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Crossing-symmetric four-body unitarity condition

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The four-body unitarity condition is studied in the case of a crossing-symmetric scattering process where the scattering is essentially iterated four times. It is found that there is a substantial enlargement of the analyticity domain in the $\cos\theta$ variable for fixed energy for the absorptive part of the scattering amplitude over what one obtains from the elastic unitarity alone.

As is well known, the unitarity of the S matrix implies a set of nonlinear relations in the sense that the absorptive part (or the discontinuity) of the scattering amplitude is expressible as folded products of two blocks of transition amplitudes suitably summed (or integrated) over the phase space of a set of appropriate intermediate states on the mass shell. It is generally recognized that unitarity is a powerful tool in enlarging the analyticity domain of ImT.[1] The use of the elastic unitarity is well known.[1-4]

The purpose of this paper is to study some of the implications of the inelastic unitarity. We have in mind, in particular, the use of four-body unitarity, although the result pertaining thereto could conceivably be extended to more general cases.

Four-body unitarity is thought to be intractable since a general study would call for the (as yet incomplete) knowledge of the analyticity properties of the (2 in)-(4 out) six-point functions (Fig. 1). Our working assumption here is that there is a process among those included in Fig. 1 where each of the two-particle to four-particle blobs is dominated in the crossed channel by one-particle exchange. This process (Fig. 2) which has manifest crossing symmetry in all channels involves only the iteration of the (2 in)-(2 out) scattering amplitude itself. This appears to be an eminently solvable problem. The solution can be compared with the known results for the analogous problem in perturbation theory on account of the close analogy between the structure of unitarity singularity and that of perturbation singularity.[5,6] Since the four-body scattering problem (corresponding to Fig. 2) in perturbation theory was solved some time ago by the present author,[7] it would seen worthwhile to undertake the study of the crossing-symmetric four-body unitarity problem.

Our primary aim here is to examine the analyticity of ImT in the scattering-angle $\cos\theta$ variable (in the center-of-mass frame) for fixed energy variables. More specifically, we wish to answer the following question. If one starts with the Lehmann ellipse analyticity in $\cos\theta$ for the scattering amplitude T, what is the outcome for ImT as a result of the action of the four-body unitarity depicted by Fig. 2? The answer to this question is given in the statement (B) below.

Our main observations are the following:

(A) In integrating over the set of intermediate states on the mass shell, the dominant (or singular) contribution comes from the extreme values of the azimuthal angles ($\phi = 0$ or π). Every such azimuthal condition implies the coplanarity of a set of three momentum three-vectors. Thus, for example, in the c.m. frame, the singular contribution comes from the case in which all the intermediate-state momentum three-vectors lie in the same scattering plane. This implies that a linear addition law holds for the polar angles as it does in plane geometry rather than in solid geometry. This result is well known for the elastic unitarity case[3,4] and obviously has general validity in the sense of pure kinematics for the phase-space volume integral.[8] One simple way to visualize such a singularity is by an appeal to the connection between physical singularity and stationary points in the classical path integral. This readily yields that stationary points correspond to the ex-

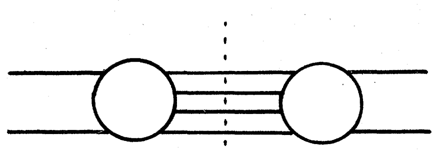

FIG. 1. Four-body unitarity in general.

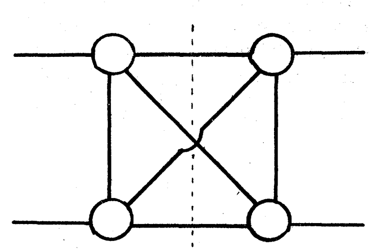

FIG. 2. Crossing-symmetric four-body unitarity.

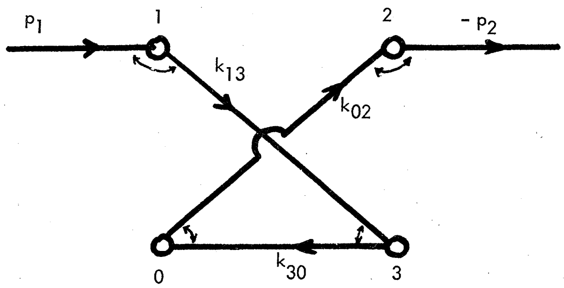

FIG. 3. A set of independent momentum vectors in the intermediate state and a convenient momentum flow diagram showing the choice of the scattering angle at each stage. θ is the overall scattering angle in the c.m. frame.

