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Relativistic energy-dependent partial-wave analysis for particles with spin

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Relativistic expansions of two-body scattering amplitudes are presented for reactions involving particles with arbitrary spins. The expansions are "complete" in that the entire dependence on the kinematical variables (energy, scattering angle, and azimuthal angle) is displayed explicitly in known functions. They are provided by the representation theory of the Lorentz group O(3,1) which in this case is identified as an "internal Lorentz group," acting in a specific manner on two-particle states. The expansion coefficients, called Lorentz amplitudes, carry the dynamics of the reaction. The phenomenological motivation for writing such expansions is to be able to analyze large bodies of data simultaneously, e.g., the scattering data measured in a specific reaction for all angles and all available energies.

I. INTRODUCTION

The purpose of this article is to present and discuss a method for treating two-body scattering processes involving particles with arbitrary spins. The method is a generalization of the ordinary partial-wave analysis and a further development of relativistic two-variable expansions proposed elsewhere.[1-7] The essence of this approach is that the scattering amplitudes are considered explicitly as functions of both intrinsic kinematic variables (in our case the center-of-mass energy and scattering direction). They are expanded in terms of known functions of these two variables. The expansion functions are essentially the finite transformation matrices for the Lorentz group O(3,1). Since they have their origin in a group-theoretical treatment of Lorentz invariance, these expansion functions will automatically display some of the kinematical properties following from the general principles of scattering theory.

The purpose of writing two-variable expansions is thus to go beyond the usual single-variable expansions of partial-wave analysis, Regge-pole theory, impact-parameter expansions, etc., in separating the kinematical and dynamical aspects of a reaction. The dynamics is carried by the expansion coefficients—the "Lorentz amplitudes." Theoretical models can then be formulated in terms of assumptions on the form of these amplitudes. From the phenomenological point of view two-variable expansions make it possible to fit scattering data simultaneously for all available scattering angles and energies. In this sense they provide a model-independent tool for an energy-dependent partial-wave analysis of scattering data. Since the energy is an integral of motion and since we wish to expand in terms of explicit functions of the energy, the group providing the expansion can clearly not be an invariance group of the scattering matrix. Hence the expansion will introduce labels that are not conserved quantum numbers.

The bulk of previous work in this direction concerned the scattering of spinless particles.[1-7] In this simple case, all the kinematics of the scattering can be described by the four-direction of the four-momentum of one final particle, e.g., $x = p_3/m_3$. Thus the scattering amplitude can be considered as a function defined on the upper sheet of a two-sheeted hyperboloid $x^2 = 1$, $x^0 \geq 1$. As such it can be developed in terms of the basis functions of the irreducible unitary representations of the Lorentz group. When spins are involved the scattering amplitude becomes a scattering matrix and we have to account for the spin indices to perform globally the energy-dependent expansion. An earlier article[8] was devoted to the case of general spin and made use of one-particle states transforming according to irreducible representations of the Poincaré group $\mathcal{P}$, using a basis corresponding to the reduction $\mathcal{P} \supset O(3,1) \supset O(3) \supset O(2)$.[1,9-11] Unfortunately, the derivation contained a flaw which led to an asymmetrical treatment of the particles involved [finite-dimensional representations of O(3, 1) for three of the particles, unitary ones for the fourth]. The obtained expansions were unnecessarily complicated and involved quantum numbers that were hard to interpret.

In this paper we shall make explicit use of the "internal Lorentz basis" for two-particle states defined in a recent article[12] (an "internal Galilean basis" has also been defined.[13] These states transform according to irreducible unitary representations of the "internal Lorentz group" contained in the direct product of the two Lorentz groups acting on the two particles. This internal Lorentz group acts in different ways on each particle and should not be confused with the kinematical Lorentz group (relating different inertial frames) which acts in the same way on both particles. The transformations of the internal group change the total energy $w=[(p_1+p_2)^2]^{1/2}$ of the two-particle system and hence do not leave the scattering matrix invariant. This is precisely the property which allows us to obtain two-variable expansions in a natural manner and makes it possible to utilize the well developed representation theory of the group O(3, 1).[14-18]

It should be stressed that both the internal and kinematical Lorentz groups have the same rotation subgroup, used to perform the usual single-variable expansions in terms of the scattering angle appearing in O(3) D-functions. The coefficients of this expansion are the partial waves which depend on the energy. Our two-variable expansion is thus a further development of the partial waves in terms of the boost matrices of the internal Lorentz group.

After some mathematical preliminaries we devote Sec. II to a discussion of the internal Lorentz basis for two-particle states. The core of the article is Sec. III, in which we derive the complete expansions of the scattering amplitudes. Some properties of the expansions are discussed in Sec. IV, in particular, threshold behavior and the requirements of parity and time-reversal invariance. In Sec. V we use the expansions to derive energy-dependent expansions of experimental quantities. Some final comments are made in Sec. VI. In the Appendix we summarize some relevant results on the representation theory of the Lorentz group O(3, 1).

II. THE INTERNAL LORENTZ GROUP AND INTERNAL LORENTZ BASIS FOR TWO-PARTICLE STATES

A. Preliminary comments

In a reference tetrad $\{e_\alpha\}$ [with $e_0=(1,0,0,0)$, $e_1=(0,1,0,0)$, $e_2=(0,0,1,0)$, and $e_3=(0,0,0,1)$] the spherical coordinates (a,θ,ζ) of a four-momentum p will be defined as

$$p=m[\cosh a e_0+\sinh a u(\theta,\zeta)], \tag{1a}$$

$$u(\theta,\zeta)=\sin\theta\cos\zeta e_1+\sin\theta\sin\zeta e_2+\cos\theta e_3, \tag{1b}$$

with $a\in[0,\infty)$, $\theta\in[0,\pi]$, and $\zeta\in[0,2\pi)$ (and the mass m satisfies $m>0$).

We consider two three-parameter families of Lorentz transformations, each transforming the vector me_0 into the four-momentum p:

$$L_p\equiv L(a,\theta,\zeta)=R(\zeta,\theta,0)B_3(a)R^{-1}(\zeta,\theta,0), \tag{2a}$$

$$\Lambda_p\equiv(a,\theta,\zeta)=R(\zeta,\theta,0)B_3(a). \tag{2b}$$

The rotations $R(\zeta,\theta,\psi)$ are parametrized by the Euler angles and $B_3(a)$ is a boost along the axis 3.

Let us consider the finite transformation matrices $D^{\rho\nu}_{jns\sigma}(L_p)$ and $D^{\rho\nu}_{jns\lambda}(\Lambda_p)$ corresponding to (2a) and (2b) in a unitary representation $(\rho\nu)$ of the Lorentz group (see Appendix for definitions). The general completeness and orthogonality relations (A20) and (A21) reduce to

$$\sum_\sigma\int\mu(p)d^3p\,D^{\rho\nu}_{jns\sigma}(L_p)D^{\rho'\nu'*}_{j'n's\sigma}(L_p)=\frac{2\pi^2(2s+1)}{\mu(\rho,\nu)}\delta(\rho-\rho')\delta_{\nu\nu'}\delta_{jj'}\delta_{nn'} \tag{3a}$$

$$\mu(p)=\sinh^2 a\sin\theta,\quad d^3p=da\,d\theta\,d\zeta,$$

$$\sum_{jn}\sum_\nu\int_0^\infty\mu(\rho,\nu)d\rho\,D^{\rho\nu}_{jns\sigma}(L_p)D^{\rho\nu*}_{jns\sigma'}(L_{p'})=\frac{2\pi^2(2s+1)}{\mu(p)}\delta^3(p-p')\delta_{\sigma\sigma'} \tag{3b}$$

$$\delta^3(p-p')\equiv\delta(a-a')\delta(\theta-\theta')\delta(\zeta-\zeta'),$$

$$\mu(\rho,\nu)=\rho^2+\nu^2.$$

The relations for the family Λ_p are identical. The two equations (3a) and (3b) exhibit the two sets of variables $\{p=(a,\theta,\zeta),\ \sigma(\text{or}\,\lambda)\}$ and $\{\rho,\nu,j,n\}$ that will be used to describe the states of a particle with mass m and spin s.

