Gauge zero modes, instanton determinants, and quantum-chromodynamic calculations

Claude Bernard
Department of Physics, University of California, Los Angeles, California 90024
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A treatment of the gauge zero modes about an instanton in a singular gauge places them on the same footing as all other zero modes and simplifies the calculation of the collective-coordinate part of the instanton determinant. This determinant is calculated first for the gauge group SU(3) and then for general SU(N). The answers differ from previously published results: For SU(3), the reason for this difference is trivial [the inclusion of certain factors of $1/\sqrt{2}$ whose absence from 't Hooft's original SU(2) calculation was recently discovered] but the effects on quantum-chromodynamic calculations may be important; for large N, the reasons are more involved, but the usual conclusion that instantons are absent in the planar limit is unaffected.

I. INTRODUCTION

A necessary ingredient in the computation of all instanton effects in a gauge theory is the value of the quadratic functional integral about the instanton. This calculation was first performed by 't Hooft[1] for an SU(2) gauge group.

One of the more subtle aspects of 't Hooft's work was his treatment of the gauge zero modes which are due to the arbitrary orientation of the instanton within the gauge group. Because of the complicated space-time dependence of the gauge modes, it was necessary to place the system in a box and to resort to an unconventional form of the Faddeev-Popov ansatz in order to compute the contribution of these modes. Here, I show that the calculation may be greatly simplified by working with the instanton field in a singular gauge, rather than the regular gauge of Ref. 1. In a singular gauge, the infinitesimal gauge transformations which generate the zero modes approach constants at large distance (essentially because of the rapid falloff of the gauge potential) and may be easily identified with changes in the potential under variations in the collective coordinates which describe the gauge orientation of the instanton. I show in Sec. II that this allows one to treat the gauge modes on exactly the same footing as the other zero modes (translations, dilatations). For definiteness, I work first in the context of SU(3) and compute the one-loop functional integral about a single SU(3) instanton.

In Sec. III, I write down the answer for a general SU(N) theory, which requires only minor generalizations from the SU(3) computation. For reasons I explain, my results differ from the recently published work of Bashilov and Pokrovsky.[2] However, the prediction of vanishing instanton effects in the large-N limit[3,4] (the planar theory) is unaffected.

Because I take into account the recently discovered[5] numerical error in the original SU(2) calculation, my result for SU(3) differs by a predictable numerical factor from the previously published answer.[6] In Sec. IV, I make some comments about the effects of this new number on some recent instanton calculations[7-9] in quantum chromodynamics (QCD). I argue that while it is still possible to find significant instanton effects, to do so one is forced to go to larger values of the coupling constant, where other nonperturbative effects may be more important.

II. GAUGE ZERO MODES; SU(3)

We wish to calculate $W^{(1)}$, the one-loop vacuum-vacuum amplitude about a single instanton divided, for normalization, by the same amplitude about the ordinary vacuum. If the potential is expanded about the classical value,

$$A_\mu = A_\mu^{\rm cl} + A_\mu^{\rm qu}, \tag{1}$$

then the quadratic action about an instanton is

$$S = S^{\rm cl} + \tfrac{1}{2} A^{\rm qu} M_A A^{\rm qu} + \phi^* M_{\rm gh}\phi + \cdots, \tag{2}$$

where $S^{\rm cl} = 8\pi^2/g^2$, and ϕ is the ghost field. Denoting the collective coordinates of the instanton by γ_i, $W^{(1)}$ is given by

$$W^{(1)} = \int \prod_i d\gamma_i J(\gamma) Q(\gamma) e^{-8\pi^2/g^2}, \tag{3a}$$

$$Q(\gamma) \equiv \frac{\det^{-1/2} M_A(\gamma) \det M_{\rm gh}(\gamma)}{(\det^{-1/2} M_A \det M_{\rm gh})_{A^{\rm cl}=0}}, \tag{3b}$$

where $J(\gamma)$ is the collective-coordinate Jacobian, and where the determinants in (3b) are, of course, taken over nonzero modes only.

