PHYSICAL REVIEW D VOLUME 19, NUMBER 10 15 MAY 1979

Equal-time commutators in continuous dimensions

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(Received 10 October 1978)

Motivated by the recent re-examination of the Gotō-Imamura-Schwinger term in continuous dimensions, we re-examine the familiar canonical equal-time commutator of the interacting field. Two methods of calculation are available: the equal-time limit taken before or after the spectral integration. The technique of dimensional regularization enables us to compare the two methods in a well-defined way. We establish how the unambiguous result is obtained.

I. INTRODUCTION

The method of continuous dimensions was invented first to regularize divergent Feynman integrals.[1] Later it was applied also to the Green's functions.[2,3] More recently the application was extended further to the calculations involving spectral representations.[4-7]

Of special interest is the application to the dual string model; a possible remedy was proposed to remove the difficulty of the critical dimensionality (26 or 10) of space-time.[6,8] The essential ingredient of the method is completely analogous to that in the calculation of the Gotō-Imamura-Schwinger (GIS) term in spinor quantum electrodynamics. For this reason the GIS term was examined carefully from the new point of view.[4,5]

It was then concluded that the GIS term vanishes in the regularized calculation, contrary to the long-held belief that it never vanishes because of the positivity of the spectral function. We analyzed the method of dimensional regularization on the basis of the concept of hyperfunctions[9] (or distributions[10]) showing that negative contributions are introduced automatically at infinite energy, the same feature shared by the Pauli-Villars regularization method.[11]

As has been argued frequently, however, the result depends crucially on how one defines the GIS term.[12] Even if one confines oneself, for the moment, to the formulation in terms of the spectral function, there are two different methods: Going to the equal-time limit *before* or *after* the spectral integration. For later convenience we call these "method B" (for *before*) and "method A" (for *after*), respectively. It is one of the advantages of the method of continuous dimensions that one can compare the two methods in a well-defined way. Our conclusion may be summarized as follows.[4,5]

In method B the vanishing GIS term follows for any dimensionality of space-time $N \neq 2, 4, 6, \ldots$. In method A, on the other hand, the equal-time limit of the current commutator does not exist (even with $N \neq$ even integer) unless N is chosen sufficiently small ($N < 2$) so that the integral in method B is convergent in the usual sense. For such a small N the results of the two calculations agree with each other. Method B may be regarded as the one providing us with a smooth extrapolation of the common result to the other values of N.

In the above analysis it is crucial that we have required the existence of the equal-time limit. Obviously the GIS term is not the only quantity in which this requirement is essential. Among other examples, let us consider the simplest: The canonical equal-time commutator of the interacting field. It is this commutator that enables us to express the wave-function renormalization constant in terms of the spectral function of the intermediate state which the field creates. The original derivation by Lehmann[13] corresponds to method B, though without any explicit use of the regularization procedure. One may ask oneself the following question: If one argues against (at least the naive use of) method B in the GIS term, why does one refrain from doing the same in the canonical equal-time commutator of the field?

In this paper we re-examine this important and familiar commutator with the same level of rigor as in the GIS term. In the light of the method of continuous dimensions, we study the simplified example of a real scalar field which creates a fermion-antifermion pair. The conclusion turns out to be essentially the same; in short, method B is justified.

According to Wilson's argument on short-distance behaviors,[14] the equal-time limits of the commutators of the fields (or their products) do not exist in general. They are usually infested with the behavior ~ln$t$. There are, however, some exceptions, notably the equal-time commutator of the conserved currents (apart from the GIS term in the conventional calculations). Our result in this paper adds another example of the well-defined equal-time commutators in the regularized calculation.

In Sec. II the necessary formulas are given. We consider the lowest-order contribution to the commutator of the scalar field in continuous dimensions. Section III contains the calculation in method B. Method A is then developed in Sec. IV. The ambiguities in the equal-time limit are discussed in detail. It is shown that the equal-time limit exists only if $N<4$, and that the result then agrees with that of the simpler method B. Section V is devoted to concluding remarks, including a brief summary of the calculation in the dual string model.