Note that when the particle is at rest we have $p=me_0$, $a=0$, and θ, ζ are not defined. The matrices reduce to

$$D^{\rho\nu}_{jns\sigma}(L(0,\theta,\zeta))=\delta_{js}\delta_{n\sigma},$$
$$D^{\rho\nu}_{jns\lambda}(\Lambda(0,\theta,\zeta))=\delta_{js}D^j_{n\lambda}(\zeta,\theta,0). \tag{4}$$

The energy-dependent expansions of the scattering amplitudes will be performed in terms of the functions $D^{\rho\nu}_{jns\sigma}(L_p)$ or $D^{\rho\nu}_{jns\lambda}(\Lambda_p)$.

B. Internal Lorentz group

An "internal Lorentz group" and an "internal Lorentz basis" were introduced explicitly in a recent article.[12] These are very suitable for deriving two-variable expansions of scattering amplitudes. Indeed, such expansions for spinless particles[1-7] at least implicitly made use of the internal Lorentz group in the relativistic case and of an internal Galilei group[13] in the nonrelativistic one.[4]

Here we shall briefly review some relevant properties of the internal Lorentz group and the internal Lorentz basis. For all details and the derivation of formulas we refer to a previous article.[12] Let us consider two particles with arbitrary (positive) masses m_i, spins s_i, and momenta p_i ($i=1,2$). Let us introduce an arbitrary unit timelike four-vector x:

$$x^2=x_0^2-\vec{x}^2=1, \quad x_0\geq 1. \tag{5}$$

Let us consider a reference frame, such that a pair of linear momenta (p_1,p_2) satisfy

$$(p_1,p_2)=(m_1x, m_2\pi x), \tag{6}$$

where π is the parity operator. The set of all such pairs of vectors is left invariant by a subgroup of the direct product of the two Lorentz groups $O^1(3,1)\times O^2(3,1)$, acting on particles 1 and 2, namely the group of transformations

$$\tilde{\Lambda}=\Lambda\times\pi\Lambda\pi^{-1}. \tag{7}$$

The transformations $\tilde{\Lambda}$ form an O(3, 1) group. We shall call it the internal Lorentz group and denote it $\tilde{O}(3,1)$. It is conjugate to the physical "diagonal Lorentz group" O(3, 1) generated by

$$\vec{J}=\vec{J}^{(1)}+\vec{J}^{(2)}, \quad \vec{K}=\vec{K}^{(1)}+\vec{K}^{(2)}, \tag{8}$$

where $\vec{J}^{(i)}$ and $\vec{K}^{(i)}$ are the rotation and boost generators for particles 1 and 2. Indeed, we have

$$\tilde{O}(3,1)=(I\times\pi)O(3,1)(I\times\pi)^{-1}, \tag{9}$$

so that the internal Lorentz group is generated by

$$\vec{J}=\vec{J}^{(1)}+\vec{J}^{(2)}, \quad \vec{A}=\vec{K}^{(1)}-\vec{K}^{(2)}. \tag{10}$$

Lorentz invariance implies that the scattering matrix should be invariant under the diagonal Lorentz group. No such property holds for the internal Lorentz group; however, the O(3) subgroups of these two Lorentz groups coincide.

Note that any pair of momenta $(m_1x, m_2\pi x)$ can be obtained from the state of two particles at rest (m_1e_0, m_2e_0) by an internal Lorentz transformation $\tilde{L}_x=(L_x\times\pi L_x\pi^{-1})$ with L_x, as in (2).

From a practical point of view the reference frame in which to perform expansions is the center-of-mass frame, rather than the frame defined by (6) [this last frame is characterized by the fact that the vector $q\equiv(p_1/m_1+p_2/m_2)/((p_1/m_1+p_2/m_2)^2)^{1/2}$ satisfies $q=e_0$]. The relation between this "q frame" and the center-of-mass system (c.m.s.) has been studied elsewhere[12] and we shall just present the formulas needed below. We shall parametrize states by the coordinates of the vector x, so let us relate x to the c.m.s. particle momenta p_1 and p_2. In the c.m.s. we have

$$\begin{aligned} p_1&=m_1(\cosh a_1 e_0+\sinh a_1 u), \\ p_2&=m_2(\cosh a_2 e_0-\sinh a_2 u), \\ m_1\sinh a_1&=m_2\sinh a_2. \end{aligned} \tag{11}$$

We define the unit timelike four-vector x as

$$x=\left(\frac{p_1}{m_1}+\pi\frac{p_2}{m_2}\right)\Big/\left[\left(\frac{p_1}{m_1}+\pi\frac{p_2}{m_2}\right)^2\right]^{1/2}, \tag{12}$$

$$x=\cosh a e_0+\sinh a u, \tag{13}$$

and it is easy to check that we have

$$a=\tfrac{1}{2}(a_1+a_2). \tag{14}$$

The invariant mass of a two-particle state depends on a only,

$$s\equiv w^2(a)=(p_1+p_2)^2=(m_1+m_2)^2+4m_1m_2\sinh^2 a, \tag{15}$$

and we have

$$(m_1+m_2)\leq w(a)<\infty \quad \text{as} \quad 0\leq a<\infty.$$

We can now parametrize our two-particle states. In a canonical basis (quantization along the third axis) we have

$$|p_1\sigma_1\rangle\otimes|p_2\sigma_2\rangle\equiv|x\sigma_1\sigma_2\rangle\equiv|a\theta\zeta\sigma_1\sigma_2\rangle, \tag{16}$$

and in a helicity basis,

$$\begin{aligned} |p_1\lambda_1\rangle\otimes(-1)^{s_2-\lambda_2}e^{-i\pi J_y}|p_2\lambda_2\rangle&\equiv|x\lambda_1\lambda_2\rangle \\ &\equiv|a\theta\zeta\lambda_1\lambda_2\rangle \end{aligned} \tag{17}$$

(we use the original Jacob and Wick phase convention[19-21]). These two-particle states are now defined on a unit hyperboloid $x^2=1$, $x_0\geq 1$ and this has been achieved for arbitrary spins and masses. Equivalently, they can be considered to be defined on the group manifold of the internal Lorentz group, i.e., as functions of the Lorentz transformations L_x or Λ_x.

C. Internal-Lorentz-group states

To proceed further we must introduce the internal-Lorentz-group two-particle states[12]

$$\begin{aligned} |(\rho\nu s)jn\rangle&=\frac{1}{2\pi^2(2s+1)}\sum_{\sigma_1\sigma_2\sigma}(s_1\sigma_1s_2\sigma_2|s\sigma)\int\mu(x)d^3x\,D^{\rho\nu *}_{jns\sigma}(L_x)|x\sigma_1\sigma_2\rangle, \\ &=\frac{1}{2\pi^2(2s+1)}\sum_{\lambda_1\lambda_2\lambda}(s_1\lambda_1s_2-\lambda_2|s\lambda)\int\mu(x)d^3x\,D^{\rho\nu *}_{jns\lambda}(\Lambda_x)|x\lambda_1\lambda_2\rangle. \end{aligned} \tag{18}$$

Here we use standard O(3) Clebsch-Gordan coefficients[22] $(s_1m_1s_2m_2|sm)$, and $D^{\rho\nu}_{jns\sigma}(\Lambda)$ is a Lorentz-group finite-transformation matrix[15,17,18] (see Appendix). The real number $\rho\in[0,\infty)$ and integer or half-integer $\nu\in[-s,-s+1,\ldots,s]$ label the irreducible unitary representations of O(3,1). The matrices are written in the Naimark canonical basis[14] corresponding to the group reduction $O(3,1)\supset O(3)\supset O(2)$. The labels j and s are the usual total angular momentum and total spin; n and σ (or λ) are their respective projections.