The focus here will be on the evaluation of the collective-coordinate Jacobian, in particular the contribution to it from the gauge zero modes. However, before evaluating $J(\gamma)$ in the case of interest, it is useful to review the usual method[10] for replacing zero modes with collective coordinates in a nongauge theory. For simplicity, we consider a scalar quantum field B which has a classical value $B=B^{\mathrm{cl}}(\gamma)$, depending on a single collective coordinate γ. Let $M(\gamma)$ be the operator that appears in the expansion of the action to quadratic order about B^{cl}:

$$B=B^{\mathrm{cl}}+B^{\mathrm{qu}},\qquad S=S^{\mathrm{cl}}+\tfrac{1}{2}B^{\mathrm{qu}}MB^{\mathrm{qu}}+\cdots. \tag{4}$$

M has a complete set of orthogonal eigenfunctions χ_i with eigenvalues ϵ_i and norms $\sqrt{u_i}$:

$$u_i\equiv\langle\chi_i|\chi_i\rangle. \tag{5}$$

There is, of course, a zero mode

$$\chi_0=\frac{\partial B^{\mathrm{cl}}}{\partial\gamma},\quad \epsilon_0=0. \tag{6}$$

If we expand B^{qu} as

$$B^{\mathrm{qu}}=\sum_i \xi_i\chi_i, \tag{7}$$

then the measure for functional integration is

$$(dB)=(dB^{\mathrm{qu}})=\prod_i\left(\frac{u_i}{2\pi}\right)^{1/2}d\xi_i. \tag{8}$$

That this is the correct measure can be seen by performing the integration over all the nonzero modes. The Gaussian integrations give

$$\int(dB)e^{-S}=\int\left(\frac{u_0}{2\pi}\right)^{1/2}d\xi_0 e^{-S^{\mathrm{cl}}}\det{}^{-1/2}M+\cdots, \tag{9}$$

where $\cdots$ represents higher loops and the effects of other classical sectors. The fact that we get precisely $\det^{-1/2}M$ without some infinite multiplicative factor indicates that we have chosen the correct normalization.[5,2]

We may now insert a factor of unity which will require the quantum field to be orthogonal to the zero mode:

$$1=u_0\int d\gamma\,\delta(\langle B-B^{\mathrm{cl}}(\gamma)|\chi_0(\gamma)\rangle)+\cdots, \tag{10}$$

where $\cdots$ represents terms of higher order in the quantum field. Inserting this factor into (9) and performing the ξ_0 integration results in

$$\int(dB)e^{-S}=\int d\gamma\left(\frac{u_0}{2\pi}\right)^{1/2}e^{-S^{\mathrm{cl}}}\det{}^{-1/2}M+\cdots. \tag{11}$$

The case of the gauge theory is slightly more subtle. The difference lies in the requirement of fixing a gauge (taken here to be the usual background gauge with respect to the classical field). The derivative of the classical field with respect to a collective coordinate will not, in general, be in the background gauge. The ith zero mode is thus given by[1]

$$\psi_\mu^{(i)}=\frac{\partial A_\mu^{\mathrm{cl}}}{\partial\gamma_i}+D_\mu^{\mathrm{cl}}\Lambda^{(i)},\qquad D_\mu^{\mathrm{cl}}\psi_\mu^{(i)}=\partial_\mu\psi_\mu^{(i)}-ig[A_\mu^{\mathrm{cl}},\psi_\mu^{(i)}]=0, \tag{12}$$

where D_μ^{cl} is the gauge-covariant derivative at the classical field and $\Lambda^{(i)}$ is the gauge transformation necessary to put the ith mode into the background gauge. Working in parallel with Eqs. (4) through (11), it is then easy to see that $J(\gamma)$, as defined in (3a), is given by

$$J(\gamma)=\left(\prod_i\frac{1}{\sqrt{2\pi}}\right)(\det V)(\det U)^{-1/2} \tag{13}$$

where the matrices V and U are defined by[11]

$$V_{ij}=\left\langle\frac{\partial A^{\mathrm{cl}}}{\partial\gamma_i}\Big|\psi^{(j)}\right\rangle,\qquad U_{ij}=\langle\psi^{(i)}|\psi^{(j)}\rangle. \tag{14}$$

Now, if

$$\Lambda^{(i)}\psi_\mu^{(j)}<O\left(\frac{1}{r^3}\right) \tag{15}$$

at large distances r, then a simple integration by parts gives $V=U$ and the familiar result

$$J(\gamma)=\left(\prod_i\frac{1}{\sqrt{2\pi}}\right)(\det U)^{1/2}. \tag{16}$$

Thus, provided that we can express each zero mode as the derivative of the classical field with respect to a collective coordinate plus an additional gauge transformation [i.e., in the form of (12)] and provided that the gauge transformation $\Lambda^{(i)}$ vanishes sufficiently rapidly at large distances [i.e., (14) is obeyed], the calculation of the collective-coordinate Jacobian is straightforward. With the instanton in the regular gauge, only the translation and dilatation zero modes obey these conditions; however, in the singular gauge, the gauge modes are also well behaved (as we will see presently), and all modes may be treated on the same footing.

