T-invariant static Yang-Mills equations

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Time-reversal invariance is considered for the Yang-Mills external-source problem. Constraints on the electric field and sources of invariant static solutions are found to result in simplified field equations resembling gauge models with smaller groups. A class of non-Coulomb solutions of the SU(2) equations is obtained numerically for a line charge source.

I. INTRODUCTION

The purpose of this paper is to examine the simplification of the static Yang-Mills equations with external charged sources which result from requiring the solutions to be time-reversal invariant.

For the analogous problem in the Maxwell theory T invariance is essentially automatic, since no currents are present. This property does not extend in general to nonlinear gauge theories in which the fields themselves can carry current. In an Abelian model such as scalar electrodynamics with external charges, where the current is gauge invariant and has the normal time-reversal behavior, it must still vanish for an invariant solution. However, in the Yang-Mills theory the nonlinear terms of the field tensor require a modification of the usual time-reversal transformation in order for it to be a symmetry of the equations of motion. A consequence is that the invariance condition for a static solution places constraints on the electric field and sources, while nonvanishing field currents are not excluded.

Given this situation one does not know whether all static solutions of interest are T invariant and imposing such a requirement is therefore an ansatz. The motivation for considering it arises of course from the intractability of the full set of Yang-Mills equations. With an appropriate choice of gauge the number of nonvanishing field components is substantially reduced and the resulting system resembles a gauge model for a lower-dimensional group, with the components of the scalar potential looking like scalar fields.

The SU(2) T-invariant equations have been solved numerically for the somewhat unphysical case of an infinite-line charge. This example was chosen for computability in that for a class of solutions invariant under translation along the source line the problem becomes one-dimensional. The results illustrate for a weakly singular source the color screening and energy lowering discussed by Mandula[1] and others.[2-4] Critical charge thresholds appear to be absent. In addition it was found that a completely sourceless solution representing an infinite-length flux tube is not present in the model.

The T-invariant equations for the SU(2) group are obtained in Sec. II. Their application to the line-charge problem is discussed in Sec. III. Section IV considers the extension of the ansatz to SU(3).

II. STATIC SU(2) SYSTEM

The equations for the classical SU(2) Yang-Mills field interacting with an external current $j_a^\mu(x)$ are[5,6]

$$\partial_\mu F_a^{\mu\nu}(x) - g\epsilon_{abc}A_{b\mu}(x)F_c^{\mu\nu}(x) = g j_a^\nu(x)\,, \tag{1}$$

$$F_a^{\mu\nu}(x) = \partial^\mu A_a^\nu(x) - \partial^\nu A_a^\mu(x) - g\epsilon_{abc}A_b^\mu(x)A_c^\nu(x)\,. \tag{2}$$

The local gauge transformations will be written in the form[7]

$$T_a j_a^{\mu\prime}(x) = R(x)T_a j_a^\mu(x)R(x)^{-1}\,, \tag{3a}$$

$$T_a A_a^{\mu\prime}(x) = R(x)T_a A_a^\mu(x)R(x)^{-1} + ig^{-1}[\partial^\mu R(x)]R(x)^{-1}\,, \tag{3b}$$

where the matrices $(T_a)_{bc} = -i\epsilon_{abc}$ are generators of the adjoint representation, to which $R(x)$ belongs

In the following $j_a^\mu(x)$ represents an external charge distribution, spatially fixed in the chosen reference frame:

$$j_a^\mu(x) = \delta_0^\mu\,\rho_a(x)\,. \tag{4}$$

The charges are still free to rotate internally according to the equation of motion[6]

$$\partial_0\rho_a(x) = g\,\epsilon_{abc}A_b^0(x)\,\rho_c(x)\,, \tag{5}$$

which follows from (1). The magnitude $\rho_a(x)\rho_a(x)$ is a constant of the motion.

The system defined by (1) and (4) has various symmetries, characterized by transformations which take any solution into another one with the same positions and magnitudes for the source charges. A solution can be defined as invariant under such a transformation if it changes at most by a gauge transformation, i.e., there is no physical change. At least for the usual symmetries this is a gauge-invariant definition.

