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# Zero-mass limit for a class of jet-related cross sections

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Certain cross sections related to jets in $e^+e^-$ annihilation are seen to be finite in the zero-mass limit to all orders in perturbation theory. The power-counting argument covers a number of quantities discussed in the literature.

It has been proposed that the predictive power of perturbative quantum chromodynamics (QCD) may extend to certain color-averaged quantities which are free of divergences in the zero-mass limit.[1] The absence of mass divergences not only means that the expansion in the effective coupling is unspoiled by large logarithms at high energy, but is also a necessary condition for the absence of long-distance contributions, for which nonperturbative effects are important. If nonperturbative contributions can be dropped, then the renormalization group immediately gives an expansion in the effective coupling. Following this reasoning, one can derive the dominance of the cross section for electron-positron annihilation into hadrons by two-jet events, as well as find predictions for energy and angular dependence for three- and higher-jet events in the same process. The freedom of these quantities from mass divergences (referred to simply as "finiteness" below), when defined in terms of angular and energy resolutions, has been shown to all orders in perturbation theory.[2]

Related to these observations are proposals of a number of "weighted" cross sections as candidates for application of the renormalization group, and therefore for predictive analysis of hadronic final states in annihilation processes.[3] A weighted cross section will refer to a cross section found by summing over all states, but with a numerical weight associated with each. The purpose of this paper is to show how a slight variation of the methods developed to show the finiteness of jet cross sections can also be used to show the finiteness of a class of weighted cross sections,[4,5] including many of those discussed in the literature.

The common features of all these quantities are the following: (a) The weighting is the same for states which differ only by the emission or absorption of zero-momentum particles, and (b) the weighting is the same for states with different numbers of parallel-moving particles with the same total energy. (a) and (b) are related to the cancellation of infrared and collinear divergences, respectively.

We will denote a weighted cross section as $\sigma[f]$, $f=\{f_n\}$, with $f_n$ the weighting function for $n$-particle phase space:

$$\sigma[f]=\sum_{n=2}^{\infty}\int \sigma_n(p_1,\ldots,p_n) \times f_n(\vec{p}_1,\ldots,\vec{p}_n)\prod_{i=1}^{n}(d^3p_i/E_i)\,. \qquad (1)$$

Each $f_n$ is chosen to satisfy (a) and (b) above. More precisely, we can begin by demanding that the $f_n$'s be smooth functions of the momenta and satisfy, for instance,

$$\lim_{|\vec{p}_i|\to 0}\frac{f_n(\vec{p}_1,\ldots,\vec{p}_i,\ldots,\vec{p}_n)-f_{n-1}(\vec{p}_1,\ldots,\vec{p}_{i-1},\vec{p}_{i+1},\ldots,\vec{p}_n)}{|\vec{p}_i|^{\alpha}}=0\,,$$
$$\lim_{|\theta_{ij}|\to 0}\frac{f_n(\ldots,\vec{p}_i,\ldots,\vec{p}_j,\ldots)-f_{n-1}(\ldots,\vec{p}_i+\vec{p}_j,\ldots,\vec{p}_{j-1},\vec{p}_{j+1},\ldots)}{|\theta_{ij}|^{\beta}}=0 \qquad (2)$$

for some $\alpha,\beta>0$, with $\theta_{ij}$ the angle between $\vec{p}_i$ and $\vec{p}_j$.

As in Ref. 2, we organize the perturbation theory calculation of $\sigma[f]$ in terms of cut vacuum polarization graphs. We will say that cut $C$ splits a graph $G$ into two vertex functions $\Gamma_L^{(C)}$ and $\Gamma_R^{(C)}$, as in Fig. 1.

We first look for mass divergences in the integrals of the contributions $\sigma_G[f]$ to $\sigma[f]$ from cuts of $G$:

$$\sigma_G[f]=\sum_C\int \Gamma_L^{(C)}d\tau_C\,\Gamma_R^{(C)*}f_{n_C}\,, \qquad (3)$$

where $d\tau_C$ is the differential phase-space element for cut $C$, including spin factors, and $n_C$ is the number of lines in cut $C$. Mass divergences in (3) can arise only from tree subdiagrams of $\Gamma_L^{(C)}$ and $\Gamma_R^{(C)}$, or from those "pinch singular points" where

 

