PHYSICAL REVIEW D VOLUME 19, NUMBER 10 15 MAY 1979

Multivortex solutions of the Ginzburg-Landau equations

Erick J. Weinberg
Department of Physics, Columbia University, New York, New York 10027
(Received 5 February 1979)

Multivortex solutions of the Ginzburg-Landau equations (or, equivalently, of the Abelian Higgs model) are considered for a special choice of parameters. It is shown that for every n there is a $2n$-parameter family of n-vortex solutions. It is conjectured that the parameters are just those needed to specify the positions of the vortices and that the vortices behave very much like noninteracting particles.

I. INTRODUCTION

The existence of vortex solutions to the Ginzburg-Landau equations of superconductivity has been known for some time.[1] These solutions are also of relevance in elementary-particle physics, where they occur as classical solutions to the Abelian Higgs model, which is mathematically similar to the Ginzburg-Landau theory.[2] Although they arise in three-dimensional theories, the vortices are invariant under translations along a fixed axis, and thus may be viewed as finite-energy solutions to a theory in two space dimensions. Associated with them is a topological invariant—the quantized conserved magnetic flux.

Although the equations depend on three parameters, two may be eliminated by an appropriate rescaling of the fields and lengths, leaving only one, λ, which is of physical significance. Depending on whether λ is less or greater than one, the superconductor will be of type I or type II. The case $\lambda=1$ is of particular interest, as it is known to have a number of rather special properties. Bogomol'nyi[3] has shown that for this case a lower bound on the energy of an n-vortex configuration can be obtained. Since the vortex number is a topological invariant, any configuration which achieves this bound will be a minimum of the energy, and thus a solution of the Ginzburg-Landau equations. Furthermore, to find fields which achieve this bound one need solve only a set of first-order differential equations, rather than the second-order Ginzburg-Landau equations. Finally, Jacobs and Rebbi[4] have recently done a detailed numerical study of the interaction energy between a pair of vortices for various values of λ. For $\lambda=1$ they find that, to a very high accuracy, the energy is independent of the vortex separation.

In this paper we study the theory with $\lambda=1$. We show that for every integer n there is a $2n$-parameter family of n-vortex solutions satisfying Bogomol'nyi's bound. Furthermore, every solution satisfying this bound must belong to such a family. We conjecture that these parameters may be chosen to be the $2n$ coordinates needed to specify the positions of the vortices. As we do not find explicit expressions for the solutions, we must use indirect means to study their properties. Assuming the existence of an arbitrary solution of minimal energy, we consider the problem of finding small fluctuations which leave the energy unchanged. We derive a special case of the Atiyah-Singer index theorem[5] and use it to show that there are precisely $2n$ independent solutions to this problem. Combining this fact with the previously established existence of solutions for n vortices superimposed at the same point[6] yields our results.

The remainder of this paper is organized as follows. In Sec. II we review some properties of the theory, including Bogomol'nyi's results. In Sec. III we derive a form of the index theorem as well as a related vanishing theorem, and apply these to the problem of small fluctuations. In Sec. IV we consider the case of n vortices at the origin, and show directly that the small-fluctuation problem has exactly $2n$ solutions; this serves as a check for the more formal calculations of Sec. III. We conclude in Sec. V with some remarks.

II. SOME PROPERTIES OF THE THEORY

The potential energy in the two-dimensional Abelian Higgs model (or, equivalently, the free energy per unit length in the Ginzburg-Landau theory) is

$$E=\int d^2\tilde{x}\left[\tfrac{1}{2}|(\tilde{\partial}_j-ie\tilde{A}_j)\tilde{\phi}|^2+\tfrac{1}{4}\tilde{F}_{jk}\tilde{F}_{jk}+h(|\tilde{\phi}|^2-v^2)^2\right]. \tag{2.1}$$

Here $\tilde{\phi}$ is a complex scalar field and

$$\tilde{F}_{jk}=\tilde{\partial}_j\tilde{A}_k-\tilde{\partial}_k\tilde{A}_j \tag{2.2}$$

is the field strength. Rescaling fields and lengths according to

$$\tilde{\phi} = v\phi\,,\quad \tilde{A}_j = vA_j\,,\quad \tilde{x}_j = (1/ev)x_j\,, \tag{2.3}$$

leads to

$$E = v^2 \int d^2x\left[\tfrac{1}{2}|(\partial_j - iA_j)\phi|^2 + \tfrac{1}{4}F_{jk}F_{jk} + \tfrac{1}{8}\lambda(|\phi|^2 - 1)^2\right], \tag{2.4}$$

where

$$\lambda = 8h/e^2\,. \tag{2.5}$$

For the remainder of the paper we shall set $\lambda = 1$.