We now specialize the calculation to an SU(3) gauge theory. The general SU(N) case is only slightly more complicated and is presented in Sec. III. An SU(3) instanton can be obtained simply by embedding the SU(2) instanton into the "upper-left-hand corner" of the fundamental representation of SU(3).[12] Thus the singular gauge instanton has the form

$$A_\mu^{\rm cl}(x)=\frac{2}{g}\,\frac{\bar\eta_{a\mu\nu}x_\nu}{x^2(x^2+\rho^2)}\,\frac{\lambda_a}{2}\,, \tag{17}$$

where λ_a ($a=1,2,3$) are the first three Gell-Mann matrices and the symbols $\bar\eta_{a\mu\nu}$ are defined in Ref. 1. Under the action of this SU(2) subgroup, the generators of SU(3) form one triplet ($\lambda_1,\lambda_2,\lambda_3$), two doublets (made from $\lambda_4,\lambda_5,\lambda_6,\lambda_7$), and one singlet ($\lambda_8$). Using this fact, it is easy to write down the twelve background gauge zero modes. First, there are the eight isospin-1 modes, which are just the ones given in Ref. 1, after conversion to the singular gauge of the instanton[13]:

$$\psi_\mu^{(\nu)}(x)=\left.\frac{\partial A_\mu^{\rm cl}(x-z)}{\partial z^\nu}\right|_{z=0}+D_\mu^{\rm cl}(A_\nu^{\rm cl}(x))\,, \tag{18a}$$

$$\psi_\mu^{(\rho)}(x)=\frac{\partial A_\mu^{\rm cl}(x)}{\partial\rho}\,, \tag{18b}$$

$$\psi_\mu^{(a)}(x)=D_\mu^{\rm cl}\left[\frac{\lambda_a}{g}\,\frac{x^2}{x^2+\rho^2}\right], \tag{18c}$$

where $\psi_\mu^{(\nu)}$ ($\nu=1,\dots,4$) are the translation modes, $\psi_\mu^{(\rho)}$ is the dilatation mode, and $\psi_\mu^{(a)}$ ($a=1,2,3$) are the gauge modes generated by $\lambda_1,\lambda_2,\lambda_3$. In addition, there are four modes which are members of isospin doublets and which can be obtained from the isospin-$\frac{1}{2}$ spinor modes of Ref. 1, since vectors and right-handed spinors obey the same equation. They are pure gauge modes, generated by λ_α ($\alpha=4,5,6,7$):

$$\psi_\mu^{(\alpha)}=D_\mu^{\rm cl}\left[\frac{\lambda_\alpha}{g}\left(\frac{x^2}{x^2+\rho^2}\right)^{1/2}\right]. \tag{19}$$

There are no normalizable isospin-0 zero modes (λ_8 does not generate a gauge mode since it commutes with the field of the instanton).

$\psi_\mu^{(\nu)}$ and $\psi_\mu^{(\rho)}$ are already in the form of (12), with $\Lambda^{(i)}$ vanishing rapidly at infinity. To put the gauge modes $\psi_\mu^{(\alpha)}$ in this form, we must simply make more explicit the collective coordinates to which they correspond. The orientation of an instanton in SU(3) is described by a group element G:

$$A_\mu^{\rm cl}[G]=G^{-1}A_\mu^{\rm cl}G\,. \tag{20}$$

If we represent an infinitesimal change in G by

$$G+\delta G=(I-idt^i\lambda_i)G\,,\quad i=1,\dots,8 \tag{21}$$

then seven gauge zero modes are given by

$$\chi_\mu^{(k)}[G]=\frac{\partial A_\mu^{\rm cl}[G]}{\partial t^k}=-iG^{-1}[A_\mu^{\rm cl},\lambda_k]G\,, \tag{22}$$

