnetic potential $\{A^{\mu}\} = (A^0, \vec{A})$ is given by

$$A^0 = -e^2/r = -A_0,$$
$$\vec{A} = (eg_p/2M_p)(\vec{\nabla}\times\vec{S}_p)\frac{1}{r}. \tag{10.7}$$

Here the vector potential $\vec{A}$ is due to the magnetic moment of the proton; M_p, $\vec{S}_p$, and g_p are the mass, the spin, and the gyromagnetic ratio of the proton, respectively. The Dirac equation (10.4) then becomes

$$[i\gamma^\mu(\partial_\mu + ieA_\mu) - \tfrac{3}{4}a_\mu\gamma^5\gamma^\mu - m]\psi_e = 0 , \tag{10.8}$$

for the electron in hydrogen atom.

For the proton at rest at the origin, the axial-vector current of (9.13) is given by

$$J_{50} = 0, \quad \vec{J}_5 = \vec{S}_p\,\delta^3(\vec{x}) . \tag{10.9}$$

Use of this in (9.11)–(9.12) shows that space-space components of the antisymmetric part of $T_{\mu\nu}$, $T_{[\alpha\beta]}$, vanish identically; therefore, we find that

$$A_{\alpha\beta} = 0 \tag{10.10a}$$

around the proton. On the other hand, the (0α) components of an antisymmetric field are given by (9.35):

$$(A_{0\alpha}) = -\frac{\lambda}{8\pi}(\vec{\nabla}\times\vec{S}_p)\frac{1}{r} . \tag{10.10b}$$

Using (10.10a)–(10.10b) in (9.6b), we obtain the axial-vector part of the torsion tensor around the proton at rest,

$$a^0 = 0, \quad \vec{a} = -\frac{\lambda}{12\pi}\vec{\nabla}\times(\vec{\nabla}\times\vec{S}_p)\frac{1}{r} . \tag{10.11}$$

In order to evaluate the effects due to an antisymmetric field, we rewrite the Dirac equation (10.8) into two-component wave equations,

$$\left(i\frac{\partial}{\partial t} + \frac{e^2}{r} + \frac{3}{4}\vec{a}\cdot\vec{\sigma}\right)\phi = \vec{\sigma}\cdot(\vec{p} + e\vec{A})\chi \tag{10.12a}$$

$$\left(i\frac{\partial}{\partial t} + 2m + \frac{e^2}{r} + \frac{3}{4}\vec{a}\cdot\vec{\sigma}\right)\chi = \vec{\sigma}\cdot(\vec{p} + e\vec{A})\phi , \tag{10.12b}$$

where we put

$$\psi_e = \begin{pmatrix}\phi\\ \chi\end{pmatrix}e^{-imt} , \tag{10.13}$$

and used the standard representation of the γ matrices.[42] Here $\vec{p}$ denotes the momentum operator; $p^\alpha = -i\partial/\partial x^\alpha$. In the Pauli approximation, in which (10.12b) may be approximated to

$$\chi = \frac{1}{2m}(\vec{p} + e\vec{A})\cdot\vec{\sigma}\phi , \tag{10.12b$'$}$$

we get

$$i\frac{\partial\phi}{\partial t} = H\phi , \tag{10.14}$$

$$H = \frac{1}{2m}(\vec{p} + e\vec{A})^2 - \frac{e^2}{r} + \left(\frac{e^2 g_p}{4mM_p} + \frac{\lambda}{16\pi}\right) \times\vec{\sigma}\cdot\{\vec{\nabla}\times(\vec{\nabla}\times\vec{S}_p)\}\frac{1}{r} , \tag{10.15}$$

The last term of (10.15), which consists of two parts, describes the spin-spin interaction of the electron and the proton: One is due to the magnetic moment of the proton, and the other due to an antisymmetric field.

The spin-spin coupling due to an antisymmetric field is not restricted to the case of the electron and the proton, but quite universal. For any two spin-$\frac{1}{2}$ particles, A and B, separated by r, we can show in the similar way that *in the nonrelativistic approximation the coupling with an antisymmetric field leads to universal spin-spin coupling,*

$$\begin{aligned}H_{\text{spin-spin}} &= \frac{\lambda}{8\pi}\vec{S}_A\cdot[\vec{\nabla}\times(\vec{\nabla}\times\vec{S}_B)]\frac{1}{r}\\ &= \frac{\lambda}{8\pi}\left[\frac{8\pi}{3}(\vec{S}_A\cdot\vec{S}_B)\delta^3(\vec{x})\right.\\ &\quad\left.-\frac{1}{r^3}\left((\vec{S}_A\cdot\vec{S}_B) - \frac{3}{r^2}(\vec{S}_A\cdot\vec{x})(\vec{S}_B\cdot\vec{x})\right)\right],\end{aligned} \tag{10.16}$$

where $\vec{S}_A$ and $\vec{S}_B$ are the spin vectors of the spin-$\frac{1}{2}$ particles, A and B, respectively. This spin-spin coupling makes a contribution to the hyperfine splitting of energy levels in atoms and muonium (the bound state of an electron and a positive muon).