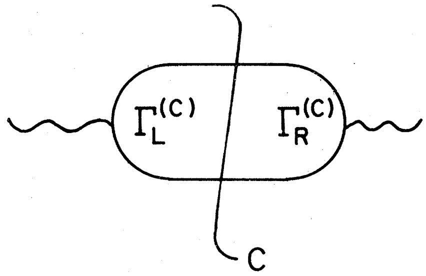


FIG. 1. Cut vacuum polarization graph.

momentum-space contour integrals in $\Gamma_L^{(C)}$ or $\Gamma_R^{(C)}$ are *trapped* by Feynman denominator singularities. With each such pinch singular point we associate a reduced diagram, in which all off-shell lines are contracted. At the pinch singular point, this reduced diagram actually represents a physical process, in which lines propagate freely between space-time points which represent vertices. The same is true of tree subdiagrams, and they can be considered as special cases of pinch singular points.

For the special case of cut vacuum polarization diagrams, the reduced diagrams of pinch singular points fall into a quite simple class, illustrated in Fig. 2. The reduced diagram consists of a set (in this case three) of "jet" subdiagrams $J_i$, made up entirely of parallel-moving lines, and a set of zero-momentum lines, $S$. The only vertices which connect finite-energy lines *not* moving in the same direction are $V_1$ and $V_2$. Figure 2 is a composite of two physical pictures. To the left of cut $C$, a set of jets is emitted at $V_1$. They interact internally, and with each other via zero-momentum lines (long-range forces), forming the final state $C$. To the right of $C$ the flow of time is reversed and the jets are absorbed at $V_2$. It should be noted that

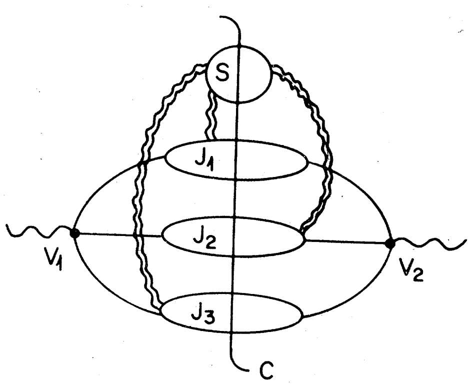


FIG. 2. Typical reduced diagram of pinch singular point. The double wavy lines represent multiple connections between the zero-momentum subdiagram $S$ and the jet subdiagrams $J_i$.

Fig. 2 is the reduced diagram of a pinch singular point only for (3) and not for the uncut graph $G$.

We now specialize to the axial gauge,[6] for which longitudinal vector degrees of freedom do not propagate on the light cone. Noncovariant unphysical poles decouple from cross sections in a manner that does not affect our reasoning.[7] The important point here is that, in the axial gauge, the power-counting degree of divergence from any surface of pinch singular points is, at worst, logarithmic.[2,5] Thus, any power suppression of the integrand at any such surface will preclude divergence in the integral. Our approach to cancellation is to show that summing over cuts of $C$ leads to just such a suppression at each pinch singular point.

Let $P$ be a pinch singular point of (3) with reduced diagram $R_P$. With $P$ we associate a subspace $X_P$ in the loop momentum space of $G$ where (a) all the energy components of the loop momenta of $G$ are integrated from minus to plus infinity, (b) spatial loop momenta vary over a finite range around their values at $P$, subject to: (c) if $P'$ is any pinch singular point in $X_P$, then there is another pinch singular point $P''$ in $X_P$ with reduced diagram $R_P$, such that $P'$ differs from $P''$ only in its energy components.[8] All the pinch singular points encountered in $X_P$ are in a sense equivalent to $P$.

Now consider the quantity

$$S_P = \sum_C \int_{X_P} I_{\Gamma_L^{(C)}} d\tau_C I^*_{\Gamma_R^{(C)}} \times f_n(\vec{p}_1^{(0)}, \ldots, \vec{p}_n^{(0)}) \prod_a d^4 l_a \prod_b d^4 l_b , \quad (4)$$

where $a$ and $b$ run over loops of $\Gamma_L^{(C)}$ and $\Gamma_R^{(C)}$, respectively, and where we define

$$\Delta = \int \prod_a d^4 l_a I_\Delta \quad (5)$$

and similarly for any other graph. We choose $f_n$ in (4) to be the weight function for any *fixed* cut of $R_P$. The integral in (4) is over the projection of $X_P$ onto the loop and phase-space momenta shown. Because $X_P$ extends over all energies in loops of $R_P$, the sum over cuts $C$ can be explicitly performed to give[2,9]

$$S_P = \int_{X_P} I_{V_1} (I_{R_P} - I^*_{R_P}) I^*_{V_2} \times f_n(\vec{p}_1^{(0)}, \ldots, \vec{p}_n^{(0)}) \prod_c d^4 l_c , \quad (6)$$

where $c$ goes over all loops of $G$, and where $V_1$ and $V_2$ are the subgraphs contracted to the right and left of $R_P$, respectively, as in Fig. 2.