The completeness properties (3) of the functions $D^{\rho\nu}_{jns\sigma}(L_x)$ make it possible to invert these defining relations (18) and to obtain O(3, 1) expansions of the usual two-particle states:

$$|x\sigma_1\sigma_2\rangle=\sum_s(s_1\sigma_1s_2\sigma_2|s\sigma)\sum_{\nu jn}\int d\rho\,\mu(\rho,\nu)D^{\rho\nu}_{jns\sigma}(L_x)|(\rho\nu s)jn\rangle\,,$$
$$|x\lambda_1\lambda_2\rangle=\sum_s(s_1\lambda_1s_2-\lambda_2|s\lambda)\sum_{\nu jn}\int d\rho\,\mu(\rho,\nu)D^{\rho\nu}_{jns\lambda}(\Lambda_x)|(\rho\nu s)jn\rangle\,. \tag{19}$$

The states $|(\rho\nu s)jn\rangle$ correspond to a definite value j of the spin operator which is a relativistic invariant (the total angular momentum of the two particles in their c.m.s.) They do not, however, transform according to an irreducible representation of the Poincaré group. Indeed, two-particle states that do transform irreducibly under the Poincaré group can be written as

$$|w(a)jn,ls\rangle=\frac{(2j+1)^{1/2}}{[4\pi(2l+1)]^{1/2}}\sum_m\sum_{\sigma_1\sigma_2\sigma}(lms\sigma|jn)(s_1\sigma_1s_2\sigma_2|s\sigma)\int_0^\pi\int_0^{2\pi}\sin\theta\,d\theta\,d\zeta\,Y_{lm}(\theta,\zeta)|a\theta\zeta\sigma_1\sigma_2\rangle \tag{20}$$

in the l-s coupling scheme, or as

$$|w(a)jn,\lambda_1\lambda_2\rangle=\frac{(2j+1)^{1/2}}{4\pi}\int_0^\pi\int_0^{2\pi}\sin\theta\,d\theta\,d\zeta\,D^{j*}_{n\lambda_1-\lambda_2}(\zeta,\theta,0)|a\theta\zeta\lambda_1\lambda_2\rangle \tag{21}$$

in the helicity coupling scheme.[19-21]

In terms of these irreducible states we can express the internal Lorentz basis states as

$$|(\rho\nu s)jn\rangle=\frac{2}{\pi(2s+1)(2j+1)^{3/2}}\sum_{l\lambda}(2l+1)(l0s\lambda|j\lambda)\int_0^\infty\sinh^2a\,da\,d^{\rho\nu*}_{js\lambda}(a)|w(a)jn,ls\rangle$$
$$=\frac{2}{\pi(2s+1)(2j+1)^{1/2}}\sum_{\lambda_1\lambda_2\lambda}(s_1\lambda_1s_2-\lambda_2|s\lambda)\int_0^\infty\sinh^2a\,da\,d^{\rho\nu*}_{js\lambda}(a)|w(a)jn,\lambda_1\lambda_2\rangle \tag{22}$$

[the boost functions $d^{\rho\nu}_{js\lambda}(a)$ are defined in the Appendix]. We see that the $|(\rho\nu s)jn\rangle$ states do not correspond to a definite value of the invariant total energy $w^2(a)=(p_1+p_2)^2$. Instead they involve an integral over all energies, from the threshold $w^2(a)=(m_1+m_2)^2$ upward. Formulas (22) can be inverted using (A19) of the Appendix and we thus obtain the irreducible states in terms of the internal Lorentz states, and their energy dependence is now displayed explicitly in known functions, namely the O(3, 1) d functions:

$$|w(a)jn,ls\rangle=\frac{1}{(2j+1)^{1/2}}\sum_\lambda(l0s\lambda|j\lambda)\sum_\nu\int\mu(\rho,\nu)d\rho\,d^{\rho\nu}_{js\lambda}(a)|(\rho\nu s)jn\rangle\,, \tag{23}$$

$$|w(a)jn,\lambda_1\lambda_2\rangle=\frac{1}{(2j+1)^{1/2}}\sum_s(s_1\lambda_1s_2-\lambda_2|s\lambda)\sum_\nu\int\mu(\rho,\nu)d\rho\,d^{\rho\nu}_{js\lambda}(a)|(\rho\nu s)jn\rangle\,. \tag{24}$$

III. ENERGY-DEPENDENT EXPANSIONS OF SCATTERING AMPLITUDES

Let us consider the reaction

$$1+2\rightarrow3+4 \tag{25}$$

in the c.m.s. Let $\vec{p}_1=-\vec{p}_2$ be parallel to the third axis of a coordinate system. The initial state is then characterized by the spherical coordinates $(a',0,0)$, the final state by (a,θ,ζ).

We wish to present explicitly the energy dependence of the usual partial-wave amplitudes, i.e, the matrix elements of the scattering matrix between Poincaré-invariant irreducible states. Thus in the canonical ls basis we obtain the canonical amplitudes $A^j_{lsl's'}(a)$,

$$\langle w(a)jn,ls|T|w(a')j'n',l's'\rangle=\delta(w(a)-w(a'))\,\delta_{jj'}\delta_{nn'}A^j_{lsl's'}(a)\,, \tag{26}$$

and in the helicity basis the helicity amplitudes

$A^j_{\lambda_1\lambda_2\lambda_3\lambda_4}(a)$,

$$\langle w(a)jn,\lambda_3\lambda_4|T|\,w(a')j'n',\lambda_1\lambda_2\rangle = \delta(w(a)-w(a'))\,\delta_{jj'}\delta_{nn'}\,A^j_{\lambda_1\lambda_2\lambda_3\lambda_4}(a)\,. \quad (27)$$

Energy conservation provides the relation $w(a) = w(a')$, i.e.,

$$(m_1+m_2)^2+4m_1m_2\sinh^2 a' = (m_3+m_4)^2+4m_3m_4\sinh^2 a\,. \quad (28)$$

We can thus expand in terms of a' or a, depending on which variable varies through a more convenient region. For $m_3+m_4>m_1+m_2$ the appropriate variable will be a, since $a=0$ then corresponds to the physical threshold $w(a=0)=m_3+m_4$. For $m_3+m_4<m_1+m_2$ the choice would be a', since the threshold then is $w(a'=0)=m_1+m_2$. For $m_1+m_2 = m_3+m_4$, in particular elastic scattering ($m_1=m_3$, $m_2=m_4$), the two coincide. For definiteness let us consider the case $m_3+m_4 \geq m_1+m_2$ which is the case most often realized experimentally.

We now use formulas (23) and (24) to introduce the internal-Lorentz-group basis into the matrix elements (26) and (27). We have

$$A^j_{ls\,l's'}(a)=\sum_{\lambda,\lambda'}(l0s\lambda|j\lambda)\,(l'0s'\lambda'|j\lambda')\,X^j_{s\lambda s'\lambda'}(a)\,,$$
$$A^j_{\lambda_1\lambda_2\lambda_3\lambda_4}(a)=\sum_{s,s'}(s_1\lambda_1s_2-\lambda_2|s'\lambda')\times(s_3\lambda_3s_4-\lambda_4|s\lambda)X^j_{s\lambda s'\lambda'}(a)\,, \quad (29)$$

where we have defined

$$X^j_{s\lambda\,s'\lambda'}(a)=\frac{1}{2j+1}\sum_{\nu',\nu''}\int\mu(\rho',\nu')d\rho'\int\mu(\rho'',\nu'')d\rho''\,d^{\rho'\nu'}_{js\lambda}{}^*(a)\,d^{\rho''\nu''}_{js'\lambda'}(a')\,\langle\rho'\nu's|T^j|\rho''\nu''s'\rangle \quad (30)$$

and used the Wigner-Eckart theorem to set:

$$\langle\rho'\nu's|T^j|\rho''\nu''s'\rangle\equiv\langle(\rho'\nu's)jn|T|(\rho''\nu''s')j'n'\rangle\delta_{jj'}\delta_{nn'}\,. \quad (31)$$

Equation (30), involving the product of two d functions of related arguments a and a' [see (28)] cannot be directly inverted. We can, however, use the orthonormality and completeness relations (A18) and (A19) to introduce the quantity

$$Q\binom{\rho'\nu'\rho''\nu''\rho\nu}{jss'\lambda'}=\frac{2}{\pi(2j+1)(2s+1)}\sum_\lambda\int_0^\infty\sinh^2 a\,da\,d^{\rho\nu}_{js\lambda}(a)\,d^{\rho'\nu'}_{js\lambda}{}^*(a)\,d^{\rho''\nu''}_{js'\lambda'}(a') \quad (32)$$

and to express

$$d^{\rho'\nu'}_{js\lambda}{}^*(a)\,d^{\rho''\nu''}_{js'\lambda'}(a')=\sum_\nu\int_0^\infty\mu(\rho,\nu)d\rho\,Q\binom{\rho'\nu'\rho''\nu''\rho\nu}{jss'\lambda'}d^{\rho\nu*}_{js\lambda}(a)\,. \quad (33)$$