Consider, for example, static or time-translation invariant solutions. With the translation defined by

$$A_a^\mu(\vec{x},t)\to A_a^{\mu\prime}(\vec{x},t)=A_a^\mu(\vec{x},t+\epsilon)\,, \tag{6a}$$

$$\rho_a(\vec{x},t)\to\rho_a'(\vec{x},t)=\rho_a(\vec{x},t+\epsilon)\,, \tag{6b}$$

then $A_a^{\mu\prime}$, ρ_a' will differ from A_a^μ, ρ_a at most by a gauge transformation:

$$T_aA_a^\mu(\vec{x},t+\epsilon)=R(x,\epsilon)T_aA_a^\mu(\vec{x},t)R(x,\epsilon)^{-1}+ig^{-1}[\partial^\mu R(x,\epsilon)]R(x,\epsilon)^{-1}\,, \tag{7a}$$

$$T_a\rho_a(\vec{x},t+\epsilon)=R(x,\epsilon)T_a\rho_a(\vec{x},t)R(x,\epsilon)^{-1}\,. \tag{7b}$$

As would be expected, these conditions imply the existence of another, not unique transformation which removes the t dependence from the fields and sources. For infinitesimal ϵ, $R(x,\epsilon)\simeq 1-i\epsilon T_aW_a(x)$ and to first order in ϵ (7a) reduces to

$$g\,\partial_0A_a^\mu(x)=\partial^\mu W_a(x)-g\epsilon_{acb}A_c^\mu(x)W_b(x)\,. \tag{8}$$

Defining $S(\vec{x},t)$ as the solution of

$$\begin{aligned}&\partial_0S(\vec{x},t)=iS(\vec{x},t)W_a(\vec{x},t)T_a\,,\\ &S(\vec{x},0)=1\,,\end{aligned} \tag{9}$$

and applying the corresponding gauge transformation $A_a^\mu(x)\to\hat{A}_a^\mu(x)$ it is readily verified using (8) that $\partial_0^a\hat{A}_a^\mu(x)=0$. Similarly $\hat{\rho}_a$ is t independent. The remaining gauge freedom, consistent with keeping the fields time independent, can be used to satisfy any of a class of gauge conditions.

A similar procedure will be followed to deduce the consequences of assuming T invariance. The invariance requirement is formulated in a gauge-invariant way. A gauge transformation is then applied to make the symmetry manifest in the form of the fields. This is rather like making a partial choice of gauge, except that it is not applicable to arbitrary fields. It offers some flexibility over the alternative of working within a fixed gauge from the outset.

Owing to the nonlinear terms in the field tensor (2), the analog of the usual Abelian time-reversal transformation

$$A_a^0(\vec{x},t)\to A_a^{0\prime}(\vec{x},t)=A_a^0(\vec{x},-t)\,, \tag{10a}$$

$$A_a^i(\vec{x},t)\to A_a^{i\prime}(\vec{x},t)=-A_a^i(\vec{x},-t) \tag{10b}$$

is not a symmetry of the equations of motion. The extra sign change required for the nonlinear terms to transform as the gradients can be accomplished with a transformation of the form

$$A_a^0(\vec{x},t)\to A_a^{0\prime}(\vec{x},t)=M_{ab}A_b^0(\vec{x},-t)\,, \tag{11a}$$

$$A_a^i(\vec{x},t)\to A_a^{i\prime}(\vec{x},t)=-M_{ab}A_b^i(\vec{x},-t)\,, \tag{11b}$$

$$\rho_a(\vec{x},t)\to\rho_a'(\vec{x},t)=M_{ab}\,\rho_b(\vec{x},t)\,, \tag{11c}$$

where M_{ab} is a constant matrix. Assuming it to be orthogonal for the sake of a canonical transformation one has

$$\epsilon_{abc}(M_{bd}A_d^\mu)(M_{cf}A_f^\nu)=(\mathrm{Det}\,M)M_{ab}(\epsilon_{bcd}A_c^\mu A_d^\nu)\,, \tag{12}$$

so that $\mathrm{Det}(M)=-1$ is needed. Equations (1) and (4) are then invariant under (11). The various possible M's are related by global rotation factors and are presumably equally valid. It is convenient to choose $M_{ab}=-\delta_{ab}$, in which case Eq. (11c) presents an interesting analogy with the time-reversal transformation for mechanical spin.