Let us first consider the hyperfine structure interval $\Delta\nu(H)$ of the ground state of the hydrogen atom. We denote by $\Delta\nu_{\text{QED}}(H)$ the theoretical value which is based on conventional quantum electrodynamics and on the assumption that the proton is a Dirac particle without internal structure. Adding possible corrections to $\Delta\nu_{\text{QED}}(H)$, we express $\Delta\nu(H)$ as

$$\Delta\nu(H) = \Delta\nu_{\text{QED}}(H)[1 + \delta_N^{(2)} + \delta_A(H)] . \tag{10.17}$$

Here $\delta_N^{(2)}$ is the correction due to internal structure of the proton: The precise value of $\delta_N^{(2)}$ is not known at present, but it is estimated to be 1–2 ppm.[43] The last term $\delta_A(H)$ is a possible correction which arises from universal spin-spin coupling of (10.16): From the expression (10.15) for the Hamiltonian, we obtain

$$\delta_A(H) = \frac{\lambda/16\pi}{e^2 g_p/4mM_p} = 0.012\times\frac{\lambda}{4\pi}\,(\text{GeV})^2 . \tag{10.18}$$

The theoretical value $\Delta\nu_{\text{QED}}(H)$ is in good agreement with the experimental value[44];

$$\frac{\Delta\nu_{\exp}(H) - \Delta\nu_{\text{QED}}(H)}{\Delta\nu_{\exp}(H)} = (2.5 \pm 4.0)\times 10^{-6} . \tag{10.19}$$

Since the correction $\delta_N^{(2)}$ is of the order of 1 ppm, we estimate the upper limit on $\delta_A(H)$ as

$$\delta_A(H) \lesssim 5\times10^{-6} \,. \tag{10.20}$$

Because of ambiguity in the correction $\delta_N^{(2)}$, it seems difficult to estimate the value of $\delta_A(H)$ with higher precision than (10.20). Combining (10.18) and (10.20), we get

$$\frac{\lambda}{4\pi} \lesssim 4\times10^{-4}\ (\mathrm{GeV})^{-2} \,. \tag{10.21}$$

Next we shall consider the hyperfine structure interval of the ground state of muonium. The theoretical value $\Delta\nu_{\mathrm{QED}}(e\mu)$ based on conventional quantum electrodynamics agrees well with the experimental value[45];

$$\frac{\Delta\nu_{\exp}(e\mu) - \Delta\nu_{\mathrm{QED}}(e\mu)}{\Delta\nu_{\exp}(e\mu)} = -(0.45\pm1.5)\times10^{-6} \,. \tag{10.22}$$

The value of $\Delta\nu_{\exp}(e\mu)$ is known with much higher precision than $\Delta\nu_{\mathrm{QED}}(e\mu)$, due to uncertainty in our knowledge of the fundamental constants μ_μ/μ_p and α; here μ_μ and μ_p are the magnetic moments of the muon and the proton, respectively, and α is the fine-structure constant. Possible fractional correction to $\Delta\nu_{\mathrm{QED}}(e\mu)$, $\delta_A(e\mu)$, which arises from universal spin-spin coupling of (10.16), is obtained from (10.18) by replacing the proton parameters, M_p and g_p, with the muon parameters, M_μ and g_μ;

$$\delta_A(e\mu) = \frac{\lambda/16\pi}{e^2 g_\mu/4mM_\mu} = 0.0037\times\frac{\lambda}{4\pi}\ (\mathrm{GeV})^2 \,. \tag{10.23}$$

Here M_μ is the muon mass and g_μ is the gyromagnetic ratio of the muon. From (10.22) we estimate the upper limit on $\delta_A(e\mu)$ as

$$\delta_A(e\mu) \lesssim 10^{-6} \,. \tag{10.24}$$

This upper limit can be improved, provided that the fundamental constants μ_μ/μ_p and α would be known with higher precision. Using (10.23) in (10.24), we obtain

$$\frac{\lambda}{4\pi} \lesssim 3\times10^{-4}\ (\mathrm{GeV})^{-2} \,. \tag{10.25}$$

Summing up, we conclude from (10.21) and (10.25) that *the square of the coupling strength of an antisymmetric field is bounded by* $\lambda/4\pi \lesssim 3\times10^{-4}\ (\mathrm{GeV})^{-2}$. This result is in agreement with the quantum-field-theoretical estimation of Miyamoto and Nakano.[7]

XI. TIME-DEPENDENT SPHERICALLY SYMMETRIC FIELDS

We now turn to a spherically symmetric, but *not necessarily static* gravitational field in vacuum. When we considered the static, isotropic gravitational field in Sec. V, we assumed that the state of a central gravitating spherical body does not change under space inversion, besides it is invariant under time reversal and space rotation. This is the case either if constituent particles of a spherical body are spinless, or if the spin of constituent particles is randomly distributed and can be ignored. If the spin of constituent particles of a spherical body happens to be polarized to outward (or inward) radial direction, however, the spin state of the gravitating body changes under space inversion: In fact, if the spin of constituent particles is polarized to *outward* radial direction, then after space inversion the spin is polarized to *inward* radial direction. Therefore, we assume here that *the parallel vector fields* $\underline{b} = \{\underline{b}_k\} = \{b_k{}^\mu\}$ *and their dual* $\underline{b}^* = \{\underline{b}^k\} = \{b^k{}_\mu\}$ *are form invariant under space rotation* (5.1c), *but not necessarily form invariant under time reversal* (5.1a) *and space inversion* (5.1b). It is shown in Appendix B that we can then take the following expression for $\underline{b}^* = \{\underline{b}^k\} = \{b^k{}_\mu\}$:

$$(b^k{}_\mu) = \begin{array}{c} \\ \downarrow \\ k \end{array}\!\!\begin{array}{c} \to\mu \\ \begin{pmatrix} C & 0 \\ Hx^a & D\delta^a_\alpha + F\epsilon_{a\alpha\beta}x^\beta \end{pmatrix} \end{array}, \tag{11.1a}$$

where C, D, F, and H are unknown functions of t and $r = (x^\alpha x^\alpha)^{1/2}$. The parallel vector fields $\underline{b} = \{\underline{b}_k\} = \{b_k{}^\mu\}$ are then represented as

$$(b_k{}^\mu) = \begin{array}{c} \\ \downarrow \\ k \end{array}\!\!\begin{array}{c} \to\mu \\ \begin{pmatrix} 1/C & -\dfrac{H}{CD}x^\alpha \\ 0 & \left(D\delta^\alpha_a + \dfrac{F^2}{D}x^a x^\alpha + F\epsilon_{a\alpha\beta}x^\beta\right)\Big/(D^2 + r^2F^2) \end{pmatrix} \end{array} \tag{11.1b}$$

and the invariant distance ds^2 is expressed in a rotationally invariant form,

$$ds^2 = -(C^2 - r^2H^2)dt^2 + 2DH\,dt(x^\alpha dx^\alpha) + (D^2 + r^2F^2)dx^\alpha dx^\alpha - F^2(x^\alpha dx^\alpha)^2 \,. \tag{11.2}$$