Substituting (33) into (30) we obtain

$$X^j_{s\lambda s'\lambda'}(a)=\frac{1}{2j+1}\sum_\nu\int\mu(\rho,\nu)d\rho\,T^{\rho\nu}_{jss'\lambda'}\,d^{\rho\nu*}_{js\lambda}(a)\,, \quad (34)$$

where

$$T^{\rho\nu}_{jss'\lambda'}=\sum_{\nu'\nu''}\int\mu(\rho',\nu')d\rho'\int\mu(\rho'',\nu'')d\rho''\,\hat{Q}\binom{\rho'\nu'\rho''\nu''\rho\nu}{jss'\lambda'}\langle\rho'\nu's|T^j|\rho''\nu''s'\rangle\,. \quad (35)$$

Equation (34) can easily be inverted and we have

$$T^{\rho\nu}_{jss'\lambda'}=\frac{2}{\pi(2s+1)}\sum_\lambda\int_0^\infty\sinh^2 a\,da\,d^{\rho\nu}_{js\lambda}(a)\,X_{s\lambda s'\lambda'}(a)\,. \quad (36)$$

The coefficients $T^{\rho\nu}_{jss'\lambda'}$ are energy independent. Indeed, these are our fundamental quantities, the Lorentz amplitudes.

Returning to the matrix elements of the transition matrix in the linear-momentum basis we have, in terms of canonical amplitudes,

$$\langle x\sigma_3\sigma_4|T|a'\sigma_1\sigma_2\rangle = \langle a\theta\zeta\sigma_3\sigma_4|T|a'00\sigma_1\sigma_2\rangle$$

$$= \delta(w(a)-w(a'))\sum_{s,s'}(s_1\sigma_1 s_2\sigma_2|s'\sigma')(s_3\sigma_3 s_4\sigma_4|s\sigma)$$

$$\times\sum_{ll'j}[4\pi(2l+1)]^{1/2}\frac{2l'+1}{2j+1}(l\sigma'-\sigma s\sigma|j\sigma')(l'0s'\sigma'|j\sigma')A^j_{lsl's'}(a)Y_{l\sigma'-\sigma}(\theta,\zeta)\,, \tag{37}$$

and, in terms of helicity amplitudes,

$$\langle x\lambda_3\lambda_4|T|a'\lambda_1\lambda_2\rangle = \langle a\theta\zeta\lambda_3\lambda_4|T|a'00\lambda_1\lambda_2\rangle = \delta(w(a)-w(a'))\sum_j(2j+1)\;A^j_{\lambda_3\lambda_4\lambda_1\lambda_2}(a)\,D^{j*}_{\lambda_1-\lambda_2,\,\lambda_3-\lambda_4}(\zeta,\theta,0)\,. \tag{38}$$

Substituting (29) into (37) and (38) and using (34), we obtain the final expansions. In the case of canonical quantization we have

$$\langle x\sigma_3\sigma_4|T|a'\sigma_1\sigma_2\rangle = \delta(w(a)-w(a'))\sum_{s,s'}(s_3\sigma_3 s_4\sigma_4|s\sigma)(s_1\sigma_1 s_2\sigma_2|s'\sigma')\sum_{\nu j}\int\mu(\rho,\nu)d\rho\,T^{\rho\nu}_{jss'\sigma'}D^{\rho\nu*}_{j\sigma's\sigma}(L_x)\,. \tag{39}$$

In the case of helicity quantization we have

$$\langle x\lambda_3\lambda_4|T|a'\lambda_1\lambda_2\rangle = \delta(w(a)-w(a'))\sum_{s,s'}(s_3\lambda_3 s_4-\lambda_4|s\lambda)(s_1\lambda_1 s_2-\lambda_2|s'\lambda')\sum_{\nu j}\int\mu(\rho,\nu)d\rho\,T^{\rho\nu}_{jss'\lambda'}D^{\rho\nu*}_{j\lambda's\lambda}(\Lambda_x)\,. \tag{40}$$

Equations (39) and (40) represent the "complete" expansions. Indeed, the entire dependence on the kinematical variables a, θ, and ζ is contained in the Lorentz-group D functions. They directly generalize the O(3,1)-group two-variable expansions for spinless particles considered earlier.[1-7]

Expansions (39) and (40) can be inverted and we obtain the Lorentz amplitudes in terms of the scattering amplitudes, integrated over the entire region of kinematical variables. We have

$$T^{\rho\nu}_{jss'\sigma'} = \frac{1}{2\pi^2(2s+1)}\sum_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma}(s_3\sigma_3 s_4\sigma_4|s\sigma)(s_1\sigma_1 s_2\sigma_2|s'\sigma')\int\mu(x)\,d^3x\,D^{\rho\nu}_{j\sigma's\sigma}(L_x)\langle x\sigma_3\sigma_4|T|a'\sigma_1\sigma_2\rangle\,, \tag{41}$$

$$T^{\rho\nu}_{jss'\lambda'} = \frac{1}{2\pi^2(2s+1)}\sum_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda}(s_3\lambda_3 s_4-\lambda_4|s\lambda)(s_1\lambda_1 s_2-\lambda_2|s'\lambda')\int\mu(x)\,d^3x\,D^{\rho\nu}_{j\lambda's\lambda}(\Lambda_x)\langle x\lambda_3\lambda_4|T|a'\lambda_1\lambda_2\rangle\,. \tag{42}$$

We have "derived" the above expansions with a complete disregard for mathematical rigor. In particular, questions of convergence of various expansions were ignored. Let us just state that the expansions in (39) and (40) will converge in the mean and that (41) and (42) will be valid if the expanded scattering amplitudes belong to a Hilbert space of functions, square integrable over the hyperboloid $x^2=1$ [$\mu(x)d^3x$ denotes the invariant measure]. In this case the continuous O(3,1) label ρ is real and the expansions involve only unitary representations of the principal series.[14] Physical amplitudes do not necessarily satisfy the mathematical constraint of square integrability, implying, e.g., that total cross sections must decrease quite rapidly as the energy increases. The expansions must then be generalized and this can be done by analytically continuing in ρ and integrating over an appropriately chosen path in the ρ plane. Such generalizations involve nonunitary representations of O(3,1) and are similar to a replacement of Fourier transforms by complex Laplace transforms for functions of one variable.

IV. PROPERTIES OF THE ENERGY-DEPENDENT EXPANSIONS

Let us list some of the properties of the obtained expansions (39) and (40) that are important for physical applications.

(1) As was stressed earlier, the expansions are *complete*, in that the entire dependence on the invariant variables s and t, as well as on the azimuthal angle ζ, is displayed explicitly in known functions, namely the Lorentz-group D-functions.

(2) The expansions *contain the usual partial-wave expansions* and can thus be interpreted as further expansions of the Poincaré-invariant partial-wave amplitudes. Thus, in the canonical coupling scheme we can express the invariant amplitudes in (37) as

$$A^j_{ls l's'}(a) = \frac{1}{(2j+1)} \sum_{\lambda,\lambda'} (l0s\lambda|j\lambda)(l'0s'\lambda'|j\lambda') \times \sum_\nu \int \mu(\rho,\nu) d\rho\, T^{\rho\nu}_{jss'\lambda'} d^{\rho\nu *}_{js\lambda}(a)\,. \tag{43}$$

Similarly, in the helicity coupling scheme we express the invariant amplitudes in (38) as

$$A^j_{\lambda_1\lambda_2\lambda_3\lambda_4}(a) = \sum_{s,s'} (s_3\lambda_3 s_4 - \lambda_4|s\lambda)(s_1\lambda_1 s_2 - \lambda_2|s'\lambda') \times \sum_\nu \int \mu(\rho,\nu) d\rho\, T^{\rho\nu}_{jss'\lambda'} d^{\rho\nu *}_{js\lambda}(a)\,. \tag{44}$$