A static solution will be invariant under (11) if there is a gauge transformation $R(x)$ such that

$$A_a^{0\prime}(\vec{x})=-A_a^0(\vec{x})=R_{ab}(\vec{x})A_b^0(\vec{x})\,, \tag{13a}$$

$$\begin{aligned}T_aA_a^{i\prime}(\vec{x})&=T_aA_a^i(\vec{x})\\ &=R(\vec{x})T_aA_a^i(\vec{x})R(\vec{x})^{-1}+ig^{-1}[\partial^iR(\vec{x})]R(\vec{x})^{-1}\,,\end{aligned} \tag{13b}$$

$$\rho_a'(\vec{x})=-\rho_a(\vec{x})=R_{ab}\rho_b(\vec{x})\,. \tag{13c}$$

In writing (13a) and (13c) the relation $RT_aR^{-1}=T_bR_{ba}$ has been used. With sources present, $A_a^0(\vec{x})$ will be nonvanishing almost everywhere so that (13a) implies that $R(\vec{x})$ is a local rotation through π. Let $S(\vec{x})$ be a rotation which takes the local rotation axis of R into a fixed direction e_a:

$$S(\vec{x})R(\vec{x})S^{-1}(\vec{x})=R_e(\pi)\,, \tag{14}$$

where $R_e(\pi)$ is a constant rotation through π about e_a. Gauge transforming by $S(\vec{x})$ and combining with (13) gives for the new fields

$$-\hat{A}_a^0(\vec{x})=R_e(\pi)_{ab}\hat{A}_b^0(\vec{x})\,, \tag{15a}$$

$$\hat{A}_a^i(\vec{x})=R_e(\pi)_{ab}\hat{A}_b^i(\vec{x})\,, \tag{15b}$$

$$-\hat{\rho}_a(\vec{x})=R_e(\pi)_{ab}\hat{\rho}_b(\vec{x})\,, \tag{15c}$$

which imply that $\hat{A}_a^i$ is collinear with e_a for $i=1$, 2, 3 while $\hat{A}_a^0$, $\hat{\rho}_a$ are orthogonal to it. If the axis is chosen along the 3-direction one has the restrictions

$$\begin{aligned}&\hat{A}_1^i=\hat{A}_2^i=0,\quad i=1,\ 2,\ 3\\ &\hat{A}_3^0=\hat{\rho}_3=0\,.\end{aligned} \tag{16}$$

For $\hat{A}_a^\mu$ the effect of time reversal has been reduced to a global rotation as in (15), not to the identity. There may be some question whether the different global orientations of a classical system should be regarded as physically indistinguishable. If they are not then it would be more correct to say that (15) represents a minimal change rather than an invariance and there would be no strictly T-invariant static solutions with sources.

Using (16) and the notation $A_1^0=\phi_1$, $A_2^0=\phi_2$, $A_3^i=(\vec{A})^i$ the static versions of (1) and (5) reduce to

$$-\nabla^2\phi_1+g(\vec{\nabla}\cdot\vec{A})\phi_2+2g(\vec{A}\cdot\vec{\nabla})\phi_2+g^2\vec{A}^2\phi_1=g\rho_1\,, \tag{17a}$$

$$-\nabla^2\phi_2-g(\vec{\nabla}\cdot\vec{A})\phi_1-2g(\vec{A}\cdot\vec{\nabla})\phi_1+g^2\vec{A}^2\phi_2=g\rho_2\,, \tag{17b}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=g\vec{J}_F=g(\phi_1\vec{\nabla}\phi_2-\phi_2\vec{\nabla}\phi_1)+g^2({\phi_1}^2+{\phi_2}^2)\vec{A}\,, \tag{17c}$$

$$\phi_1(\vec{x})\rho_2(\vec{x})-\phi_2(\vec{x})\rho_1(\vec{x})=0\,, \tag{17d}$$

where $\vec{J}_F$ is the field current. Except for the sign of the last term in (17c) these equations are formally identical to massless scalar electrodynamics with $A_0=0$, a scalar field $\phi_1+i\phi_2$, and a scalar source $\rho_1+i\rho_2$. The gauge freedom consistent with (16) is the Abelian subgroup of rotations around the 3 axis, which preserve $\vec{B}$ and $\vec{J}_F$, and their products with $R_1(\pi)$, $R_2(\pi)$.

Examples of T-invariant solutions with smooth, localized source distributions are provided by the work of Sikivie and Weiss,[3] whose magnetic-dipole ansatz gives equations which are essentially a special case of (17). An example of a singular source will be discussed in the following section.