In empty space the antisymmetric part (7.3) of the gravitational field equation reads

$$\partial_\rho(\sqrt{-g}\,\epsilon^{ijmn}b_m{}^\rho b_n{}^\sigma a_\sigma) = 0 \,. \tag{11.3}$$

The axial-vector part $\{a^\mu\}$ of the torsion tensor is expressed in terms of the unknown functions, C, D, F, and H, as

$$\sqrt{-g}\,a^\mu = \begin{cases} P, & \text{for } \mu = 0 \,, \\ Qx^\alpha, & \text{for } \mu = \alpha \,, \end{cases} \tag{11.4}$$

with P and Q defined by

$$P = 2DF + \tfrac{2}{3}r(DF' - D'F)\,,$$
$$Q = -\tfrac{4}{3}HF + \tfrac{2}{3}(\dot{D}F - D\dot{F})\,, \tag{11.5}$$

where a dot and a prime denote $\partial/\partial t$ and $\partial/\partial r$, respectively. Using (11.4) in (11.3), we get

$$2F\,\frac{HP+DQ}{D^2+r^2F^2}\,x^a = 0 \tag{11.6}$$

for $(i,j) = (0,a)$, and

$$\partial_0\left(\frac{1}{C}(HP+DQ)\epsilon_{abc}x^c\right) - \partial_\alpha\left(\frac{H}{CD}(HP+DQ)\epsilon_{abc}x^c x^\alpha\right) + \frac{\epsilon_{abc}x^c}{r}\left[\left(\frac{C}{D}P\right)' + 2r\,\frac{F^2}{D^2+r^2F^2}\left(\frac{C}{D}P\right)\right] = 0 \tag{11.7}$$

for $(i,j) = (a,b)$. Equation (11.6) gives

$$F = 0\,, \tag{11.8a}$$

or

$$HP + DQ = 0\,. \tag{11.8b}$$

It follows from (11.5) that if $F = 0$, then P and Q vanish identically. On the other hand, if (11.8b) is satisfied, (11.7) gives

$$\left(\frac{C}{D}P\right)' + 2r\,\frac{F^2}{D^2+r^2F^2}\,\frac{C}{D}P = 0\,, \tag{11.9}$$

which can be readily integrated to give

$$\frac{C}{D}P = f(t)\exp\left(-\int_{r_0}^{r}\frac{2rF^2}{D^2+r^2F^2}\,dr\right), \tag{11.10}$$

with $f(t)$ an unknown function of t. We impose the boundary condition at spatial infinity as

$$\lim_{r\to\infty} b^k{}_\mu = \delta^k{}_\mu\,, \quad \lim_{r\to\infty} b_k{}^\mu = \delta_k{}^\mu\,. \tag{11.11}$$

Then unknown functions, C, D, F and H, satisfy

$$\lim_{r\to\infty} C(t,r) = \lim_{r\to\infty} D(t,r) = 1\,, \tag{11.12}$$

$$\lim_{r\to\infty} rH(t,r) = \lim_{r\to\infty} rF(t,r) = 0\,, \tag{11.13}$$

and hence from (11.5) if follows that

$$\lim_{r\to\infty} rP(t,r) = \lim_{r\to\infty} rQ(t,r) = 0\,. \tag{11.14}$$

Because of the boundary condition (11.13) for F, the integral in the exponent of (11.10) converges for $r\to\infty$, and so the exponential factor of (11.10) approaches a finite positive value for $r\to\infty$. Therefore, in order to satisfy the boundary condition (11.14), the unknown function $f(t)$ must vanish, and hence we get

$$P(t,r) = 0\,, \quad Q(t,r) = 0\,, \tag{11.15}$$

by virtue of (11.8b) and (11.10). It then follows from (11.4) that *the axial-vector part of the torsion tensor must vanish,*

$$a^\mu = 0 \tag{11.16}$$

for a spherically symmetric gravitational field in vacuum. The symmetric part (7.2) of the gravitational field equation now becomes the Einstein equation in vacuum,

$$G_{\mu\nu}(\{\ \}) = 0\,. \tag{11.17}$$

According to the Birkhoff theorem[45] in general relativity, a spherically symmetric solution of (11.17) must be static and is given by the Schwarzschild solution.

We have thus shown that *a spherically symmetric solution of the gravitational field equations* (7.2)–(7.3) *with source terms absent must coincides with the static, isotropic field in vacuum studied in Secs.* V *and* VII, i.e., the Schwarzschild solution. This is just *the Birkhoff theorem* of new general relativity.

XII. CONCLUSION

We have formulated new general relativity and proved the following:

(1) The equations of motion for spin-$\frac{1}{2}$ fundamental particles and photons are approximated by the WKB approximation method to yield, in the classical limit, the geodesics of the metric $\underline{g}$, the extremal curve. This is the "corresponding principle" in new general relativity.