II. PRELIMINARIES

The vacuum expectation value of the commutator of the (renormalized) real scalar field is defined by

$$i\Delta'(x)=\langle 0|[\phi(x),\phi(0)]|0\rangle\,, \tag{2.1}$$

which allows the spectral representation[13]

$$\Delta'(x)=\int\bar{\rho}(s)\Delta(x;s)ds\,, \tag{2.2}$$

where $\Delta(x;s)$ is normalized as

$$\dot{\Delta}(x;s)|_{t=0}=-\delta(\vec{x})\,. \tag{2.3}$$

It was shown that relation (2.3) remains true in continuous dimensions[4,6]; the right-hand side is the spatial δ function in $n=N-1$ dimensions.[2,4,6]

Consider that $\phi(x)$ couples to the spinor field with the Yukawa coupling constant g. In N dimensions the spectral function is calculated to the lowest order,

$$\bar{\rho}(s)=\delta(s-M^2)+\rho(s)\,,$$
$$\rho(s)=C\theta(s-4m^2)(s-M^2)^{-2}s^{-1/2}(s-4m^2)^{\nu-1/2}\,, \tag{2.4}$$

where $\nu=N/2$, M and m being the masses of the scalar field and the spinor field, respectively. The constant C is given by

$$C=g^2 2^{2-3\nu}\pi^{1/2-\nu}/\Gamma(\nu-\tfrac{1}{2})\,, \tag{2.5}$$

which tends to $g^2/8\pi^2$ as $N\to 4$.

From (2.1)

$$\dot{\Delta}'(x)|_{t=0}=-i\langle 0|[\pi(x),\phi(0)]|0\rangle_{t=0}=-Z^{-1}\delta(\vec{x})\,, \tag{2.6}$$

where Z is the wave-function renormalization constant of the scalar field. It is then expected that

$$\frac{\partial}{\partial t}\int\rho(s)\Delta(x;s)ds\bigg|_{t=0}=(1-Z^{-1})\delta(\vec{x})\,. \tag{2.7}$$

As mentioned in Sec. I, there are two ways to compute the left-hand side. In method B (A) the equal-time limit $t\to 0$ is taken before (after) the s integration is carried out. Different results follow in general from these two procedures even in the regularized calculation.

III. METHOD B: $t\to 0$ BEFORE THE s INTEGRATION

We use the relation (2.3) inside the s integration in (2.7). We then obtain

$$Z^{-1}-1=\int\rho(s)ds\,. \tag{3.1}$$

In what follows we choose $M=0$ without loss of generality in the lowest-order calculation. Substituting from (2.4) we have

$$\begin{aligned}Z^{-1}-1&=C\int_{4m^2}^{\infty}s^{-5/2}(s-4m^2)^{\nu-1/2}ds\\&=C(4m^2)^{\nu-2}B(2-\nu,\nu+\tfrac{1}{2})\\&=\tfrac{1}{3}(2\pi)^{-\nu}g^2(\nu-\tfrac{1}{2})m^{N-4}\Gamma(2-\nu)\,.\end{aligned} \tag{3.2}$$

The calculation is well defined for any $\nu\neq 2,3,4,\ldots$. For $N\simeq 4$ we have

$$Z^{-1}-1=\frac{g^2}{8\pi^2}\lim_{\nu\to 2}\frac{1}{2-\nu}\,. \tag{3.3}$$

It is pointed out that this agrees exactly with the usual calculation of Z in the Feynman-Dyson method.