The Lorentz amplitudes characterize a given reaction as such, i.e., for all energies and angles. Notice that the same Lorentz amplitudes $T^{\rho\nu}_{jss'\lambda'}$ figure in both expansions, independently of the coupling scheme. The quantum number j is the usual total angular momentum in the initial and final states, the quantum numbers l, s, λ, l', s', λ' and λ_i have their usual and obvious meanings. The quantum numbers ρ and ν, introduced by the internal Lorentz group are somewhat harder to interpret. In the nonrelativistic and spinless case the number ρ has been related to the distance from the scatterer.[4]

(3) *Threshold behavior.* The kinematic behavior of, e.g., the partial-wave helicity amplitudes at the initial and final-state thresholds and pseudothresholds is expected to be[23]

$$A^j_{\lambda_1\lambda_2\lambda_3\lambda_4}(a) \simeq [w^2(a)-(m_1+m_2)^2]^{l_{T_i}/2}[w^2(a)-(m_1-m_2)^2]^{l_{P_i}/2} \times [w^2(a)-(m_3+m_4)^2]^{l_{T_f}/2}[w^2(a)-(m_3-m_4)^2]^{l_{P_f}/2}\tilde{A}^j_{\lambda_1\lambda_2\lambda_3\lambda_4}(a)\,, \tag{45}$$

where $\tilde{A}^j_{\lambda_1\lambda_2\lambda_3\lambda_4}$ is regular and nonzero at the thresholds and pseudothresholds and l_{T_i}, l_{P_i}, l_{T_f}, and l_{P_f} are the minimal possible values of the angular momentum at the appropriate points.

We have assumed that $m_3+m_4 \geq m_1+m_2$ and hence only the final-state threshold touches the physical region at $w(a)=m_3+m_4$. At this threshold we have

$$a=0\,,\quad \sinh^2 a' = \frac{(m_3+m_4)^2-(m_1+m_2)^2}{4m_1m_2} \tag{46}$$

and the property

$$d^{\rho\nu}_{js\lambda}(0) = \delta_{js} \tag{47}$$

of the d functions insures that only the waves with $j=s$, i.e., $l_{T_f}=0$ contribute:

$$A^j_{lsl's'}(0) = \delta_{js}\frac{1}{2s+1}\sum_{\lambda,\lambda'}(l0s\lambda|s\lambda)(l'0s'\lambda'|s\lambda') \times \sum_\nu \int \mu(\rho,\nu)d\rho\, T^{\rho\nu}_{sss'\lambda'}\,, \tag{48}$$

$$A^j_{\lambda_1\lambda_2\lambda_3\lambda_4}(0) = \delta_{js}\sum_{s,s'}(s_3\lambda_3 s_4-\lambda_4|s\lambda)(s_1\lambda_1 s_2-\lambda_2|s'\lambda') \times \sum_\nu \int \mu(\rho,\nu)d\rho\, T^{\rho\nu}_{sss'\lambda'}\,.$$

For a more detailed study of the threshold behavior we would need to investigate the behavior of the d functions for small values of a. The behavior $d^{\rho\nu}_{js\lambda}(a) \sim (\sinh a)^{|j-s|}$ for $a \to 0$ has been conjectured[8] and would indeed ensure the behavior (45) at the physical threshold.

We have not studied the analyticity properties of the expansions (43) and (44), nor their convergence outside the physical region. The requirement that the partial-wave amplitudes with $l > l_{\min}$ should vanish at the nonphysical threshold and at the pseudothresholds would impose constraints on the Lorentz amplitudes:

$$\sum_\nu \int \mu(\rho,\nu)d\rho\, T^{\rho\nu}_{jss'\lambda'} d^{\rho\nu *}_{js\lambda}(a_{NP}) = C(s,s',\lambda,\lambda')\delta_{js_{NP}}\,. \tag{49}$$

Here a_{NP} and s_{NP} are the energy and spin at the corresponding nonphysical point and $C(s,s',\lambda,\lambda')$ is finite and nonzero. Notice that the constraint (49) must actually be satisfied at the nonphysical threshold $a'=0$. Indeed, this behavior is present in formulas (29) and (30) and its explicit "visibility" was lost in the redefinition (35) of the Lorentz amplitudes.

(4) Consequences of *parity conservation.* If parity is conserved in the reaction, then, e.g., the partial-wave helicity amplitudes (44) must satisfy[21]

$$A^j_{\lambda_1\lambda_2\lambda_3\lambda_4} = \eta(-1)^{s_3+s_4-s_1-s_2}A^j_{-\lambda_1-\lambda_2-\lambda_3-\lambda_4}\,,\qquad \eta=\eta_1\eta_2\eta_3\eta_4\,, \tag{50}$$

where η_i are the intrinsic parities. For the Lorentz amplitudes this implies

$$T^{\rho-\nu}_{jss'-\lambda} = \eta(-1)^{s-s'}T^{\rho\nu}_{jss'\lambda}\,. \tag{51}$$

(5) Consequences of *time-reversal invariance* for elastic scattering. If the reaction is elastic and time-reversal invariance holds, then the partial-wave helicity amplitudes (44) satisfy

$$A^j_{\lambda_1\lambda_2\lambda_3\lambda_4}(a) = A^j_{\lambda_3\lambda_4\lambda_1\lambda_2}(a). \tag{52}$$

For the Lorentz amplitudes this implies a linear relationship that can be written as

$$T^{\rho\nu}_{jss'\lambda'} = \frac{2}{\pi(2j+1)(2s+1)} \sum_{\lambda\nu'} \int \mu(\rho',\nu')d\rho' \int_0^\infty \sinh^2 a\, da\, d^{\rho'\nu'}_{js'\lambda'}{}^*(a) d^{\rho\nu}_{js\lambda}(a)\, T^{\rho'\nu'}_{js's\lambda}. \tag{53}$$

(6) *Identical particles.* If the two particles in the initial state and the two particles in the final state are identical, then the partial-wave helicity amplitudes satisfy

$$A^j_{\lambda_1\lambda_2\lambda_3\lambda_4}(a) = A^j_{\lambda_2\lambda_1\lambda_4\lambda_3}(a). \tag{54}$$

For the Lorentz amplitudes this implies

$$T^{\rho\nu}_{jss'\lambda'} = T^{\rho-\nu}_{jss'-\lambda'}. \tag{55}$$

V. ENERGY-DEPENDENT EXPANSIONS OF THE OBSERVABLES

For a given initial density matrix the unnormalized final-state density matrix is obtained in terms of the transition matrix elements in any quantization scheme as

$$(\rho_f)^{\sigma_3\sigma_4}_{\sigma_3'\sigma_4'}(x) = \sum_{\substack{\sigma_1\sigma_2\\ \sigma_1'\sigma_2'}} \langle x\sigma_3\sigma_4|T|\sigma_1\sigma_2\rangle (\rho_i)^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'} \langle x\sigma_3'\sigma_4'|T|\sigma_1'\sigma_2'\rangle^*. \tag{56}$$

In order to avoid a proliferation of O(3) Clebsch-Gordan coefficients we define the following auxiliary quantities:

$$\begin{aligned}(\rho_f)^{s\sigma}_{s'\sigma'}(x) &= \sum_{\substack{\sigma_3\sigma_4\\ \sigma_3'\sigma_4'}} (s_3\sigma_3 s_4\sigma_4|s\sigma)(s_3\sigma_3' s_4\sigma_4'|s'\sigma')(\rho_f)^{\sigma_3\sigma_4}_{\sigma_3'\sigma_4'}(x),\\ (\rho_i)^{t\tau}_{t'\tau'} &= \sum_{\substack{\sigma_1\sigma_2\\ \sigma_1'\sigma_2'}} (s_1\sigma_1 s_2\sigma_2|t\tau)(s_1\sigma_1' s_2\sigma_2'|t'\tau')(\rho_i)^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'}.\end{aligned} \tag{57}$$

The transformation (57) is unitary and can easily be inverted using the orthogonality properties of the O(3) Clebsch-Gordan coefficients.