(2) In the case of $c_1 = 0 = c_2$ the gravitational action is of the form

$$I = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa}R(\{\ \}) + \frac{9}{4\lambda}(a_\mu a^\mu) + L_M\right). \tag{12.1}$$

Here κ is the Einstein gravitational constant, $\kappa = 8\pi G/c^4 = 8\pi G$, and λ is a new parameter, bounded by $\lambda/4\pi < 10^{-4}\,\hbar c/(\mathrm{GeV})^2$ from precise experiments in quantum electrodynamics. (We leave open the possibility that λ would be equal to κ, i.e., $\lambda = \kappa$.) What differs from general relativity is the second term, which consists of the axial-vector part $\{a^\mu\}$ of the torsion tensor. From this action follows the gravitational field equation,

$$G^{\mu\nu}(\{\ \}) + L^{\mu\nu} = \kappa T^{\mu\nu}\,, \tag{12.2}$$

where

$$L^{\mu\nu} = \frac{\kappa}{2\lambda}\{a_\lambda[\epsilon^{\mu\rho\sigma\lambda}(T^{\nu}{}_{\cdot\rho\sigma} - T_{\rho\sigma}{}^{\cdot\cdot\nu}) + \epsilon^{\nu\rho\sigma\lambda}(T^{\mu}{}_{\cdot\rho\sigma} - T_{\rho\sigma}{}^{\cdot\cdot\mu})] - 3a^\mu a^\nu - \tfrac{3}{2}g^{\mu\nu}a_\rho a^\rho + 3\epsilon^{\mu\nu\rho\sigma}(b^i{}_\rho a_{i,\sigma} + a_\rho v_\sigma)\}\,. \tag{12.3}$$

Here $a_i = b_i{}^\mu a_\mu$ is a vector with respect to global Lorentz transformations, but a scalar with respect to general coordinate transformations. Obviously, the field equation is invariant under global Lorentz transformations but violates the local Lorentz invariance in general.

(3) In the static, isotropic gravitational field the axial-vector part of the torsion tensor is identically vanishing, and the solution is given by the Schwarzschild solution.

(4) New general relativity agrees with all the experiments which have so far been carried out, as general relativity does.

(5) In the weak-field approximation to the gravitational field equation, it splits into two separate equations; one is for the symmetric field $\{\bar{h}_{\mu\nu}\}$, and the other is for the antisymmetric field $\{A_{\mu\nu}\}$,

$$\Box\bar{h}_{\mu\nu} = -2\kappa T_{(\mu\nu)}\,, \tag{12.4}$$

$$\Box A_{\mu\nu} = -\lambda T_{[\mu\nu]}\,, \tag{12.5}$$

with the conditions, $\partial_\nu \bar{h}^{\mu\nu} = 0$ and $\partial_\nu A^{\mu\nu} = 0$. The first equation describes the propagation of a graviton with zero mass and spin 2, and the second means the propagation of a zero-mass and zero-spin particle, which exerts spin-dependent force among spin-$\frac{1}{2}$ fundamental particles.

(6) In microscopic processes the equivalence principle is violated by means of the antisymmetric field described by (12.5), which is coupled to spin-$\frac{1}{2}$ fundamental particles. However, in the macroscopic scale the equivalence principle is recovered.

(7) In new general relativity the Birkhoff theorem, that a spherically symmetric gravitational field in empty space must be static, with a metric given by the Schwarzschild solution, is proved.

At this point we summarize several important features of new general relativity in comparison with general relativity; see Table II.

Finally, we emphasize that new general relativity, originally due to Einstein in 1928, is a gravitational theory that is acceptable on the experimental and theoretical grounds.

At present it seems impossible to detect the differences between general relativity and new general relativity. Among other things, it is highly expected to see what are Kerr-like solutions, i.e., stationary and axially symmetric solutions, in new general relativity, since the Kerr solution in general relativity has the *total angular momentum*, to which the axial-vector part $\{a^\mu\}$ of the torsion tensor in new general relativity may contribute.

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APPENDIX A: PARITY-VIOLATING TERMS, $(v^\mu a_\mu)$ AND $(\epsilon_{\mu\nu\rho\sigma} t^{\lambda\mu\nu} t_\lambda{}^{\cdot\rho\sigma})$

In Sec. IV we constructed the gravitational Lagrangian density L_G of (4.12) by postulating the basic principles (1)–(4). We now lift up the postulate of (3), then we can add to L_G of (4.12) parity-violating terms like $(v^\mu a_\mu)$ and $(\epsilon_{\mu\nu\rho\sigma} t^{\lambda\mu\nu} t_\lambda{}^{\cdot\rho\sigma})$:

$$\begin{aligned} L_G = {} & a_1(t^{\lambda\mu\nu} t_{\lambda\mu\nu}) + a_2(v^\mu v_\mu) + a_3(a^\mu a_\mu) \\ & + a_4(v^\mu a_\mu) + a_5(\epsilon_{\mu\nu\rho\sigma} t^{\lambda\mu\nu} t_\lambda{}^{\cdot\rho\sigma})\,, \end{aligned} \tag{A1}$$

where a cosmological term is neglected. Because of the identity,

$$\sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma} t^{\lambda\mu\nu} t_\lambda{}^{\cdot\rho\sigma} = \tfrac{9}{2}\sqrt{-g}\,v^\mu a_\mu - \tfrac{27}{4}(\sqrt{-g}\,a^\mu)_{,\mu}\,, \tag{A2}$$

we can drop the a_5 term, absorbing it into the a_4 term. Accordingly, the gravitational action of (4.18) involves one additional parity violating term;

$$\begin{aligned} I_G &= \int d^4x \sqrt{-g}\, L_G \\ &= \int d^4x \sqrt{-g}\left(\frac{1}{2\kappa} R(\{\ \}) + c_1(t^{\lambda\mu\nu} t_{\lambda\mu\nu}) \right. \\ & \qquad \left. + c_2(v^\mu v_\mu) + c_3(a^\mu a_\mu) + c_4(v^\mu a_\mu)\right)\,, \end{aligned} \tag{A3}$$