III. LINE-CHARGE SOLUTIONS

The line charge is taken along the z axis,

$$\rho_a(\vec{x})=\mu\xi_a(z)\delta(\rho)/\rho,\quad a=1,2 \tag{18}$$

where $2\pi\mu g$ is the charge magnitude per unit length and $\xi_a(z)$ is a unit vector specifying the local orientation.

In addition to T invariance several further symmetry assumptions were made. For rotational invariance the field components ϕ_a, A_ρ, A_θ, A_z are independent of angle coordinate. Plane reflections preserving the z axis are symmetries and for invariant solutions there are two minimal possibilities, depending on whether or not the induced gauge transformation contains a factor such as $R_1(\pi)$. If it does, the fields can be made to satisfy $A_\rho=A_z=\phi_2=\rho_2=0$; if not, then $A_\theta=0$ can be assumed. Models of the former type have been considered elsewhere.[3,8] In this calculation $A_\theta=0$ and the magnetic field will be circumferential around the source axis.

The remaining local gauge freedom can be used to satisfy $A_\rho=0$, $A_z(\rho=0)=0$. This choice is convenient for solutions invariant under translations along the source, which is the final ansatz. The resulting form for the fields is

$$\phi_a(\vec{x})=R(\rho)\xi_a(z),\quad A_z=A_z(\rho)\,,\qquad \xi_a(z)=(-\sin Kz,\ \cos Kz)\,, \tag{19}$$

where $K\geq 0$ and all other solutions satisfying the symmetry assumptions are related to (19) by global rotations. The source and scalar potential have a corkscrew configuration, and it is evident that a z translation produces just a rotation around the internal 3 axis.

With the definition $S=K+gA_z$ the field equations reduce to

$$-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{dR}{d\rho}+S^2R=g\mu\delta(\rho)/\rho\,, \tag{20a}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{dS}{d\rho}+g^2R^2S=0\,. \tag{20b}$$

For $K=0$ one solution is the Coulomb potential $R=-\mu g\ln\mu\rho$, $S=0$. For $K\neq 0$ there exist solutions in which the magnetic field cylindrically confines the electric field, the factor of S^2 in (20a) acting like a nonconstant mass term. At small ρ the behavior is

$$R=-\mu g\ln\mu\rho+r+O(\rho^2\ln\mu\rho),\quad \rho\to 0\,, \tag{21a}$$

$$S=K+O(\rho^2\ln^2\mu\rho),\quad \rho\to 0\,, \tag{21b}$$

while R tends rapidly to zero for large ρ and S becomes logarithmic:

$$R=0,\quad \rho\to\infty\,, \tag{22a}$$

$$S=c\ln\mu\rho+d,\quad \rho\to\infty\,. \tag{22b}$$

The consistency of (22) with (20) can be seen by applying the Wentzel-Kramers-Brillouin (WKB) approximation.

The boundary problem is to find r in (21a) such that (22a) is satisfied, and it appears that this is possible for any μ, g, K. However, there is actually only a one-parameter family of distinct solutions, from which the others can be obtained by scaling. With the definitions

$$a=g\exp(r/\mu g),\quad x=\mu g\rho/a\,,\qquad \tilde{S}=aS/\mu g\,,\quad \tilde{R}=aR/\mu\,, \tag{23}$$

the field equations become

$$-\frac{1}{x}\frac{d}{dx}x\frac{d\tilde{R}}{dx}+\tilde{S}^2\tilde{R}=ag\,\delta(x)/x\,, \tag{24a}$$

$$\frac{1}{x}\frac{d}{dx}x\frac{d\tilde{S}}{dx}+\tilde{R}^2\tilde{S}=0\,, \tag{24b}$$

with the boundary conditions

$$\tilde{R}=-ag\ln x+O(x^2\ln x)\,,\quad x\to 0\,, \tag{25a}$$

$$\tilde{S}=aK/\mu g+O(x^2\ln^2 x)\,,\quad x\to 0\,, \tag{25b}$$

$$\tilde{R}=0,\quad x\to\infty\,. \tag{25c}$$

The rescaled system involves only the two parameters $\tilde{S}_0\equiv aK/\mu g$ and $G\equiv ag$, with $\tilde{S}_0$ given by some function $f(G)$. Once f is known the value of a corresponding to given μ, g, K is obtained by solving

$$\frac{aK}{\mu g}=f(ag)\,, \tag{26}$$

and the fields are given by

$$R(\rho)=\frac{\mu}{a}\tilde{R}_G(\mu g\rho/a)\,,$$
$$S(\rho)=\frac{\mu g}{a}\tilde{S}_G(\mu g\rho/a)\,, \tag{27}$$

$\tilde{R}_G$, $\tilde{S}_G$ being the solutions of (24) for $G=ag$.