Using the expansion (39) of the transition matrix elements, we obtain

$$(\rho_f)^{s\sigma}_{s'\sigma'}(x) = \sum_{\substack{tt'\\ \tau\tau'}} \sum_{j\nu'} \int \mu(\rho',\nu')d\rho' \sum_{j',\nu''} \int \mu(\rho'',\nu'')d\rho''\, T^{\rho'\nu'}_{jst\tau}(\rho_i)^{t\tau}_{t'\tau'} T^{\rho''\nu''}_{j's't'\tau'}{}^* D^{\rho'\nu'}_{j\tau s\sigma}{}^*(L_x) D^{\rho''\nu''}_{j'\tau's'\sigma'}(L_x). \tag{58}$$

The product of O(3,1) matrices in (58) can be expanded using the O(3,1) Clebsch-Gordan series [see (A22) of the Appendix]. We have

$$\begin{aligned}D^{\rho'\nu'}_{j\tau s\sigma}{}^*(L_x) D^{\rho''\nu''}_{j'\tau's'\sigma'}(L_x) &= (-1)^{\tau-\sigma} D^{\rho'\nu'}_{j-\tau s\sigma}(L_x) D^{\rho''\nu''}_{j'\tau's'\sigma'}(L_x)\\ &= (-1)^{\tau-\sigma} \sum_\nu \int \mu(\rho,\nu)d\rho \sum_{JS} (j-\tau j'\tau'|JM)(s-\sigma s'\sigma'|SN) \left[\begin{array}{cc|c} \rho'\nu' & \rho''\nu'' & \rho\nu \\ js & j's' & JS \end{array}\right] D^{\rho\nu}_{JM\,SN}(L_x).\end{aligned} \tag{59}$$

The expression in square brackets is defined in (A26) as a product of two reduced O(3,1) Clebsch-Gordan coefficients. Substituting (59) into (58) we have

$$(\rho_f)^{s\sigma}_{s'\sigma'}(x) = (-1)^{s-\sigma} \sum_S (s-\sigma s'\sigma'|S\sigma'-\sigma) \sum_{\nu J} \int \mu(\rho,\nu)\, d\rho \sum_{\substack{tt'\\ \tau\tau'}} D^{\rho\nu}_{J\tau'-\tau S\sigma'-\sigma}(L_x) K^{\rho\nu JS}_{st\tau s't'\tau'} (\rho)^{t\tau}_{t'\tau'}, \tag{60}$$

where we have defined

$$K^{\rho\nu JS}_{st\tau s't'\tau'} = (-1)^{\tau-s}\sum_{\nu' j}\int \mu(\rho',\nu')d\rho' \sum_{\nu'' j}\int \mu(\rho'',\nu'')d\rho''(j-\tau j'\tau' | J\tau'-\tau) \left[\begin{array}{cc|c} \rho'\nu' & \rho''\nu'' & \rho\nu \\ js & j's' & JS \end{array}\right] T^{\rho'\nu'}_{jst\tau}\, T^{*\rho''\nu''}_{j's't'\tau'} . \tag{61}$$

The expansion (60) can obviously be simplified at the threshold, but we shall not go into that here.

Let us consider a special case of interest, namely the scattering $0+\frac{1}{2}\to 0+s$, i.e., the case $s_1=s_3=0$, $s_2=\frac{1}{2}$, and $s_4=s$. The general expansion (39) of the scattering amplitude in the canonical quantization reduces to

$$\langle x\, s\sigma | T | a'\, \tfrac{1}{2}\,\epsilon\rangle = \delta(w(a)-w(a'))\sum_{\nu j}\int \mu(\rho,\nu)\,d\rho\; T^{\rho\nu}_{j\epsilon} D^{\rho\nu *}_{J\epsilon s\sigma}(L_x) . \tag{62}$$

The expression (60) for the matrix elements of the unnormalized final-state density matrix reduces to

$$(\rho_f)^{\sigma}_{\sigma'}(x) = (-1)^{s-\sigma} \sum_{S}(s-\sigma s\sigma' | S\sigma'-\sigma)\sum_{\nu J}\int \mu(\rho,\nu)\,d\rho \sum_{\epsilon,\epsilon'} D^{\rho\nu}_{J\epsilon'-\epsilon\, S\sigma'-\sigma}(L_x) K^{\rho\nu JS}_{\epsilon\epsilon'}(\rho_i)^{\epsilon}_{\epsilon'} . \tag{63}$$

Further, we can express the normalized initial-state density matrix in terms of the initial spinor-particle polarization vector $\vec{P}$

$$\rho_i = \tfrac{1}{2}(1+\vec{\sigma}\cdot\vec{P}) , \tag{64}$$

and the unnormalized final-state density matrix in terms of the differential cross section and the statistical polarization tensors[21]

$$[\rho_f(x)]^{\sigma}_{\sigma'} = \frac{d\sigma}{d\Omega}(x)\sum_{L,N}\frac{2L+1}{2s+1}\langle s\sigma LN | s\sigma'\rangle\, t^{L}_{N}{}^{*}(x) . \tag{65}$$

The expansion (63) for the experimental quantities now reduces to

$$\frac{d\sigma}{d\Omega}\, t^{L}_{N}{}^{*}(x) = (-1)^{L-2s}\left(\frac{2s+1}{2L+1}\right)^{1/2}\sum_{\nu J\epsilon\epsilon'}\int \mu(\rho,\nu)\,d\rho\, K^{\rho\nu JL}_{\epsilon\epsilon'} D^{\rho\nu}_{J\epsilon'-\epsilon L N}(L_x)\,\tfrac{1}{2}\,[\delta^{\epsilon}_{\epsilon'}+\vec{P}(\vec{\sigma})^{\epsilon}_{\epsilon'}] . \tag{66}$$

VI. CONCLUSIONS

The main results of this paper are the expansions (39) and (40) of the total scattering amplitudes for two-body reactions involving particles with arbitrary spins and (positive) masses, or the corresponding expansions (43) and (44) of the partial-wave amplitudes. The main properties of these expansions were discussed in Sec. IV, namely they are complete expansions, in that all the kinematic variables figure in known functions, the Lorentz-group transformation matrix elements. The formalism generalizes that of the usual partial-wave analysis, and the expansions can be interpreted as partial-wave expansions, supplemented by further expansions of the invariant partial-wave amplitudes. The partial waves automatically have the correct behavior at the physical threshold. All experimental quantities can now be expressed in terms of the expansion coefficients which we call Lorentz amplitudes (indeed we have expressed the final-state density matrix in terms of these amplitudes and the initial density matrix).

The expansions are a direct generalization of previously considered O(3, 1) two-variable expansions for particles of spin zero.[1-7] They were obtained by making use of an internal Lorentz group acting on two-particle states. The derivation made use of standard methods of group representation theory in quantum mechanics. Indeed, we started from single-particle states, constructed two-particle states as direct products, then expressed these product states in terms of irreducible ones and calculated matrix elements of the scattering matrix between such states.

We envisage a continuation of this program in several directions. One is a further exploration of the formalism, including an investigation of convergence and of the use of nonunitary representations to incorporate more general classes of scattering amplitudes. The second is a study of the implications of further physical principles, such as unitarity and analyticity of the scattering matrix for our formalism. The third direction is generalizations to many-body reactions and decays on the one hand and to expansions involving other subgroups of O(3, 1) on the other. As in the spinless case, the subgroup chain $O(3,1)\supset O(2,1)$

$\supset O(2)$ would provide an extension of complex-angular-momentum theory, the $O(3,1)\supset E(2)\supset O(2)$ chain an extension of eikonal expansions.

The most important, according to our opinion, continuation of this paper concerns phenomenological applications. Thus, we plan to consider specific reactions and to use the complete expansions to analyze the obtained experimental data simultaneously for all available energies and angles. Obvious candidates for such analyses are pion-nucleon and nucleon-nucleon scattering, where large bodies of data exist and still more are becoming available. In particular we hope that this formalism will make it possible to analyze simultaneously the abundant and detailed nucleon-nucleon data from the three existing pion factories, from the Saturne II accelerator at Saclay, and from the ZGS at Argonne. From this point of view the present formalism should be compared to the "locally energy-dependent partial-wave analysis" that has been recently applied to treat nucleon-nucleon scattering data for more restricted energy regions.[24]

Two difficulties must be overcome before the present expansions can be efficiently applied to phenomenological analyses. The first is that they involve a continuous variable, namely the Lorentz-group representation label ρ. This difficulty has been overcome for the case of spinless particles.[6] The expansions have been made discrete and the integral over ρ replaced by a sum over a new discrete variable. Work on a similar discretization for particles with spin is in progress. The second difficulty is related to the well known fact that standard partial-wave analysis for nucleon-nucleon scattering is known to be unstable. It becomes stable if modified partial-wave analysis is used, i.e., if the higher partial waves are not set equal to zero but calculated on the basis of the one-pion-exchange model.[25] Work is currently being performed to establish whether a similar stabilizing effect is provided either by the one-pion-exchange model or some more sophisticated approach, e.g. using the "Paris nucleon-nucleon potential" (Ref. 26) to calculate the higher Lorentz amplitudes.