where the parameters, c_1, c_2, and c_3, are given by (4.19) and c_4 is given by

$$c_4 = a_4\,. \tag{A4}$$

The gravitational field equations are then given by (4.22) with the tensor $\{F_{\mu\nu\lambda}\}$ of (4.24) involving an additional term,

$$\begin{aligned} F^{\mu\nu\lambda} &= c_1(t^{\mu\nu\lambda} - t^{\mu\lambda\nu}) + c_2(g^{\mu\nu} v^\lambda - g^{\mu\lambda} v^\nu) \\ & \quad - \frac{c_3}{3}\epsilon^{\mu\nu\lambda\rho} a_\rho + \frac{c_4}{2}(g^{\mu\nu} a^\lambda - g^{\mu\lambda} a^\nu - \tfrac{1}{3}\epsilon^{\mu\nu\lambda\rho} v_\rho) \\ &= -F^{\mu\lambda\nu}\,. \end{aligned} \tag{A5}$$

The tensor $\{H^{\mu\nu}\}$ is still defined by (4.25), and L' is

$$\begin{aligned} L' = {} & c_1(t^{\lambda\mu\nu} t_{\lambda\mu\nu}) + c_2(v^\mu v_\mu) + c_3(a^\mu a_\mu) \\ & + c_4(v^\mu a_\mu)\,. \end{aligned} \tag{A6}$$

It is to be noticed that the choice of parameters in Ref. 4 corresponds to the case of $c_3 = 0 \neq c_4$.

In a static, isotropic gravitational field, for which the parallel vector fields $\underline{b}$ are of a diagonal form (5.2), there is no appearance of the parameters, c_3 and c_4, in the gravitational field equation. Therefore, all the results of Sec. V still hold true independently of the parity-violating

TABLE II. Comparison of new general relativity with general relativity.

	General relativity	New general relativity
Space-time	Riemann space-time	Weitzenböck space-time -absolute parallelism-
Connection	Levi-Civita connection $\Gamma^{\lambda}_{\mu\nu}=\{^{\lambda}_{\mu\nu}\}$	Nonsymmetric affine connection $\Gamma^{*\lambda}_{\mu\nu}=b_k{}^{\lambda}\partial_\nu b^k{}_\mu$
Basic structure	Metric tensor $\underline{g}=\{g_{\mu\nu}\}$	Parallel vector fields $\underline{b}=\{b_k{}^{\mu}\}\rightarrow\underline{g}=\{g_{\mu\nu}\}$
Gravitation	Riemann-Christoffel curvature tensor	Torsion tensor $T^{\lambda}_{\cdot\mu\nu}=b_k{}^{\lambda}(\partial_\nu b^k{}_\mu-\partial_\mu b^k{}_\nu)$
Transformation group	General coordinate transformation group (Local Lorentz group)	General coordinate transformation group Global Lorentz group
The Birkhoff theorem	Yes	Yes
Static, isotropic gravitational field	The Schwarzschild solution	The Schwarzschild solution
Static, axially symmetric gravitational field	The Kerr solution	Not yet found
Very strong, static, isotropic field	Black holes	Black holes
Newtonian approximation	Yes	Yes
Weak-field approximation	Symmetric field $\{\bar{h}_{\mu\nu}\}$; $\Box\bar{h}_{\mu\nu}=-2\kappa T_{\mu\nu}$ with $\partial_\nu\bar{h}^{\mu\nu}=0$	Symmetric field $\{\bar{h}_{\mu\nu}\}$; $\Box\bar{h}_{\mu\nu}=-2\kappa T_{(\mu\nu)}$ with $\partial_\nu\bar{h}^{\mu\nu}=0$ Antisymmetric field $\{A_{\mu\nu}\}$; $\Box A_{\mu\nu}=-\lambda T_{[\mu\nu]}$ with $\partial_\nu A^{\mu\nu}=0$
Quantum	Graviton; spin 2 and massless	Graviton; spin 2 and massless Scalar particle; positive energy, spinless and massless
Theory	Macroscopic	Microscopic
Equivalence principle	Yes	Yes, for macroscopic phenomena No, for microscopic phenomena

c_4 term of L_G. In particular, the values of the parameters, c_1 and c_2, are severely restricted by the solar-system experiments, as is shown in (6.16).

We shall thus assume henceforth in this appendix that $c_1=0=c_2$. Furthermore, in order to elucidate effects of the parity-violating c_4 term of L_G, we here apply the gravitational field equation to weak-field situations, where (9.1a) is satisfied. Using the notation introduced in Sec. IX, we find that the symmetric and antisymmetric parts of the gravitational field equation are given by

$$G^{(1)}_{\mu\nu}(\{\ \})-\tfrac{1}{3}\kappa c_4\partial^\lambda(\partial_\mu\bar{A}_{\nu\lambda}+\partial_\nu\bar{A}_{\mu\lambda})=\kappa T_{(\mu\nu)}\,, \qquad \text{(A7)}$$

$$-\tfrac{4}{9}c_3[\Box A_{\mu\nu}+\partial^\lambda(\partial_\mu A_{\nu\lambda}-\partial_\nu A_{\mu\lambda})] + c_4[-\tfrac{1}{3}\partial^\lambda(\partial_\mu\bar{A}_{\nu\lambda}-\partial_\nu\bar{A}_{\mu\lambda}) + \tfrac{1}{3}\epsilon_{\mu\nu\lambda\rho}\partial^\lambda\partial_\sigma(\tfrac{1}{2}\bar{h}^{\rho\sigma}-A^{\rho\sigma})]=T_{[\mu\nu]}\,, \qquad \text{(A8)}$$

where $\{\bar{A}_{\mu\nu}\}$ is the dual of $\{A_{\mu\nu}\}$,

$$\bar{A}_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}A^{\rho\sigma}\,. \qquad \text{(A9)}$$

The nonsymmetric energy-momentum tensor $\{T_{\mu\nu}\}$ satisfies the ordinary conservation law (9.9).