It is now easy to show that $S_P$ is finite, because the loop momenta of $S_P$ are no longer trapped at $P$ or any of the other pinch singular points encountered in the interior of $X_P$. To see this, pick any two jet subdiagrams, $J_1$ and $J_2$, in $R_P$. Let $q^{(12)}$ be a loop momentum passing through both jets, and let $l^{(i,k)}$ be the $k$th internal momentum of jet $i$. At $P$ the total momentum carried by jet $i$ can be written as $\pm q^{(12)}+K^{(i)}$. We parametrize the line momenta $k^{(\xi)}$ of each jet so that the $l^{(i,k)}$ all vanish at $P$:

$$k^{(\xi)}=\gamma^{(\xi)}(\pm q^{(12)}+K^{(i)}) +I^{\xi n}l^{(i,n)}, \quad 0<\gamma^{(\xi)}\leq 1. \tag{7}$$

$I^{\xi n}$ is an "incidence matrix" specifying the loops which pass through line $k^{(\xi)}$. Now the Landau equations for the momentum $q^{(12)}$ are

$$\sum_{\xi\in J_1}\alpha_\xi\gamma^{(\xi)}k_\mu^{(\xi)}-\sum_{\eta\in J_2}\alpha_\eta\gamma^{(\eta)}k_\mu^{(\eta)}=0, \tag{8}$$

where sums are over all line momenta in jets 1 and 2. But in $X_P$, each line momentum is proportional to the corresponding total jet momentum so (8) can only be satisfied if $\alpha_\xi=\alpha_\eta=0$, for all $\xi$ and $\eta$. Our ability to deform the $q^{(12)}$ contours may be checked explicitly by a direct examination of the form of the Feynman integrals.

The quantities in which we are really interested are not the $S_P$'s, but ones such as

$$T_P=\sum_C\int_{X_P}I_{\Gamma_L^{(C)}}d\tau_C I^*_{\Gamma_R^{(C)}} \times f_{n_C}(\ldots,\vec{p}^{(0)}_{n_C})\prod_a d^4l_a\prod_b d^4l_b. \tag{9}$$

But consider

$$S_P-T_P=\sum_C\int_{X_P}I_{\Gamma_L^{(C)}}d\tau_C I^*_{\Gamma_R^{(C)}} \times(f_n-f_{n_C})\prod_a d^4l_a\prod_b d^4l_b. \tag{10}$$

Divergences in $T_P$ are at worst logarithmic on a point-by-point basis in the axial gauge. By construction, however, $f_{n_C}-f_n$ vanishes as a power for any singular point with reduced diagram $R_P$. This is enough to ensure that (10) is finite.

In summary, the integrals $T_P$, constructed to include arbitrary pinch singular points $P$, are seen to differ by finite amounts from the quantities $S_P$, which are themselves finite. On the other hand, every divergent contribution to $\sigma[f]$ must come from pinch singular points such as $P$. Since all such divergences cancel in the sum over cuts, $\sigma[f]$ is also finite.

The above reasoning depends crucially on properties (2) of the weight functions $f$. An interesting class of weighted cross sections has been proposed by Basham, Brown, Ellis, and Love,[3] in which the $f_n$'s involve angular $\delta$ functions. The simplest of these (which they named the "antenna pattern") can be defined by

$$f_n^{(\mathrm{AP})}=C_n\sum_{b=1}^{n}\frac{E_b}{E_{\mathrm{tot}}}\delta^2(\Omega_b-\Omega). \tag{11}$$

$C_n$ is a combinatoric factor and $\Omega_b$ defines the direction of particle $b$. Substituted into (1), this $f_n$ gives a differential cross section $d\Sigma/d\Omega$ for energy flow in direction $\Omega$. A slight modification of the reasoning above will show that this cross section is also free of mass divergences in each order of perturbation theory.