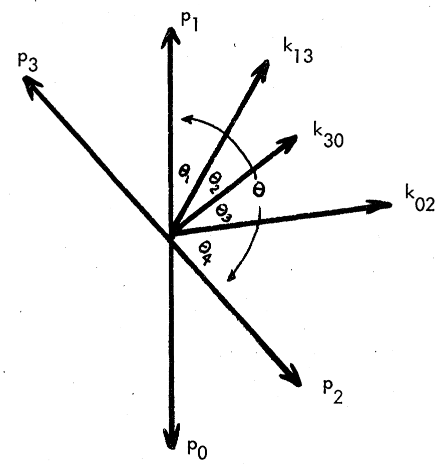

FIG. 4. Vectors $\vec{k}_{13}$, $\vec{k}_{30}$, $\vec{k}_{02}$ drawn in the c.m. frame $\vec{p}_0+\vec{p}_1=0$. Singular contribution from four-body unitarity comes from the complete coplanarity of all three-vectors in the scattering plane, hence $\theta=\sum\theta_k$.

treme values of $\phi=0$ or π.

Thus, in the present crossing-symmetric four-body unitarity case, one such relation for the polar-angle addition law reads generically

$$\theta=\sum_{k=1}^{4}\theta_k\,, \tag{1}$$

where each θ_k is a suitable scattering angle associated with the intermediate scattering.

(B) For the case under study, the intermediate process is arranged in such a way that there is in effect a fourfold iteration of the (2 in)-(2 out) scattering amplitude, once at each vertex. If the scattering amplitude at each stage $T_k=T(\cos\theta_k)$ has the analyticity in the $\cos\theta_k$ variable in the (small) Lehmann ellipse parametrized by

$$\cos\theta_k=\cosh(\xi-i\rho_k),\quad 0\leqslant\rho_k\leqslant 2\pi \tag{2}$$

with $a\equiv\cosh\xi$, the semimajor axis of the ellipse with unit foci, our analysis shows that as a consequence of Eq. (1), ImT is analytic in a much larger ellipse in $\cos\theta$ (for fixed energy $s\geqslant 16m^2$) given by

$$\cos\theta=\cosh(4\xi-i\rho),\quad 0\leqslant\rho\leqslant 2\pi \tag{3}$$

with the new semimajor axis

$$a^{(4)}=\cosh 4\xi=8a^4-8a^2+1\,. \tag{4}$$

Recall that the action of elastic unitarity yields that ImT is analytic inside the large Lehmann ellipse with the semimajor axis[2,3]

$$a^{(2)}=2a^2-1\,. \tag{5}$$

We briefly sketch here the steps that led to the above statement (A).

Statement (A) can be seen explicity by a straightforward integration scheme in the c.m. frame. We have, from Fig. 2, after dropping inessential kinematical and numerical factors,

$$\begin{aligned}\mathrm{Im}T=&\int dk_{13}dk_{30}dk_{02}dk_{12}\,\delta(k_{13}{}^2-m^2)\delta(k_{03}{}^2-m^2)\delta(k_{02}{}^2-m^2)\delta(k_{12}{}^2-m^2)\\&\times dk_{01}dk_{23}(D_{01}D_{23})^{-1}T_0^\dagger T_1^\dagger T_2T_3\prod_{l=0}^{3}\delta\Big(p_l+\sum_{m\neq l}k_{lm}\Big)\\=&\,\mathrm{const}\times\delta(\textstyle\sum p)\displaystyle\int\frac{d^3k_{13}}{E_{13}}\frac{d^3k_{30}}{E_{03}}\frac{d^3k_{02}}{E_{02}E_{12}}\delta(E_{12}+E_{13}+E_{03}+E_{02}-\sqrt{s}\,)\frac{T_0^\dagger T_1^\dagger T_2T_3}{D_{01}D_{23}}\,,\end{aligned} \tag{6}$$

where

$$D_{01}\equiv k_{01}{}^2-m^2=(p_0+k_{02}+k_{03})^2-m^2\,,$$

$$D_{23}\equiv k_{23}{}^2-m^2=(p_3-k_{03}-k_{13})^2-m^2\,.$$

We have chosen as independent variables the momenta k_{12}, k_{30}, k_{02}. It would be helpful to keep in mind a flowpath (Fig. 3) $\vec{p}_1\xrightarrow{T_1}\vec{k}_{13}\xrightarrow{T_3}\vec{k}_{30}\xrightarrow{T_0}\vec{k}_{02}\xrightarrow{T_2}\vec{p}_2$. We label all the angles involved by the corresponding indices, thus $\theta_{13,\,1}=\sphericalangle(\vec{k}_{13},\vec{p}_1)$, $\theta_{30,\,13}=\sphericalangle(\vec{k}_{30},\vec{k}_{13})$, etc., and $\theta=\sphericalangle(\vec{p}_1,\vec{p}_2)$ is the overall scattering angle in the c.m. frame $\vec{p}_0+\vec{p}_1=0$. The scattering angle at the intermediate stage for each T_k will be chosen as the angle between the incoming and the outgoing momenta in the above chain (Fig. 4).