Corresponding to the invariance of the gravitational field equation (4.22) under general coordinate transformations, the linearized field equation (A7)–(A8) is invariant under gauge transformation,

$$h'_{\mu\nu}=h_{\mu\nu}-(\partial_\mu\Lambda_\nu+\partial_\nu\Lambda_\mu),$$
$$A'_{\mu\nu}=A_{\mu\nu}+\tfrac{1}{2}(\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu)\,, \tag{A10}$$

where Λ_μ are four small but otherwise arbitrary functions that leave $\{h_{\mu\nu}\}$ and $\{A_{\mu\nu}\}$ weak. For the symmetric field $\{h_{\mu\nu}\}$, the most convenient choice of gauge is to put the harmonic condition (9.15), which we shall assume henceforth.

It is to be noticed that the field equation (A7)–(A8) is not invariant under space inversion and time reversal, when c_4 does not vanish. This is, of course, a direct consequence of the fact that the gravitational Lagrangian density of (A3) involves the term $c_4(v^\mu a_\mu)$, which changes sign under space inversion and time reversal. This apparent parity violation, however, does not lead to any observable effects in the weak-field approximation, as will be shown below.

Multiplying ∂^ν on (A7) [or (A8)], we get

$$-\tfrac{2}{3}c_4\Box\partial^\nu\bar{A}_{\mu\nu}=\partial^\nu T_{\nu\mu}\,, \tag{A.11}$$

of which the retarded solution is given by

$$\partial^\nu\bar{A}_{\mu\nu}(\vec{x},t)=\frac{3}{8\pi c_4}\int d^3x'\,\frac{\partial^\nu T_{\nu\mu}(\vec{x}',t-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}\,, \tag{A.12}$$

since c_4 is assumed to be nonvanishing. Using (9.4), (9.15), and (A12) in (A7), the field equation of $h_{\mu\nu}$ reads

$$\Box\bar{h}_{\mu\nu}=-2\kappa T_{(\mu\nu)}-\frac{\kappa}{2\pi}\int d^3x'\,\frac{\partial^\lambda T_{\lambda(\mu,\nu)}(\vec{x}',t-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}\,, \tag{A13}$$

where

$$T_{\lambda(\mu,\nu)}\equiv\tfrac{1}{2}(\partial_\mu T_{\lambda\nu}+\partial_\nu T_{\lambda\mu})\,. \tag{A14}$$

The retarded solution of (A13) is

$$\Box\bar{h}_{\mu\nu}(\vec{x},t)=\frac{\kappa}{2\pi}\int d^3x'\,\frac{T_{(\mu\nu)}(\vec{x}',t-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}+\frac{\kappa}{8\pi^2}\int\int d^3x'd^3x''\,\frac{\partial^\lambda T_{\lambda(\mu,\nu)}(\vec{x}'',t-|\vec{x}-\vec{x}'|-|\vec{x}'-\vec{x}''|)}{|\vec{x}-\vec{x}'|\,|\vec{x}'-\vec{x}''|}\,, \tag{A15}$$

which is interpreted as the gravitational radiation produced by the source $\{T_{\mu\nu}\}$. Inspection of (A15) shows that if $\{\partial_\lambda T^{\lambda\mu}\}$ does not identically vanish, the field $\{h_{\mu\nu}\}$ propagates *inside* the light cone as if it is massive.

It seems natural, however, to restrict the theoretical framework of gravitation by requiring that gravitational radiation should propagate *on* the light cone with the speed of light. In view of this criterion, the case of $c_4\neq 0$ should be disregarded unless the energy-momentum tensor satisfies

$$\partial_\lambda T^{\lambda\mu}=0\,, \tag{A.16}$$

in addition to the ordinary energy-momentum conservation law (9.9). Therefore, *we shall assume* (A16) *hereafter*. Then the spin tensor $\{S^{\lambda\mu\nu}\}$ is totally antisymmetric with respect to its three indices, and is represented as (9.12).

It follows from (A13) and (A16) that the symmetric field $\{h_{\mu\nu}\}$ satisfies the field equation

$$\Box\bar{h}_{\mu\nu}=-2\kappa T_{(\mu\nu)}\,, \tag{A.17}$$

which is nothing but the field equation (9.17) in the case of $c_4=0$. Consequently, we find that *the symmetric field* $\{h_{\mu\nu}\}$ *is not influenced at all by the parity-violating* c_4 *term of* L_G.

From (A12) and (A16) it follows that

$$\partial_\nu\bar{A}^{\mu\nu}=0\,, \tag{A.18a}$$

which, in view of (9.6b), is equivalent to the vanishing of the axial-vector part of the torsion tensor;

$$a^\mu=0\,. \tag{A.18b}$$

Therefore, $\{A_{\mu\nu}\}$ can be represented as curl of a vector field $\{B_\mu\}$,

$$A_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu. \tag{A.19}$$

Using (9.11)–(9.12), (A18a), and (A19) in (A8), we find that the field equation of $\{A_{\mu\nu}\}$ is rewritten as

$$\Box A_{\mu\nu}=\frac{3}{2c_4}(\partial_\mu J_{5\nu}-\partial_\nu J_{5\mu})\,, \tag{A.20}$$

the retarded solution of which is given by (A19), with B_μ defined by

$$B_\mu=-\frac{3}{8\pi c_4}\int d^3x'\,\frac{J_{5\mu}(\vec{x}',t-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}\,. \tag{A21}$$

It follows from (A18b) that an antisymmetric field does not couple with spin-$\tfrac{1}{2}$ fundamental particles [see the Dirac equation (3.13b)]. On the other hand, the electromagnetic field is decoupled from an antisymmetric field, since the former interacts with the gravitational field through the metric tensor $\{g_{\mu\nu}\}$. Consequently, an antisymmetric field (A19) does not interact with fundamental particles and fields, and so *it is entirely devoid of physical reality*.