Figure 1 shows the function f, which apparently tends to a nonzero limit at $G=0$, indicating that there is no critical value of μg below which (26) cannot be satisfied. The absence of a critical charge would perhaps be expected for the weaker line singularity compared to the point-charge case.[1] The structure in f is associated with a discrete set of solutions for which S tends to a constant for large ρ. The first few such solutions occur at

$$G_n \doteq 3.01,\ 5.40,\ 7.75,\ 10.09,\ \ldots\,. \tag{28}$$

Figures 2 and 3 show the fields R, S for $n=1$ and 2 with $\mu=g=1$. For given μ and g, $K=\mu g^2 f(G)/G$ and the sources unwind as G increases. Over its known range $f(G)$ grows less than linearly, suggesting that $G\to\infty$ is a Coulomb limit.

In all of these solutions the field current flows only in the z direction, with a density of gR^2S. Using (20b) the total current is

$$I=\int_0^\infty \rho\, d\rho\, gR^2S=-c/g\,, \tag{29}$$

which gives an asymptotic magnetic field $B_\theta=-g^{-1}dS/d\rho \simeq I/\rho$. $I=0$ for the discrete set (28).

The energy of the system per unit length is given by

$$E=\tfrac{1}{2}\int \rho\, d\rho\,(E_1^iE_1^i+E_2^iE_2^i+B_\theta^2)$$
$$=\tfrac{1}{2}\int_\epsilon^\lambda \rho\, d\rho\,[g^{-2}(dS/d\rho)^2+(dR/d\rho)^2+(RS)^2]\,. \tag{30}$$

The integral diverges in general at both limits so that cutoffs have been inserted. The "energy" of the Coulomb solutions is then

$$E_c=\tfrac{1}{2}(g\mu)^2\ln(1/\epsilon\mu)+\tfrac{1}{2}(g\mu)^2\ln\lambda\mu\,. \tag{31}$$

For the magnetic solutions it is useful to integrate by parts in (30) and use the boundary conditions to obtain

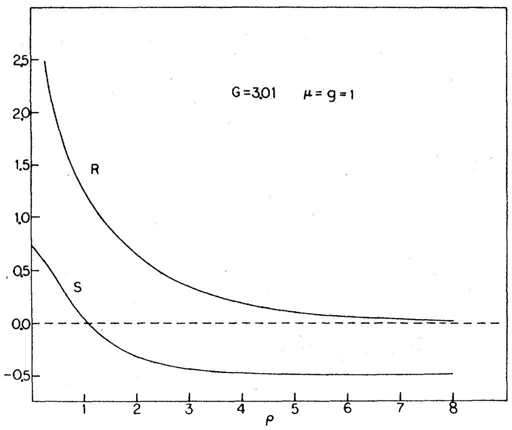

FIG. 2. The line-charge fields $R(\rho)$ for $\mu=g=1$ and $G=3.01$, the smallest value of G for which $S(\rho)$ tends to a constant at infinity.

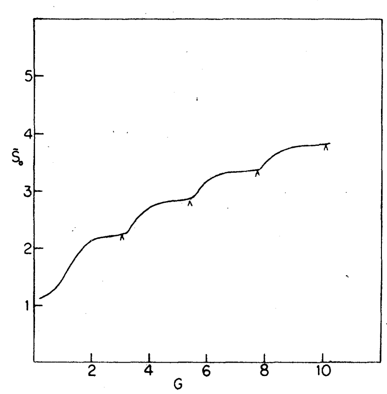

FIG. 1. The function $f(G)$ giving the values of S_0 determined by the boundary condition $R(\infty)=0$, for G between 0 and 10. At the indicated points $S(x)$ tends to a constant as $x\to\infty$.