From the point of view of any cut graph $G$, the only pinch singular points that can contribute at all to $d\Sigma/d\Omega$ are those which have a jet moving in precisely direction $\Omega$. (A jet may consist of only one line.) For the purposes of argument, we can pick $\Omega$ to be in the 3 direction. Then for line $k^{(\xi)}$

$$\delta^2(\Omega_b-\Omega)=|k^{(\xi)}|^2\delta^2(\vec{k}^{(\xi)}), \tag{12}$$

with $\vec{a}$ the transverse components of line momentum $a$. For any pinch singular point $P$ we can pick $q^{(12)}$ as above, choosing jet 1 to be moving in the 3 direction, and we construct subspace $X_P$ and the quantity $S_P$ in just the same way.

We now use the $\delta$ functions (12) to evaluate the two $\vec{q}^{(12)}$ integrals in $X_P$. This leaves free the $q_0^{(12)}$ and $q_3^{(12)}$ integrals, and the corresponding contours must still be trapped if $S_P$ is to have any divergence. A necessary condition for trapping is still Eq. (8), but now for $\mu=0, 3$ only. Despite the fact that we now have only two equations instead of four, the only solution is again $\alpha_\xi=\alpha_\eta=0$, and $P$ is not a pinch singular point of $S_P$. $S_P$, and thus $T_P$, is then finite. Now there is no enhancement associated with having jet 1 directed in precisely the 3 direction. Therefore, the dependence of the integrand of $d\Sigma/d\Omega$ on $\vec{q}^{(12)}$ is smooth, and effectively the $\delta$ functions (12) may be treated as normal functions in discussing the integral near $X_P$. $d\Sigma/d\Omega$ is seen to be finite by a reapplication of the arguments given above.

The same arguments apply to simple generalizations of (11), found by multiplying $f_n^{(\mathrm{AP})}$ by other functions satisfying (2). Similar methods may also be used to show that the double energy cross section $d^2\Sigma/d\Omega\,d\Omega'$, defined by

$$f_n^{(\mathrm{DE})}=D_n\sum_{b,b'=1}^{n}\frac{E_bE_{b'}}{E_{\mathrm{tot}}^2}\delta^2(\Omega_b-\Omega)\,\delta^2(\Omega_{b'}-\Omega), \tag{13}$$

is finite, for $\Omega$ neither parallel nor antiparallel to $\Omega'$. With these conditions, the only dangerous pinch singular points are those with three or more jets, out of which one will be in the $\Omega$, one in the $\Omega'$, direction. Label these jets 1 and 3, respectively. By analogy to our procedure above, we choose two loops, $q^{(12)}$ and $q^{(32)}$, passing through jets 1 and 2, and 3 and 2, respectively. In place of (7), we now have at $P$

$$\begin{aligned} k^{(\xi_1)} &= \gamma^{(\xi_1)}(q^{(12)} + K^{(1)}) + I^{\xi_1 n} l^{(1,n)}, \\ k^{(\xi_2)} &= \gamma^{(\xi_2)}(-q^{(12)} - q^{(32)} + K^{(2)}) + I^{\xi_2 n} l^{(2,n)}, \quad (14) \\ k^{(\xi_3)} &= \gamma^{(\xi_3)}(q^{(32)} + K^{(3)}) + I^{\xi_3 n} l^{(3,n)}, \end{aligned}$$

where the $\xi_i$ range over all the lines in jet $i$. The reasoning then proceeds essentially unchanged from the case of the antenna pattern.

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[4]G. Tiktopoulos has recently given some interesting all-order arguments for the finiteness of a similar set of weighted cross sections in Nucl. Phys. B147, 371 (1979). The author wishes to thank Sherwin Love for bringing this work to his attention. The proof given here is based on a considerably different approach. In particular, it does not use the finiteness of two-particle irreducible diagrams in the zero-mass limit. This condition has been discussed in some detail in Refs. 5, but a complete technical proof of the result has not yet appeared in the literature for non-Abelian gauge theories.

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[8]$X_P$ can be characterized as a "wedge-shaped" region. That is, condition (c) means that there will be pinch singular points arbitrarily close to $P$ which are not in $X_P$. Our choice of $P$ is arbitrary, however, so the same arguments can be carried out for these pinch singular points as well. It is the basic power-counting assumption of this and similar arguments that such an approach is adequate to treat the behavior of Feynman integrals. See Refs. 2 and 5, as well as S. Libby and G. Sterman, Phys. Rev. D 18, 4737 (1978) for more discussion of this point.

[9]S. Libby and G. Sterman, Ref. 5.