The present case of $c_4 \neq 0$ is invariant under the gauge transformation (A10). Using this gauge freedom, we have put the harmonic condition (9.15), which is necessary to eliminate unphysical components of the symmetric field $\{h_{\mu\nu}\}$. We are still left with the freedom to perform a gauge transformation (A10) with Λ_μ satisfying the d'Alembert equation,

$$\Box \Lambda_\mu = 0 . \tag{A22}$$

It follows from (A21), however, that $\{B_\mu\}$ satisfies the *inhomogeneous* d'Alembert equation,

$$\Box B_\mu = \frac{3}{2c_4} J_{5\mu}, \tag{A23}$$

if matter exists. Accordingly, a gauge transformation (A10) with (A22) is *insufficient* to make an antisymmetric field (A19) vanishing in that space-time region where there exists nonvanishing source, $\{J_{5\mu}\} \neq 0$.

Therefore, an antisymmetric field (A19), although it is unphysical, cannot be eliminated by a symmetry transformation of the present case of $c_4 \neq 0$. This situation is to be contrasted with that of the electromagnetic field, in which unphysical components of the electromagnetic potential $\{A_\mu\}$ can be eliminated by choosing an appropriate gauge. It is unreasonable to accept a theory involving such unphysical degrees of freedom that cannot be removed by a symmetry transformation of a theory. Consequently, we should disregard the case of $c_4 \neq 0$.

APPENDIX B: SPHERICALLY SYMMETRIC PARALLEL VECTOR FIELDS

Consider the parallel vector fields for a spherically symmetric (but time-dependent in general) system. We mean by "spherically symmetric" that it is possible to choose a "quasi-Minkowskian" coordinates, $x^1, x^2, x^3, x^0 = t$, such that the parallel vector fields $\underline{b}^* = \{\underline{b}^k\} = \{b^k{}_\mu\}$ are *form invariant* under space rotation

$$x^\alpha \to R_{\alpha\beta} x^\beta, \quad \underline{b}^{(a)} \to R_{ac} \underline{b}^{(c)}, \tag{B1}$$

where $R = (R_{ac}) = (R_{\alpha\beta})$ is an orthogonal 3×3 matrix

$$RR^t = R^t R = I, \quad \det R = 1 .$$

The most general expression of $\{b^k{}_\mu\}$ can then be given by

$$(b^k{}_\mu) = \begin{array}{c} \\ k\downarrow \end{array}\!\!\overset{\quad\to\mu}{\begin{pmatrix} C & G x^\alpha \\ H x^a & D\delta_{a\alpha} + E x^a x^\alpha + F \epsilon_{a\alpha\beta} x^\beta \end{pmatrix}}, \tag{B2}$$

where C, D, E, F, G, and H are unknown functions of t and $r = (x^\alpha x^\alpha)^{1/2}$. We are, however, still free to redefine the time coordinate and the radius by

$$t' = \phi(t, r), \quad x'^\alpha = \psi(t, r) x^\alpha , \tag{B3}$$

with ϕ and ψ arbitrary functions of t and r.

Under arbitrary coordinate transformation $x^\mu \to x'^\mu$, the parallel vector fields $\{b^k{}_\mu\}$ transform like covariant vectors

$$b'^k{}_\mu(x') = (\partial x^\nu / \partial x'^\mu) b^k{}_\nu(x) . \tag{B4}$$

For a redefinition (B3) of t and r, the transformation coefficients $(\partial x^\nu / \partial x'^\mu)$ are given by

$$\begin{aligned} &\partial t/\partial t' = (\psi + r\psi')/\Delta , \\ &\partial x^\alpha/\partial t' = -(\dot\psi/\Delta) x^\alpha, \\ &\partial t/\partial x'^\alpha = -(\phi'/r\Delta) x^\alpha , \\ &\partial x^\alpha/\partial x'^\beta = \frac{1}{\psi}\left(\delta^\alpha_\beta + \frac{1}{\Delta}(\phi'\dot\psi - \dot\phi\psi')\frac{x^\alpha x^\beta}{r}\right) , \end{aligned} \tag{B5}$$

where

$$\begin{aligned} &\dot\phi = \partial\phi/\partial t, \quad \phi' = \partial\phi/\partial r , \\ &\dot\psi = \partial\psi/\partial t, \quad \psi' = \partial\psi/\partial r , \\ &\Delta \equiv \frac{\partial(t', r')}{\partial(t, r)} = \psi\dot\phi + r(\dot\phi\psi' - \phi'\dot\psi) . \end{aligned} \tag{B6}$$

Using (B5) in (B4), we obtain

$$b'^{(0)}{}_0 = \frac{1}{\Delta}\left[(\psi + r\psi')C - r^2\dot\psi G\right], \tag{B7a}$$

$$b'^{(0)}{}_\alpha = \frac{1}{\Delta}\left(-\frac{\phi'}{r} C + \dot\phi G\right) x^\alpha , \tag{B7b}$$

$$b'^{(a)}{}_0 = \frac{1}{\Delta}\left[(\psi + r\psi')H - \dot\psi(D + r^2 E)\right] x^a , \tag{B7c}$$

$$\begin{aligned} b'^{(a)}{}_\alpha = {} & \frac{D}{\psi}\delta^a{}_\alpha + \frac{1}{\Delta}\left(\dot\phi E - \frac{\phi'}{r} H - \frac{\dot\phi\psi' - \phi'\dot\psi}{r\psi} D\right) x^a x^\alpha \\ & + \frac{F}{\psi}\epsilon_{a\alpha\beta} x^\beta . \end{aligned} \tag{B7d}$$