where $k=1,\dots,7$. These modes are not in the background gauge. However, from (18c) and (19), we easily find the necessary gauge transformation[13] for $G=I$,[14]

$$\psi_\mu^{(k)}=\frac{\partial A_\mu^{\rm cl}}{\partial t^k}+D_\mu\Lambda^{(k)}\,, \tag{23}$$

where

$$\Lambda^{(k)}=\begin{cases}\dfrac{1}{g}\left(\dfrac{-\rho^2}{x^2+\rho^2}\right), & k=1,2,3\\[2ex] \dfrac{1}{g}\left[\left(\dfrac{x^2}{x^2+\rho^2}\right)^{1/2}-1\right], & k=4,5,6,7\,.\end{cases} \tag{24}$$

Now all the modes are in the form of (12), with $\Lambda^{(i)}$ vanishing sufficiently rapidly at infinity so that (15) is obeyed. [This would not have been possible in the regular gauge since in that gauge the gauge transformations appearing in (18c) and (19) do not approach constants at infinity.] We can thus apply (16) to compute $J(\gamma)$. The modes are orthogonal and their normalization is easily calculated to be

$$\|\psi_\mu^{(\nu)}\|=\frac{1}{\sqrt2}\,\|\psi_\mu^{(\rho)}\|=\frac{1}{\rho\sqrt2}\,\|\psi_\mu^{(a)}\|=\frac{1}{\rho}\,\|\psi_\mu^{(\alpha)}\|=\frac{2\sqrt2\,\pi}{g}\,. \tag{25}$$

This implies

$$J(\gamma)=\frac{2^{14}\pi^6\rho^7}{g^{12}}\,. \tag{26}$$

To complete the calculation of $W^{(1)}$ [Eq. (3)], it is necessary to compute $Q(\gamma)$, which is the contribution of the nonzero modes. 't Hooft's calculation,[1] which has been verified by others,[15] gives the nonzero-mode determinants with Pauli-Villars regularization for arbitrary spin and isospin. Here we have vector (gauge) fields and scalar (ghost) fields each forming one isotriplet, two isodoublets, and one isoscalar. In addition, we must recall that the regulator fields contribute one factor of μ_0, the regulator mass, for each zero mode of the true fields. The results of Ref. 1 then immediately imply

$$Q(\gamma)=\mu_0^{12}\exp[-\ln(\mu_0\rho)-\alpha(1)-2\alpha(\tfrac{1}{2})]\,, \tag{27}$$

where the coefficients $\alpha(t)$ give the contribution of each isospin t and are tabulated in Ref. 1 [$\alpha(0)=0$].

We may now insert (26) and (27) into (3). Since the integrand in (3) is independent of the gauge orientation of the instanton, the integration over those collective coordinates may be performed. From (21), one learns that integration over all eight parameters t^i would simply give the volume of SU(3), calculated with the right-invariant Haar measure. However, only the seven t^k of (22) are collective coordinates; integration over them gives the volume of SU(3)/U(1), where the U(1) is generated by λ_8. SU(3)/U(1) is the set of equivalence classes on SU(3) given by

$$G'\simeq G \text{ if } G'=e^{i\theta\lambda_8}G\,. \tag{28}$$

[In other words, two elements of SU(3) are counted as equivalent if they produce the same instanton orientation.] The volume of SU(3)/U(1) is calculated in the Appendix to be $\pi^4/2$. Combining this information with (13), (26), and (27) gives the result

$$W^{(1)}=2^{13}\pi^{10}e^{-\alpha(1)-2\alpha(1/2)}\int\frac{d^4z d\rho}{\rho^5}\frac{e^{-8\pi^2/g^2(\rho)}}{g^{12}}, \qquad (29)$$

where z is the space-time location of the instanton, and

$$\frac{8\pi^2}{g^2(\rho)}=\frac{8\pi^2}{g^2}-11\ln(\rho\mu_0)=-\ln(\rho\mu) \qquad (30)$$

is just the usual renormalization-group result for $g(\rho)$. The second equality in (30) is simply a definition of the scale μ (the quantity that can be determined by electroproduction scaling violations).