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APPENDIX: SOME RESULTS ON REPRESENTATIONS OF O(3,1)

We consider representations of O(3,1) in the Naimark canonical basis[14] corresponding to the group reduction $O(3,1)\supset O(3)\supset O(2)$. The basis functions $|\rho\nu JM\rangle$ satisfy

$$\begin{aligned}\Delta\,|\rho\nu JM\rangle &= (\nu^2-\rho^2-1)\,|\rho\nu JM\rangle\,,\\ \Delta'\,|\rho\nu JM\rangle &= \rho\nu\,|\rho\nu JM\rangle\,,\\ \vec{L}^2\,|\rho\nu JM\rangle &= J(J+1)\,|\rho\nu JM\rangle\,,\\ L_3\,|\rho\nu JM\rangle &= M\,|\rho\nu JM\rangle\,,\end{aligned}\tag{A1}$$

with $\Delta=\vec{L}^2-\vec{K}^2$, $\Delta'=\vec{L}\cdot\vec{K}$. The generators of rotations $\vec{L}$ and pure Lorentz transformations $\vec{K}$ satisfy

$$\begin{aligned}&[L_i,L_j]=i\epsilon_{ijk}L_k\,,\quad [L_i,K_j]=i\epsilon_{ijk}K_k\,,\\ &[K_i,K_k]=-i\epsilon_{ijk}L_k\,.\end{aligned}\tag{A2}$$

Relations (A1) do not determine the phases of the basis functions. To specify their dependence on J and M completely, Naimark postulates the action of the generators on $|\rho\nu JM\rangle$:

$$\begin{aligned}L_\mu\,|\rho\nu JM\rangle &= [J(J+1)]^{1/2}(J\,1\,M\,\mu|J\,M+\mu)\,|\rho\,\nu\,J\,M+\mu\rangle\,,\\ K_\mu\,|\rho\nu JM\rangle &= -\sum_k R^{\nu\rho J}_{J+k}(-1)^k(J\,1\,M\,\mu|J+k\,M+\mu)\\ &\qquad\times|\rho\,\nu\,J+k\,M+\mu\rangle\,.\end{aligned}\tag{A3}$$

Here μ runs through the values ±1 and 0:

$$\begin{aligned}L_\pm &= \mp\frac{L_1\pm iL_2}{\sqrt{2}}\,,\quad L_0=L_3\,,\\ K_\pm &= \mp\frac{K_1\pm iK_2}{\sqrt{2}}\,,\quad K_0=K_3\,,\end{aligned}\tag{A4}$$

and

$$\begin{aligned}R^{\nu\rho J}_{J-1} &= \tfrac{1}{2}\left(\frac{J}{2J-1}\right)^{1/2}R^{\nu\rho J}\,,\\ R^{\nu\rho J}_{J+1} &= \tfrac{1}{2}\left(\frac{J+1}{2J+3}\right)^{1/2}R^{\nu\rho\,J+1}\,,\\ R^{\nu\rho J} &= -\frac{2i}{J}\,[(J^2-\nu^2)(J^2+\rho^2)]^{1/2}\,,\\ R^{\nu\rho J}_{J} &= \nu\rho[J(J+1)]^{-1/2}\,.\end{aligned}\tag{A5}$$

Unitary representations of the principal series can be realized in a Hilbert space of functions $\phi(u)$ defined over the group SU(2), satisfying [we are considering the covering group $SL(2,C)$ of O(3.1)]

$$\begin{aligned}&\phi(\gamma u)=e^{im\omega}\phi(u)\,,\quad u=\begin{pmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{pmatrix},\\ &\gamma=\begin{pmatrix}e^{-i\omega}&0\\0&e^{i\omega}\end{pmatrix},\quad u_{11}u_{22}-u_{12}u_{21}=1\end{aligned}\tag{A6}$$

and

$$\int |\phi(u)|^2 du < \infty \tag{A7}$$

[du is the invariant measure on SU(2)]. The basis functions satisfying (A1) and (A3) in this space can be shown to be[14]

$$\phi^{\rho\nu}_{JM}(u) = K^J_{\nu\rho}(2J+1)^{1/2} D^J_{\nu M}(u)\,, \tag{A8}$$

$$K^J_{\nu\rho} = \prod_{r=|\nu|}^{J} \frac{-r+i\rho}{(r^2+\rho^2)^{1/2}} = (-1)^{J-|\nu|+1}\left[\frac{\Gamma(J-i\rho+1)\Gamma(|\nu|+i\rho)}{\Gamma(J+i\rho+1)\Gamma(|\nu|-i\rho)}\right]^{1/2}. \tag{A9}$$

Here $D^J_{\nu M}(u)$ is an SU(2) Wigner D function and the normalization is such that

$$\int \phi^{\rho\nu *}_{JM}(u)\,\phi^{\rho\nu}_{J'M'}(u)\,du = \delta_{JJ'}\delta_{MM'}. \tag{A10}$$

It is worth mentioning here that the basis used by Rühl[15] is obtained from (A8) by omitting the phase factor $K^J_{\nu\rho}$. The basis used by Joos[9] and also by Chakrabarti *et al.*[10] is obtained by multiplying (A8) by $e^{-i(\pi/2)J}$ (all bases are specified up to a phase factor that can depend on ρ and ν only).

We are interested in the Lorentz-group D functions in the canonical basis (A8):

$$D^{\rho\nu}_{J_1M_1J_2M_2}(\Lambda) = \langle \rho\nu J_1 M_1 |\, T(\Lambda)\,| \rho\nu J_2 M_2\rangle\,, \tag{A11}$$

where $T(\Lambda)$ is an operator representing the general Lorentz transformation Λ in the irreducible representation (ρ, ν). Using the Euler parametrization

$$\Lambda = u_1 d u_2\,, \tag{A12}$$

where u_1 and u_2 are rotations and $d = \exp(iaK_3)$ is a Lorentz transformation along the z axis, we find

$$D^{\rho\nu}_{J_1M_1J_2M_2}(\Lambda) = \sum_\lambda D^{J_1}_{M_1\lambda}(u_1)\, d^{\rho\nu}_{J_1J_2\lambda}(a)\, D^{J_2}_{\lambda M_2}(u_2)\,, \tag{A13}$$

where

$$d^{\rho\nu}_{J_1J_2\lambda}(a) = K^{J_1 *}_{\nu\rho} K^{J_2}_{\nu\rho}[(2J_1+1)(2J_2+1)]^{1/2} \times \int_0^1 dt\, d^{J_1}_{\nu\lambda}(2t-1)\, d^{J_2}_{\nu\lambda}(2t_a-1) \times [te^{-a}+(1-t)e^a]^{i\rho-1}\,,$$
$$t_a = te^{-a}[te^{-a}+(1-t)e^a]^{-1}\,. \tag{A14}$$