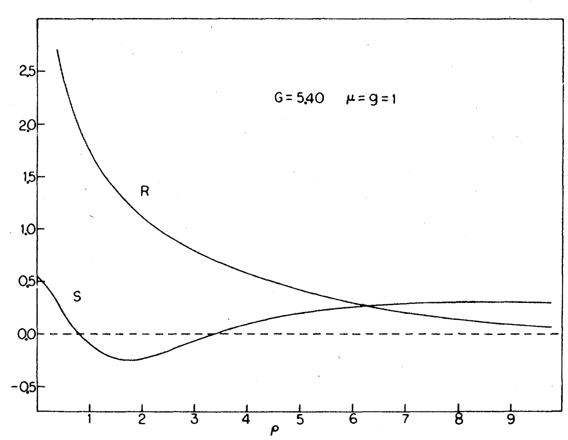

FIG. 3. The line-charge fields $R(\rho)$, $S(\rho)$ for $\mu=g=1$ and $G=5.40$, the second value of G for which $S(\rho)$ tends to a constant at infinity.

$$E = \tfrac{1}{2}(g\mu)^2 \ln(1/\epsilon\mu) + \tfrac{1}{2}\int_0^\infty \rho\, d\rho (RS)^2 + \tfrac{1}{2} g\mu r + \frac{c^2}{2g^2}\ln \mu\lambda . \quad (32)$$

The last term is the divergent part of the magnetic field energy. Over the range of G examined the coefficient of the ln is smaller than for the infrared term in (31) (for the same μ, g) so that if such comparisons have any meaning in this context one could say that the energy is lower than that of the Coulomb solutions. For the discrete set (28), however, $c=0$ and the electric field is cylindrically confined with only a finite increase in magnetic field energy per unit length. The finite contribution in (32) was evaluated and found to increase monotonically with n for n up to 4, the limit of the calculation.

Owing to the unlocalized nature of this system, the total charge (per unit length) is gauge dependent. In the present gauge it is proportional to $\xi_a(z)$ and nonvanishing because of the presence of a z-dependent z component of the electric field. Total charge is zero in, for example, the gauge $\phi_1 = 0$.

A search was made for a sourceless solution of (20) having R finite at $\rho = 0$. If it existed it could correspond to a flux tube connecting infinitely separated charges. Unfortunately, expansions similar (21) show that such an R must be an increasing function for small ρ and the S^2R term in (20a) forces it to continue growing. Confinement of the electric field as in the line-charge solutions is therefore not possible.

IV. STATIC SU(3) SYSTEM

For SU(3) the time-reversal transformation is again of the form (11). The covariance condition (12) becomes

$$f_{abc} M_{bd} M_{cf} = -M_{ab} f_{bdf} , \quad (33)$$

where f_{abc} are the structure constants. The general form of M has not been determined, but $M_{ab} = -\delta_{ab}$ is a solution as before ($M_{ab} = \delta_{ab}$ is not) and will be adopted in the following.

It is convenient to use the 3×3 matrix representation of the potentials $A^\mu = A_a^\mu \lambda_a/2$. The invariance requirement is then the existence of $U(x)$ belonging to SU(3) such that

$$-A^0 = UA^0U^{-1} , \quad (34a)$$

$$A^i = UA^iU^{-1} + ig^{-1}(\partial^i U)U^{-1} . \quad (34b)$$

(34a) and (34b) are form invariant under t-independent gauge transformations $V(\vec{x})$, with U going into VUV^{-1}. Hence U can be assumed diagonal:

$$U_{\alpha\beta} = \delta_{\alpha\beta} u_\alpha ; \quad |u_\alpha| = 1, \quad u_1 u_2 u_3 = 1 . \quad (35)$$

Substitution into (34) then results in the constraints

$$u_\alpha u_\beta^* A^0_{\alpha\beta} = -A^0_{\alpha\beta} , \quad (36a)$$

$$u_\alpha u_\beta^* A^i_{\alpha\beta} = A^i_{\alpha\beta} , \quad (36b)$$

$$\partial_i u_\alpha = 0 . \quad (36c)$$

Equations (36a) and (36b) imply that $A^0_{\alpha\alpha} = 0$ and that at least one off-diagonal element of A^0 vanishes. With a global rotation, if necessary, $A^0_{12} = 0$ can be assumed. Then from (36b) $A^i_{13} = A^i_{23} = 0$. The surviving components are $A^0_4 - A^0_7$, $A^i_1 - A^i_3$, and A^i_8.

The field equations are similar to a static SU(2) $\times$ U(1) model with a charged doublet ($A^0_4 - iA^0_5$, $A^0_6 - iA^0_7$), a triplet (A^i_1, A^i_2, A^i_3), and the Abelian field A^i_8.

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