III. SU(N)

The generalization from SU(3) to SU(N) is fairly straightforward. The only subtlety involves the integration over the collective coordinates which describe the gauge orientation of the instanton—this integration is performed in the Appendix.

If we embed the instanton in the standard way into the SU(2) in the "upper-left hand corner" of the fundamental representation of SU(N), the generators of SU(N) form one triplet (the analog of $\lambda_1, \lambda_2, \lambda_3$) and $2(N-2)$ doublets (the analogs of $\lambda_4, \ldots, \lambda_7$) under the action of this SU(2).[12] All other generators are singlets. This implies that, in addition to the eight zero modes of the form (18), there will be $4(N-2)$ zero modes of the form (19). Following the steps that led to (26), we now have

$$J(\gamma)=\frac{4}{\rho^5}\left(\frac{2\rho\sqrt{\pi}}{g}\right)^{4N}. \qquad (31)$$

Similarly, following the steps leading to (27) gives

$$Q(\gamma)=\mu_0^{4N}\exp[-\tfrac{1}{3}N\ln(\mu_0\rho)-\alpha(1)-2(N-2)\alpha(\tfrac{1}{2})]. \qquad (32)$$

Equations (31) and (32) may then be inserted in (3a). The integral over the group orientation is defined in the Appendix as $V(C_N)$ and is given by (A14). The result is

$$W^{(1)}=\frac{4}{\pi^2}\frac{\exp[-\alpha(1)-2(N-2)\alpha(\frac{1}{2})]}{(N-1)!(N-2)!}\times\int\frac{d^4z d\rho}{\rho^5}\left(\frac{4\pi^2}{g^2}\right)^{2N}e^{-8\pi^2/g^2(\rho)}, \qquad (33)$$

where, according to the renormalization group,

$$\frac{8\pi^2}{g^2(\rho)}=\frac{8\pi^2}{g^2}-\frac{11N}{3}\ln(\mu_0\rho). \qquad (34)$$

Equation (33) differs by a factor of $1/\sqrt{2N}$ from previously published results,[2] for reasons explained in the Appendix. However, the large-N limit is controlled by the factorial and power behavior of (33), and the conclusion that instantons are unimportant in the planar limit[3,4] ($N\to\infty$, Ng^2 fixed) is unchanged.

IV. COMMENTS ON QCD INSTANTON CALCULATIONS

We may rewrite our answer for SU(3), Eq. (29), in terms of the mean density of instanton of scale size ρ in the dilute-gas approximation:

$$\frac{d\rho}{\rho^5}D(\rho)=b\frac{d\rho}{\rho^5}x^6e^{-x(\rho)}, \qquad b=0.0015, \qquad (35)$$

where $x\equiv 8\pi^2/g^2$, $x(\rho)\equiv 8\pi^2/g^2(\rho)$. The number b in (35) differs by a factor of $\frac{1}{64}$ from the previously published[6] 0.1, which has been used in most QCD instanton calculations to date. The reason for this discrepancy is easily found: In calculating this number, we have taken cognizance of the recently discovered[5] error in 't Hooft's original calculation and have therefore been careful to normalize the functional measure correctly. [See Eq. (8) and the remarks following it.] Compared to the original incorrect normalization, this introduces a factor of $1/\sqrt{2}$ for each zero mode. The fact that the previous result must be corrected by a factor of $\frac{1}{64}$ is by now known to most specialists in this field[16]; however, a few comments are in order on the required modifications of some recent calculations[7-9] of physical instanton effects.

Examining (35) we see that a change in b can be absorbed into an additive constant in $x(\rho)$; from (30) this just implies a change in the scale $\mu\rho$. In fact, to absorb a factor of 64, we must change $x(\rho)$ by $\ln 64\sim 4$, which means changing $\mu\rho$ by a factor of ~ 1.5. Of course, if we assume, in the usual way,[6] that the factor of x^6 in (35) is converted simply to $[x(\rho)]^6$ by the effect of higher loops, then the above argument is not strictly correct. Still, phenomena that occur for $x(\rho)$ considerably larger than 4 (for example, the interesting effects in Ref. 9 occur in the range $14<x<20$) are expected to take roughly the same form as before, but occur at somewhat larger values of the coupling constant [smaller $x(\rho)$] and correspondingly larger values of $\mu\rho$. We must keep in mind, though, that while pure dilute-gas instanton effects may be relatively unaffected, the larger coupling constants involved can bring other, nonperturbative effects into play, thus blurring some previous conclusions.