[$d^J_{\nu\lambda}(\cos\theta)$ is an O(3) d function]. Using explicit expressions[22] for $d^J_{\nu\lambda}(\cos\theta)$, we can evaluate (A14) in different manners[17,18] and obtain, e.g.,

$$d^{\rho\nu}_{J_1J_2\lambda}(a) = \sum_{d=\max(0,-\nu-\lambda)}^{\min(J_1-\nu,\,J_1-\lambda)} \; \sum_{d'=\max(0,-\nu-\lambda)}^{\min(J_2-\nu,\,J_2-\lambda)} B^{\nu\rho}_{J_1J_2\lambda dd'} e^{-a(2d'+\lambda+\nu+1-i\rho)} \times {}_2F_1(1+J_2-i\rho,\, \nu+\lambda+d+d'+1;\, J_1+J_2+2,\, 1-e^{-2a})\,, \tag{A15}$$

where

$$B^{\nu\rho}_{J_1J_2\lambda dd'} = K^{J_1 *}_{\nu\rho}K^{J_2}_{\nu\rho}(-1)^{J_1+J_2-2\lambda+d+d'} \times [(2J_1+1)(J_1-\nu)!(J_1+\nu)!(J_1-\lambda)!(J_1+\lambda)!(2J_2+1)(J_2-\nu)!(J_2+\nu)!(J_2-\lambda)!(J_2+\lambda)!]^{1/2} \times \frac{(\nu+\lambda+d+d')!(J_1+J_2-\nu-\lambda-d-d')!}{(J_1+J_2+1)!d!(J_1-\lambda-d)!(J_1-\nu-d)!(\lambda+\nu+d)!d'!(J_2-\lambda-d')!(J_2-\nu-d')!(\lambda+\nu+d')!} \tag{A16}$$

Thus, $d^{\rho\nu}_{J_1J_2\lambda}(a)$ is expressed as a finite double sum involving hypergeometric functions. It can also be expressed in terms of elementary functions.[17]

Some relevant symmetry properties of the d functions (A15) are

$$\begin{aligned} &d^{\rho\nu}_{J_1J_2\lambda}(a) = d^{-\rho-\nu}_{J_1J_2\lambda}(a)\,, \quad d^{\rho\nu *}_{J_1J_2\lambda}(a) = d^{\rho\nu}_{J_2J_1\lambda}(-a)\,,\\ &d^{\rho\nu *}_{J_1J_2\lambda}(a) = d^{-\rho\nu}_{J_1J_2\lambda}(a)\,, \quad d^{\rho\nu}_{J_1J_2\lambda}(-a) = (-1)^{J_1-J_2} d^{\rho\nu}_{J_1J_2-\lambda}(a)\,,\\ &d^{\rho\nu}_{J_1J_2\lambda}(a) = d^{\rho\lambda}_{J_1J_2\nu}(a)\,, \quad d^{\rho-\nu}_{J_1J_2-\lambda}(a) = d^{\rho\nu}_{J_1J_2\lambda}(a)\,. \end{aligned} \tag{A17}$$

The orthonormality and completeness relations for the two sets of matrices $d^{\rho\nu}_{j_1j_2m}$ and $D^{\rho\nu}_{j_1m_1j_2m_2}$ are ($\mu(\rho,\nu)=\rho^2+\nu^2$)

$$\sum_m \int_0^\infty \sinh^2 a\, da\, d^{\rho\nu}_{j_1j_2m}(a)\, d^{\rho'\nu'}_{j_1j_2m}(a)^* = \frac{\pi}{2}\frac{(2j_1+1)(2j_2+1)}{\mu(\rho,\nu)}\,\delta(\rho-\rho')\delta_{\nu\nu'}\,, \tag{A18}$$

$$\sum_{\nu}\int_0^{\infty}\mu(\rho,\nu)d\rho\, d^{\rho\nu}_{j_1j_2m}(a)d^{\rho\nu}_{j_1j_2m'}(a')^* = \frac{\pi}{2}\frac{(2j_1+1)(2j_2+1)}{\sinh^2 a}\delta(a-a')\delta_{mm'}, \tag{A19}$$

$$\int\mu(\Lambda)d\Lambda D^{\rho\nu}_{j_1m_1j_2m_2}(\Lambda)D^{\rho'\nu'}_{j_1'm_1'j_2'm_2'}(\Lambda)^* = \frac{(2\pi)^4}{\mu(\rho,\nu)}\delta(\rho-\rho')\delta_{\nu\nu'}\delta_{j_1j_1'}\delta_{j_2j_2'}\delta_{m_1m_1'}\delta_{m_2m_2'},$$

$$\mu(\Lambda)=\sin\theta\,\sinh^2 a\,\sin\beta,\quad d\Lambda = d\varphi\, d\theta\, da\, d\alpha\, d\beta\, d\gamma, \tag{A20}$$

$$\sum_{\nu=-\infty}^{+\infty}\int_0^{\infty}\mu(\rho,\nu)d\rho \sum_{j_1,j_2=|\nu|}^{\infty}\sum_{m_1m_2} D^{\rho\nu}_{j_1m_1j_2m_2}(\Lambda)D^{\rho\nu}_{j_1m_1j_2m_2}(\Lambda')^* = \frac{(2\pi)^4}{\mu(\Lambda)}\delta(\Lambda-\Lambda'),$$

$$\delta(\Lambda-\Lambda')=\delta(\varphi-\varphi')\delta(\theta-\theta')\delta(a-a')\delta(\alpha-\alpha')\delta(\beta-\beta')\delta(\gamma-\gamma'). \tag{A21}$$

The Clebsch-Gordan coefficients[18] of O(3, 1) are defined by the tensor product reduction

$$D^{\rho_1\nu_1}_{j_1m_1j_1'm_1'}(\Lambda)D^{\rho_2\nu_2}_{j_2m_2j_2'm_2'}(\Lambda) = \sum_{\nu=-\infty}^{+\infty}\int_0^{\infty}\mu(\rho,\nu)d\rho\sum_{j,j'=|\nu|}^{\infty}\begin{pmatrix}\rho_1\nu_1\rho_2\nu_2 & \rho\nu\\ j_1m_1j_2m_2 & jm\end{pmatrix}\begin{pmatrix}\rho_1\nu_1\rho_2\nu_2 & \rho\nu\\ j_1'm_1'j_2'm_2' & j'm'\end{pmatrix}^* D^{\rho\nu}_{jmj'm'}(\Lambda). \tag{A22}$$

They can be written as the product of an O(3) Clebsch-Gordan coefficient and an analytically continued O(3) $9j$ symbol,

$$\begin{pmatrix}\rho_1\nu_1\rho_2\nu_2 & \rho\nu\\ j_1m_1j_2m_2 & jm\end{pmatrix} = (j_1m_1j_2m_2|jm)\begin{bmatrix}\rho_1\nu_1\rho_2\nu_2 & \rho\nu\\ j_1j_2 & j\end{bmatrix}. \tag{A23}$$

They satisfy orthogonality and completeness relations

$$\sum_{\substack{j_1m_1\\ j_2m_2}}\begin{pmatrix}\rho_1\nu_1\rho_2\nu_2 & \rho\nu\\ j_1m_1j_2m_2 & jm\end{pmatrix}\begin{pmatrix}\rho_1\nu_1\rho_2\nu_2 & \rho'\nu'\\ j_1m_1j_2m_2 & j'm'\end{pmatrix}^* = \mu(\rho,\nu)^{-1}\delta(\rho-\rho')\delta_{\nu\nu'}\delta_{jj'}\delta_{mm'}, \tag{A24}$$

$$\sum_{\nu=-\infty}^{+\infty}\int_0^{\infty}\mu(\rho,\nu)d\rho\sum_{j=|\nu|}^{\infty}\sum_{m}\begin{pmatrix}\rho_1\nu_1\rho_2\nu_2 & \rho\nu\\ j_1m_1j_2m_2 & jm\end{pmatrix}\begin{pmatrix}\rho_1\nu_1\rho_2\nu_2 & \rho\nu\\ j_1'm_1'j_2'm_2' & jm\end{pmatrix}^* = \delta_{j_1j_1'}\delta_{m_1m_1'}\delta_{j_2j_2'}\delta_{m_2m_2'}. \tag{A25}$$

For brevity in the main text we make use of the quantities

$$\begin{pmatrix}\rho_1\nu_1\rho_2\nu_2 & \rho\nu\\ j_1j_1'j_2j_2' & jj'\end{pmatrix}\equiv\begin{bmatrix}\rho_1\nu_1\rho_2\nu_2 & \rho\nu\\ j_1j_2 & j\end{bmatrix}\times\begin{bmatrix}\rho_1\nu_1\rho_2\nu_2 & \rho\nu\\ j_1'j_2' & j'\end{bmatrix}^*. \tag{A26}$$

When using these formulas it should be noted that the symbol ρ in Refs. 14 and 18 has been replaced by 2ρ in this article. The symbol K of Ref. 18 has been set equal to $K=1$.

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