The removal of the energy conservation $\delta(\sum E - \sqrt{s})$ factor can be done quite efficiently, for example, by the method of Byers and Yang[8] in their treatment of the phase-space volume integral. We get from (6)

$$\mathrm{Im}T = \delta(\textstyle\sum p)\displaystyle\int dE_{13}dE_{03}dE_{02}|\vec{k}_{13}|d\cos\theta_{13,1}d\cos\theta_{30,13}d\cos\theta_{02,30}d\phi_{13,1}d\phi_{30,13}[X(\theta_{30,13},\theta_{02,30},\theta_{02,13})]^{-1/2}[D_{01}D_{23}]^{-1}$$
$$\times T_1^\dagger(\cos\theta_{13,1})T_3(\cos\theta_{30,13})T_0^\dagger(\cos\theta_{02,30})T_2(\cos\theta_{02,2})\,. \tag{7}$$

We now insert under the integral the identities

$$1 \equiv \int d\cos\theta_{13,2}\delta(\cos\theta_{13,2} - \cos\theta_{13,1}\cos\theta - \sin\theta_{13,1}\sin\theta\cos\phi_{13,1})$$
$$\times d\cos\theta_{02,2}\delta(\cos\theta_{02,2} - \cos\theta_{02,30}\cos\theta_{03,2} - \sin\theta_{02,30}\sin\theta_{03,2}\cos\phi_{02,1})\,. \tag{8}$$

The integration over the azimuthal angles in (7) can be easily done using

$$\int_0^{2\pi} d\phi\,\delta(\cos\alpha - \cos\beta\cos\gamma - \sin\beta\sin\gamma\cos\phi) = 2\,\Theta(X)X^{-1/2}\,, \tag{9}$$

where

$$X(\cos\alpha,\cos\beta,\cos\gamma) \equiv \begin{vmatrix} 1 & \cos\alpha & \cos\gamma \\ \cos\alpha & 1 & \cos\beta \\ \cos\gamma & \cos\beta & 1 \end{vmatrix}$$
$$= -[\cos\gamma - \cos(\alpha+\beta)][\cos\gamma - \cos(\alpha-\beta)]\,, \tag{10}$$

and $\Theta(X)$ is the step function which sets the range for the physical values of the angles. These factors are understood in (7) and (11). Equation (7) now reads

$$\mathrm{Im}T = \delta(\textstyle\sum p)\displaystyle\int dE_{13}dE_{03}dE_{02}|\vec{k}_{13}|d\cos\theta_{13,1}T_1^\dagger(\cos\theta_{13,1})d\cos\theta_{30,13}T_3(\cos\theta_{30,13})$$
$$\times d\cos\theta_{02,30}T_0^\dagger(\cos\theta_{02,30})d\cos\theta_{02,2}T_2(\cos\theta_{02,2})$$
$$\times d\cos\theta_{13,2}[D_{01}D_{23}]^{-1}[X(\theta_{30,13},\theta_{02,30},\theta_{02,13})X(\theta_{02,2},\theta_{02,30},\theta_{30,2})X(\theta_{13,2},\theta_{13,1},\theta)]^{-1/2}\,. \tag{11}$$

What we have done can be summarized as follows. The eightfold phase-space integration in Eq. (6) has been parametrized into a threefold integration over the energy variables and a fivefold integration over the angles. By formal manipulation we have converted the integration over the azimuthal angles into that over some other polar angles, resulting in the appearance of the crucial kinematical factors $X(\cos\alpha, \cos\beta, \cos\gamma)$. Of the three sets of X factors in (11), the vanishing of a single X factor amounts to the requirement of a certain sub-two-body unitarity condition. The action of the crossing-symmetric four-body unitarity condition gives the singular contribution to ImT when the three sets of the X factors vanish *simultaneously*. This requirement can be also verified, for example, by an appeal to the formal analogy between the *unitarity* singularity and the perturbation singularity. In perturbation theory it is obvious that the Landau singularity calls for a set of simultaneous loop equations. When restricted to the c.m. frame, these loop equations imply the simultaneous coplanarity of three sets of momentum three-vectors. This completes the proof of our statement (A).

Once we have derived the addition law (1) as the singular contribution to the unitarity integral as a consequence of the crossing-symmetric four-body unitarity condition, the enlargement of the analyticity domain for ImT follows in the case of Lehmann ellipse analyticity as stated in the statement (B) above. One way to do this would be to follow the technique used by Mandelstam[3] in three steps: (i) Write down a Cauchy integral formula in the $\cos\theta_k$ variable for each T_k inside the analyticity domain in terms of the boundary points on the ellipse. (ii) Integrating over appropriate angles gives the composition law for the angles, Eq. (1). (iii) This implies that if each constituent angle $\cos\theta_k$ lies on an ellipse, the overall $\cos\theta$ will lie on a much larger ellipse, given via the addition law, Eq. (3).

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