To be more specific, let us first consider the effects of instantons on the short-distance hadron currents that control e^+e^- annihilation.[7,8,17] In Ref. 7, the instanton correction to the photon self-energy $\Pi(q^2)$ for q^2 large and Euclidean is found to depend on all instanton sizes up to a maximum, cutoff scale. Thus we must ask whether we should change the cutoff with b. In the past, the cutoff could be taken, without significant difference, to be either the scale, ρ_D, at which the dilute-gas approximation breaks down [i.e., when the integrated instanton density is 1—which is given by $x(\rho_D)\simeq 14$ for $b=0.1$] or the scale, ρ_M, at which other nonperturbative effects become important [instantons are believed to ionize[6] into mesons at $x(\rho_M)\simeq 17$]. The meron calculation is purely a comparison of action and entropy and is not dependent on b (Ref. 18); however, ρ_D changes drastically with b. [In fact, a trival calculation gives $x(\rho_D)=0$ for $b=0.0015$—though of course one is hardly justified in using the renormalization group down to such values of $x(\rho)$.] Now one could take the point of view that the instanton dilute-gas approximation, while not quantitatively accurate below the point where merons appear, still gives a reasonably good qualitative picture of the effects of the whole nonperturbative sector. With this viewpoint, one may extend the cutoff to lower x, limited only by the validity of the renormalization-group calculation (as explained above, lack of diluteness is not a problem). In this way, one would find instanton effects of roughly the same numerical magnitude (within a factor of 2 or so) as those found by Andrei and Gross. On the other hand, with the point of view that instanton calculations cannot be trusted when instantons ionize into merons, we must divide by 64 the numbers R_1 and R_2 which compare instanton effects to perturbation theory—thereby making instanton effects much less important at the quoted values of q/μ. This difference can be made up by going to lower q/μ (larger coupling); however, one would again have to go beyond the point where merons appear for instanton corrections to be comparable to the perturbative ones.

In Ref. 8, Baulieu *et al.* argue that the contribution of instantons to the e^+e^- total cross section may be found by taking the imaginary part of the naive continuation of $\Pi(q^2)$ to timelike q^2. Their result depends only on instantons of scales $\rho\sim 1/q$, so there is no freedom to adjust this answer by changing a cutoff—numerical results must be divided by 64. Of course, in both Refs. 7 and 8, predictions for definite values of q (in GeV) depend on the identification of μ (in GeV), which is taken either from experiment (Ref. 8, $\mu\sim 300$–700 MeV) or from theory [Ref. 7, $\mu\sim\frac{1}{4}$ (hadron mass)—from the scale of meron ionization[6]]. To the extent that these numbers are uncertain, numerical predictions can change. (In Ref. 8, such high powers of momentum enter that our factor of 64 is lost in uncertainties in μ.)

We now turn to the recent work of Callan *et al.*[9] on the role of instantons in quark confinement and the formation of a hadron bag. These authors find that instantons act as permanent color magnetic dipoles which lead to a transition, at a critical value of an external color field, between a dilute phase with low paramagnetic permeability (the inside of the bag) and a dense phase of vacuum fluctuations with very high permeability (the outside of the bag). This phase transition is signalled by an instability in the phase diagram of color electric displacement D vs color electric field E—namely, a "nose" on the curve, where $\partial D/\partial E$ changes sign. In their calculations, with $b=0.1$, the "nose" occurs at a scale where $x(\rho)\simeq 19$. As explained at the beginning of this section, the effect of changing b should be to keep the form of this result essentially unchanged (i.e., to preserve the basic shape of the D vs E curve) but to displace the curve to somewhat larger scales and smaller values of $x(\rho)$. This is precisely what happens: When I repeat[19] their calculation using $b=0.0015$, I still find a "nose" in D vs E, but now at $x(\rho)\simeq 11$. The instanton gas is still dilute at this scale; furthermore, the coupling constant is still reasonably small so that ordinary perturbative corrections are not expected to be large. However, $x(\rho)=11$ is considerably below the point where instantons ionize into merons [recall that this occurs at $x(\rho)\simeq 17$, independent of b]. Thus the meaning we assign to this calculation will again depend on our philosophy of the nonperturbative sector. If instantons are believed to be representative of the whole sector, then we have a good qualitative understanding of an instability which leads to confinement. On the other hand, if we insist that we must at present stop calculating when instantons ionize, then our ability to see ends before things begin to look interesting. It is certainly true that no calculation has yet been done that indicates how a *meron* gas acts under an external color field. Thus, finding confinement in this picture is a tricky business. It is important to keep in mind, however, that another result of Ref. 9—that instanton corrections in the presence of large color fields inside hadrons are controlled and calculable—is unaffected.