IV. METHOD A: $t\to 0$ AFTER THE s INTEGRATION

We use the representation[3,4,6,15]

$$\Delta(x;s)=-2^{-\nu}\pi^{1-\nu}\epsilon(t)\theta(-x^2)(-x^2)^{1/2-\nu/2}s^{\nu/2-1/2}\times J_{1-\nu}(s^{1/2}(-x^2)^{1/2})\,. \tag{4.1}$$

We calculate

$$\begin{aligned}\tilde{\Delta}'(x)&\equiv\int_{4m^2}^{\infty}\rho(s)\Delta(x;s)\\&=-C\pi(2\pi)^{-\nu}\epsilon(t)\theta(-x^2)(-x^2)^{1/2-\nu/2}\\&\quad\times\int_{4m^2}^{\infty}s^{\nu/2-3}(s-4m^2)^{\nu-1/2}J_{1-\nu}(s^{1/2}(-x^2)^{1/2})ds\end{aligned} \tag{4.2}$$

for any $t=x^0$. The s integration in (4.2) can be performed analytically to give[16]

$$(4m^2)^{\nu+1/2}2^{\nu-6}\Gamma(\nu+\tfrac{1}{2})(-x^2)^{3-\nu/2}\times G_{13}^{20}\left(-m^2x^2\,\middle|\,\begin{matrix}0\\-\nu-\frac{1}{2},-\frac{5}{2},\nu-\frac{1}{2}\end{matrix}\right), \tag{4.3}$$

where G_{13}^{20} is Meijer's G function.

Expressing this function in terms of the hypergeometric series[18] we put (4.2) into the form

$$\tilde{\Delta}'(x) = -2^{-3-\nu}(\nu-\tfrac{1}{2})\pi^{3/2-N}g^2\epsilon(t)\theta(-x^2)$$
$$\times\left\{\frac{\Gamma(\nu-2)}{\Gamma(4-N)\Gamma(\nu+\frac{1}{2})}(-x^2)^{3-N}{}_1F_2[\tfrac{1}{2}-\nu;3-\nu,4-N;m^2x^2]+\frac{4}{3\sqrt{\pi}}m^{N-4}(-x^2)^{1-\nu}{}_1F_2[-\tfrac{3}{2};\nu-1,2-\nu;m^2x^2]\right\}. \tag{4.4}$$

The hypergeometric series ${}_1F_2$ is expanded into a power series,[19]

$${}_1F_2[a;b,c;z]=\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)}{b(b+1)\cdots(b+n-1)c(c+1)\cdots(c+n-1)}\frac{z^n}{n!}. \tag{4.5}$$

The series will be finite unless some of the coefficients are infinite for certain values of ν. A close inspection of (4.4) shows that $\tilde{\Delta}'(x)$ is finite for $|-x^2|\neq 0$ unless $\nu=2,3,4,\ldots$. This situation may be compared with the well-known result that any Feynman integral is finite unless ν is a certain integer. This finiteness of $\tilde{\Delta}'(x)$, however, no longer holds true in general in the limit $t\to 0$.

To see this we substitute (4.5) into (4.4) and further use the expansion[2,6]

$$(-x^2)_+^{\alpha}=\tfrac{1}{2}\omega_n\sum_{k=0}^{\infty}\frac{1}{a_k(n)}B(\tfrac{1}{2}n+k,\alpha+1)\times(\nabla^2)^k\delta(\vec{x})|t|^{n+2\alpha+2k}$$
$$=\tfrac{1}{2}\omega_nB(\tfrac{1}{2}n,\alpha+1)\delta(\vec{x})|t|^{n+2\alpha}+\ldots$$
$$[a_k(n)=2^kk!n(n+2)\cdots(n+2k-2)], \tag{4.6}$$

where $\omega_n=2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $n=N-1$ dimensions. In (4.6) we have interpreted $\theta(-x^2)(-x^2)^{\alpha}$ as a hyperfunction or a distribution $(-x^2)_+^{\alpha}$.[20] It turns out that it is sufficient to keep only the first term ${}_1F_2=1$ in the expansion (4.5); all the other terms result in higher powers in $|t|$. The result is

$$\tilde{\Delta}'(x)=-2^{-3-\nu}(\nu-\tfrac{1}{2})\pi^{1-\nu}g^2\times\left[\frac{\Gamma(\nu-2)}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{7}{2}-\nu)}\epsilon(t)|t|^{5-N}+\frac{8}{3\pi}m^{N-4}\Gamma(2-\nu)t\right]\delta(\vec{x})+\cdots. \tag{4.7}$$

The ignored terms are of order t^2 or higher as $N\to 4$, multiplied by derivatives of $\delta(\vec{x})$.