Inspection of (B7b) shows that the (0α) components, $b^{(0)}{}_\alpha$, can be eliminated by setting

$$\frac{C}{r}\phi' - G\dot\phi = 0, \quad \psi = 1 . \tag{B8}$$

In particular, if C, D, E, F, G, and H are all time independent, ϕ can be taken as

$$\phi = t + \tilde\phi(t), \quad \frac{C}{r}\frac{d\tilde\phi}{dr} - G = 0 . \tag{B9}$$

where $\tilde\phi(r)$ is a function of r. Now we assume G to be zero, then the E term in $b^{(a)}{}_\alpha$ can be eliminated by putting

$$\phi = t, \quad E - \frac{D}{r}\frac{\psi'}{\psi} = 0 . \tag{B10}$$

The parallel vector fields (B2) then take the following form:

$$(b^k{}_\mu)=\begin{pmatrix} C & 0 \\ Hx^a & D\delta_{a\alpha}+F\epsilon_{a\alpha\beta}x^\beta \end{pmatrix}. \tag{B11}$$

Further reduction of $\{b^k{}_\mu\}$ is impossible: Any of C, D, F, and H cannot be put to zero in addition to E and G. This is evident from (B7a) and (B7d) for C, D, and F. To prove this for H, we assume that the $(a0)$ components, $b^{(a)}{}_0$, were eliminated from (B11) by a suitable redefinition of t and r, then Eqs. (B7b)–(B7d) show that the function ψ must satisfy

$$\psi'=0,\quad \psi H-\dot{\psi}D=0\,.$$

However, these two conditions of ψ are not compatible with each other, since D and H are, in general, functions of r.

Now we turn to a static, spherically symmetric system. We assume furthermore that the spin of constituent particles of the system, if it exists, can be completely neglected: This means that there is no physical distinction between the left- and right-handed coordinate system to describe it. Then the parallel vector fields are *form invariant* under time reversal and space inversion,

$$t\to -t,\quad \underline{b}_{(0)}\to -\underline{b}_{(0)}\ \text{(time reversal)} \tag{B12}$$

$$x^\alpha\to -x^\alpha,\quad \underline{b}_{(a)}\to -\underline{b}_{(a)}\ \text{(space inversion)} \tag{B13}$$

in addition to space rotation (B1). Hence the $(a0)$ components, $b^{(a)}{}_0$, and the F term in the $(a\alpha)$ components, $b^{(a)}{}_\alpha$, must vanish. The parallel vector fields (B11) then become

$$(b^k{}_\mu)=\begin{pmatrix} C & 0 \\ 0 & D\delta_{a\alpha} \end{pmatrix}, \tag{B14}$$

where C and D are unknown functions of r alone.

APPENDIX C: PROOF OF $1+(c_1+4c_2)\neq 0$

The field equations for the static isotropic gravitational field become

$$-\kappa(c_1+c_2)A''-[1-\kappa(c_1-2c_2)]B'' -\frac{2}{r}\{\kappa(c_1+c_2)A'+[1-\kappa(c_1-2c_2)]B'\}=T^{00}, \tag{C1a}$$

$$[1-\kappa(c_1-2c_2)]A'+[1+\kappa(c_1+4c_2)]B'=0 \tag{C1b}$$

in the Newtonian limit. (Dividing (C1) by $[1+\kappa(c_1+4c_2)]$ gives (5.7).) Assume that $1+\kappa(c_1+4c_2)=0$, then (C1b) gives $[1-\kappa(c_1-2c_2)]A'=0$. If $A'=0$, the equation of motion (5.11a) for a nonrelativistic test particle becomes

$$\frac{d^2x^\alpha}{dt^2}=\frac{1}{2}\,\frac{\partial}{\partial x^\alpha}\,g_{00}(x)=\frac{1}{2}\,\frac{\partial}{\partial x^\alpha}\,A=0\,,$$

which contradicts the Newton equation of motion. If $1-\kappa(c_1-2c_2)=0$, on the other hand, c_1 and c_2 are given by $c_1=1/3\kappa$ and $c_2=-1/3\kappa$, respectively. Using these values of c_1 and c_2 in (C1a) gives

$$0=\kappa T^{00}\,,$$

which is a contradiction. Therefore, $[1+\kappa(c_1+4c_2)]$ should not be zero.

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[11]R. Weitzenböck, *Invariantentheorie* (Noordhoff, Groningen, 1923); Chap. XIII, Sec. 7.

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[13]Throughout this paper we mean by "geodesics" the shortest (or longest) possible path between two points, "length" being measured by the metric $\underline{g}$.

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[15]Our convention of the γ matrices is as follows:

$$\{\gamma^i,\ \gamma^j\}=-2\eta^{ij},\quad S^{ij}=(i/4)[\gamma^i,\ \gamma^j],$$

$$\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3.$$

In the spinor representation (2.20) of ψ, the γ matrices are

$$\gamma^0=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad \gamma^a=\begin{pmatrix}0 & -\sigma^a\\ \sigma^a & 0\end{pmatrix},\quad \gamma^5=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}.$$

[16]C. Møller, *The Theory of Relativity* (Clarendon, Oxford, 1952).

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$$\partial_\nu M^{\lambda\mu\nu} = 0,$$

with $M^{\lambda\mu\nu}$ defined by

$$M^{\lambda\mu\nu} = x^\lambda T^{\mu\nu} - x^\mu T^{\lambda\nu} + S^{\lambda\mu\nu}.$$

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[42]Here we use the standard representation of the γ matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma^a & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

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