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APPENDIX

Here we calculate the integral over the collective coordinates which describe the orientation of an instanton in the group SU(N). This is just the volume of the coset space C_N, defined by

$$C_N = \mathrm{SU}(N)/T_N\,, \tag{A1}$$

where T_N is the stability group of the instanton [the subgroup of SU(N) which leaves the instanton invariant].

As a preliminary, we compute the volume of SU(3). The calculation is simplified enormously[20] by considering the action of the fundamental representation on the vector

$$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

There is an SU(2) subgroup, generated by $\lambda_1, \lambda_2, \lambda_3$, which leaves v invariant; the rest of the group just takes v into an arbitrary complex 3-vector of length 1. Thus the set of equivalence classes of SU(3) under the above-mentioned SU(2) is in one-to-one correspondence with the points on the five-dimensional sphere S_5. However, the volume element of SU(3)/SU(2) is not numerically equal to the volume element on S_5; to get the relation between the two we consider how group elements near the identity act on v. Writing $I + \delta G = I - i\lambda_i dt^i$, we have

$$\delta G(v) = \begin{bmatrix} -idt^4 - dt^5 - \dfrac{i}{\sqrt{3}}\, dt^8 \\ -idt^6 - dt^7 - \dfrac{i}{\sqrt{3}}\, dt^8 \\ \dfrac{2idt^8}{\sqrt{3}} \end{bmatrix}. \tag{A2}$$

On the other hand, if we describe a point in S_5 as a complex 3-vector and denote the locally flat coordinates in the neighborhood of v by $x^1, \ldots, x^5$, then the infinitesimal change in v under displacement by these coordinates is given by

$$\delta v = \begin{pmatrix} dx^1 + idx^2 \\ dx^3 + idx^4 \\ idx^5 \end{pmatrix}. \tag{A3}$$

Comparing (A2) and (A3) allows us to relate $dx^1, \ldots, dx^5$ to $dt^4, \ldots, dt^8$. We have, for the volume elements,

$$dt^4 dt^5 dt^6 dt^7 dt^8 = \frac{\sqrt{3}}{2}\, dx^1 dx^2 dx^3 dx^4 dx^5\,. \tag{A4}$$

Thus, the volume of SU(3) is given by[21]

$$V(\mathrm{SU}(3)) = \int \prod_i dt^i = \frac{\sqrt{3}}{2}\, V(S_5) V(\mathrm{SU}(2))\,. \tag{A5}$$

Using $V(S_5) = \pi^3$, $V(\mathrm{SU}(2)) = V(S_3) = 2\pi^2$, we have

$$V(\mathrm{SU}(3)) = \sqrt{3}\,\pi^5\,. \tag{A6}$$

We can now compute $V(C_3)$, the volume of C_3. This is just the volume of SU(3) divided by the volume of T_3, where T_3 is the U(1) generated by λ_8. If we write the elements of this U(1) as $e^{i\theta\lambda_8}$ then θ has the range $0 < \theta < 2\pi\sqrt{3}$. We thus have

$$V(C_3) = \frac{V(\mathrm{SU}(3))}{V(T_3)} = \frac{\pi^4}{2}\,. \tag{A7}$$

The calculation for general N follows the same lines. In parallel with (A2)–(A5), SU(N)/SU($N-1$) can be related to S_{2N-1}, giving[21]

$$V(\mathrm{SU}(N)) = \left(\frac{N}{2(N-1)}\right)^{1/2} V(S_{2N-1}) V(\mathrm{SU}(N-1))\,. \tag{A8}$$

Using

$$V(S_{2N-1}) = \frac{2\pi^N}{(N-1)!}\,, \tag{A9}$$

we have

$$V(\mathrm{SU}(N)) = \sqrt{N} \prod_{k=2}^{N} \frac{\sqrt{2}\,\pi^k}{(k-1)!}\,. \tag{A10}$$