We are now going to dfferentiate (4.7) with respect to t. In the first term in the square brackets of (4.7) we interpret $\epsilon(t)|t|^{\beta}$ as hyperfunction $t_+{}^{\beta}-(-t)_+{}^{\beta}$, and use the formula[20]

$$\frac{d}{dt}t_+{}^{\beta}=\beta t_+{}^{\beta-1}, \tag{4.8}$$

which yields

$$\frac{d}{dt}[\epsilon(t)|t|^{5-N})=(5-N)|t|^{4-N}. \tag{4.9}$$

In this way we arrive at

$$\dot{\tilde{\Delta}}'(x)=-2^{-3-\nu}(\nu-\tfrac{1}{2})\pi^{1-\nu}g^2\times\left[\frac{\Gamma(\nu-2)(5-N)}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{7}{2}-\nu)}|t|^{4-N}+\frac{8}{3\pi}m^{N-4}\Gamma(2-\nu)\right]\delta(\vec{x})+\cdots. \tag{4.10}$$

Now we discuss the limit $t\to 0$. The second term in the square brackets in (4.10) allows a smooth limit $t\to 0$; it is in fact a constant. We confirm that this term is exactly the same as (3.3) obtained in method B. In the first term in the square brackets of (4.10), on the other hand, $|t|^{4-N}$ is ambiguous for $N\simeq 4$.

One may go to the limit $t\to 0$ keeping N held fixed off the value 4. We find

$$\lim_{t\to 0}|t|^{4-N}=\begin{cases}0, & \text{if } N<4,\\ \infty, & \text{if } N>4.\end{cases} \tag{4.11}$$

The equal-time limit of $\dot{\tilde{\Delta}}(x)$ exists only if one goes to the limit by keeping $N<4$. The result then agrees with that of method B. It is noticed that $N<4$ is the condition which renders the integral in (3.1) convergent in the usual sense.

It is accidental, in a sense, that this critical value of N agrees with the physical dimensionality. One may imagine that our space-time is six-dimensional, for example. In this case, according to (4.10), the equal-time limit does not exist for N which is close to the "physical value" 6; one has to go down to the lower value of N (< 4) and obtains the result of method B followed by the analytic continuation in N.

Our procedure is parallel to the usual calculation of the Feynman integrals in continuous dimensions based on the analytic continuation; in quantum electrodynamics, for example, the non-gauge term in the photon self-energy part is calculated first for $N<2$, followed by the extrapolation of the vanishing result to larger values of N.

We have so far considered the limiting

$$\lim_{N\to 4}\lim_{t\to 0}\,. \tag{4.12}$$

One may also try another limiting procedure

$$\lim_{t\to 0}\lim_{N\to 4}\,. \tag{4.13}$$

On going to the limit $N\to 4$ with t kept finite, we have

$$\begin{aligned}|t|^{4-N} &= \exp[(2-\nu)\ln t^2] \\ &= 1+(2-\nu)\ln t^2 + O(\nu-2)\,,\end{aligned} \tag{4.14}$$

and hence

$$\Gamma(\nu-2)\,|t|^{4-N} = \frac{1}{\nu-2} - \gamma - \ln t^2 + O(\nu-2)\,. \tag{4.15}$$

In the same way we obtain

$$\Gamma(2-\nu)m^{N-4} = -\frac{1}{\nu-2} - \gamma - \ln m^2 + O(\nu-2)\,. \tag{4.16}$$

Substituting (4.15) and (4.16) into (4.10) we find that the pole terms $(\nu-2)^{-1}$ cancel each other. We must go to the next approximation in $\nu-2$. We finally obtain

$$\lim_{N\to 4}\overset{2}{\Delta}'(x) = \frac{g^2}{8\pi^2}\,(2\gamma+\tfrac{8}{3}+\ln m^2t^2)\delta(\vec{x}) + O(t), \tag{4.17}$$

which implies that the equal-time limit does not exist, from whichever direction one goes to the limit $N\to 4$. This is in accordance with the well-known result (Ref. 14) that the limit does not exist in the unregularized calculation with the fixed dimensionality $N=4$.

It may be useful, however, to notice that by replacing t with Λ^{-1}, (4.17) reproduces exactly the usual expression of Z calculated in the Feynman-Dyson method with the cutoff Λ. This exemplifies how the result of the cutoff method which differs from the dimensional method in principle is obtained by choosing a special limiting procedure.

V. CONCLUDING REMARKS

From the study of our simplified example, we may infer this general conclusion: In method A the equal-time limit of the canonical commutator does not exist in general. The limit exists, as required almost imperatively in the standard field theory, only if N is chosen sufficiently small so that the naive power counting indicates a convergence in the usual sense. This result then agrees with that of method B, which is by itself a well-defined method to give an equal-time limit; the wave-function renormalization constant calculated in terms of the spectral function agrees also with that obtained in the Feynman-Dyson method in N dimensions.

This conclusion is essentially the same as what has been learned in the analysis of the GIS term. We reiterate here the most important points in discussing the GIS term.

In the naive application of method A the GIS term appears to blow up for $N\to 4$ like $\lim_{t\to 0}|t|^{2-N}$.[21] On this basis it has been often argued that the GIS term is divergent. The result $|t|^{2-N}$, however, simply shows that the equal-time limit does not exist in this method. If the GIS term is *defined* by the equal-time commutator, one must have the calculation according to which the equal-time limit does exist. In our method A this is achieved by first going to the region $N<2$ and continuing analytically back to the point $N=4$. The procedure results in the vanishing GIS term in accordance with method B.

One may have a suspicion that any quantity could turn out to be trivially zero in this type of calculation. That this is not true has been demonstrated by the example of the canonical equal-time commutator of the field.

One can also define the GIS term in the different ways. Both the point-separation technique and the Bjorken-Johnson-Low (BJL) limit yield infinity, as long as N is fixed to 4. In the method of continuous dimensions it is likely that the result is rather close to that of method A; the point-separation technique will give $\lim_{\delta\to 0}|\delta|^{2-N}$, while the BJL limit has been shown to give $\lim_{E\to\infty}E^{N-2}$.[5] The calculations are made to be well defined if $N<2$, resulting again in the vanishing GIS term.[22]

Our conclusion of the vanishing GIS term does not contradict the statement that the integrand approaches a constant as $s\to\infty$; in other words, the e^+e^- cross section behaves as $1/s$ in the high-energy limit. The negative contribution is present only at $s=\infty$, the *mathematical* infinity. For this reason the vanishing GIS term has no immediate physical consequences.

The same type of calculation, on the other hand, does have a physical relevance in the dual string model. The Virasoro algebra is known to have the anomaly. By viewing the string theory as the field theory in the internal two-dimensional space-time (mapped from the world sheet), it was shown that the above anomaly is the vacuum expectation value of the equal-time commutator of the internal energy-momentum tensor.[23] The evaluation in the fixed dimensionality $N=2$ of this internal space-time results in the usual anomaly. If, however, N is considered to be continuous, the anomaly disappears exactly in the same context as in the GIS term.[6,8]

The proof of the Lorentz invariance of the string theory is more involved. In the usual formulation in the lightlike gauge[24] the Lorentz invariance follows as a result of a cancellation among the terms each of which has the same structure as the Virasoro algebra anomaly. The cancellation occurs only for $D=26$ (or 10), the critical dimensionality of the real space-time. Our detailed analysis[6] shows that the relevant terms vanish *separately* for general N. The conclusion of $D=26$ is hence no longer valid.[25] The analysis in the present paper will be helpful in recognizing an intriguing link between the problem in the string theory and the fundamental structure in the relativistic quantum field theory.

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