The identification of T_N is slightly subtle. If we place the instanton in the upper-left-hand corner of the fundamental representation, then the generators of T_N are those which commute with that SU(2), namely the generators of the SU($N-2$) in the lower right and the generator

$$\lambda = \left(\frac{N-2}{N}\right)^{1/2} \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & -2/(N-2) & & & & \\ & & & -2/(N-2) & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & -2/(N-2) \end{bmatrix}, \tag{A11}$$

which commutes with SU(2) and SU($N-2$). We may therefore parametrize T_N by writing $h \in T_N$ as

$$h = e^{i\theta\lambda} g\,, \tag{A12}$$

where $g \in \mathrm{SU}(N-2)$. The range of θ is $0 \le \theta < 2\pi[N/(N-2)]^{1/2}$. (Note that for $N \ge 5$, this is less than the range of θ necessary for $e^{i\theta\lambda}$ to repeat; i.e., T_N is not the same as $\mathrm{SU}(N-2) \times \mathrm{U}(1)$.

This is because $\theta_1-\theta_2=2\pi[N/(N-2)]^{1/2}$ implies $e^{i\theta_1\lambda}$ and $e^{i\theta_2\lambda}$ differ by an element of SU($N-2$)—specifically, an element in the *center* of SU($N-2$).) We thus have

$$V(T_N)=2\pi\left(\frac{N}{N-2}\right)^{1/2}V(\mathrm{SU}(N-2)), \qquad \text{(A13)}$$

and[22]

$$V(C_N)=\frac{V(\mathrm{SU}(N))}{V(T_N)}=\frac{\pi^{2N-2}}{(N-1)!\,(N-2)!}. \qquad \text{(A14)}$$

This differs from previously published results.[2,3] Aside from trivial differences in the normalization of the generators, my disagreement with these authors is based on their identification of T_N as SU($N-2$) (Ref. 3) or U($N-2$) (Ref. 2).

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[11]The scalar product, in conformity with that of Ref. 1, is defined by

$$\langle\psi^{(i)}\,|\,\psi^{(j)}\rangle=2\int d^4x\,\mathrm{tr}(\psi_\mu^{(i)}(x)\,\psi_\mu^{(j)}(x))$$

$$=\int d^4x\,\psi_{\mu a}^{(i)}\,\psi_{\mu a}^{(j)},$$

where

$$\psi_\mu^{(i)}\equiv\tfrac{1}{2}\lambda_a\psi_{\mu a}^{(i)}.$$

[12]For a description of instantons in a general gauge group, see C. Bernard, N. Christ, A. Guth, and E. Weinberg, Phys. Rev. D 15, 2967 (1977).

[13]C. Bernard, Phys. Rev. D 18, 2026 (1978). A_ν^{cl} is the correct gauge transformation parameter in (18a) since this gives $\psi_\mu^{(\nu)}=F_{\mu\nu}^{cl}$, which clearly obeys $D_\mu^{cl}\psi_\mu^{(\nu)}=0$.

[14]For $G\neq I$ just rotate both sides of (23).

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[17]Other treatments, which we do not discuss here, are those of R. D. Carlitz and C. Lee, Phys. Rev. D 17, 3238 (1978); M. Suzuki, Phys. Lett. 76B, 466 (1978); and T. Appelquist and R. Shankar, Phys. Rev. D 18, 2952 (1978).

[18]While not dependent on b, the value of $x(\rho)$ at which merons ionize is of course dependent on the definition of the coupling constant in renormalization. Here, all numbers refer to the Pauli-Villars regularization scheme of Ref. 1.

[19]The calculation takes into account instanton interactions by placing each instanton in a spherical cavity of radius $R=2.2\rho$ in the permeable medium formed by the other instantons. As in Ref. 9, other reasonable choices for R do not change the result significantly.

[20]I thank R. Blattner for explaining this method to me. A straightforward calculation of the volume by parametrizing the group and integrating yields the same answer.

[21]The normalization of the volume is determined by the normalization of the generators. Here, I use $\mathrm{tr}(\lambda_i\lambda_j)=2\delta_{ij}$.

[22]While derived for $N>3$, this formula is in fact true for $N=2, 3$ also.