If the energy is to be finite, then as r approaches infinity, $|\phi|$ must tend to 1 and $(\partial_j - iA_j)\phi$ must vanish. Thus, asymptotically

$$\phi \approx e^{i\chi(\theta)}\,,\quad A_j \approx \partial_j\chi\,. \tag{2.6}$$

Since ϕ must be single-valued and continuous, χ must satisfy

$$\chi(\theta + 2\pi) = \chi(\theta) + 2\pi n \tag{2.7}$$

for some integer n. Continuous variations of the fields, subject only to the constraint of finite energy, cannot change n; it is therefore a topological invariant. From Eqs. (2.6) and (2.7) it follows that

$$n = -\frac{i}{2\pi}\oint_c d\ln\phi = \frac{1}{2\pi}\oint_c d\vec{l}\,\hat{l}\cdot\vec{A} = \frac{1}{2\pi}\int d^2x\,F_{12}\,, \tag{2.8}$$

where the line integrals are to be taken around a contour at $r=\infty$. Because of the continuity of ϕ, the contour may be deformed into a sum of contours enclosing the zeros of ϕ. If there are $n_\pm$ points where ϕ vanishes like $[(x \pm iy) - (x_0 \pm iy_0)]$, then $n = n_+ - n_-$. We shall say that there are n_+ vortices and n_- antivortices.

For the case $\lambda = 1$ a lower bound on the energy is obtained[3] by using an integration by parts to rewrite Eq. (2.4) as

$$E = v^2 \int d^2x\left\{\tfrac{1}{2}[(\partial_1\phi_1 + A_1\phi_2) \mp (\partial_2\phi_2 - A_2\phi_1)]^2 + \tfrac{1}{2}[(\partial_2\phi_1 + A_2\phi_2) \pm (\partial_1\phi_2 - A_1\phi_1)]^2 + \tfrac{1}{2}[F_{12} \pm \tfrac{1}{2}(\phi_1^2 + \phi_2^2 - 1)]^2\right\} \pm \frac{v^2}{2}\int d^2x\,F_{12}\,, \tag{2.9}$$

where ϕ_1 and ϕ_2 are the real and imaginary parts of the scalar field ϕ. The first integral is positive-semidefinite while the second is simply a multiple of the vortex number n. Taking the upper or lower sign according to whether n is positive or negative yields

$$E \geq |n|(\pi v^2) \tag{2.10}$$

with equality if

$$\begin{aligned} 0 &= (\partial_1\phi_1 + A_1\phi_2) \mp (\partial_2\phi_2 - A_2\phi_1)\,,\\ 0 &= (\partial_2\phi_1 + A_2\phi_2) \pm (\partial_1\phi_2 - A_1\phi_1)\,,\\ 0 &= F_{12} \pm \tfrac{1}{2}(\phi_1^2 + \phi_2^2 - 1)\,. \end{aligned} \tag{2.11}$$

III. AN INDEX THEOREM

Let us assume that we are given an n-vortex solution of Eqs. (2.11) (for the sake of definiteness, take $n > 0$),[7] and count the modes of fluctuation about this solution which leave the energy unchanged.[8] Many of these are simply gauge transformations and are of no interest; these can be eliminated by imposing a gauge condition, e.g., the Coulomb gauge[9]

$$0 = \partial_1 A_1 + \partial_2 A_2\,. \tag{3.1}$$

Expanding Eqs. (2.11) and (3.1) about the solution and keeping terms linear in the fluctuation, we obtain

$$0 = \mathfrak{D}\eta\,, \tag{3.2}$$

where

$$\eta = (\delta\phi_1, \delta\phi_2, \delta A_1, \delta A_2)^t$$

and

$$\mathfrak{D} = \begin{bmatrix} (\partial_1 + A_2) & (-\partial_2 + A_1) & \phi_2 & \phi_1 \\ (\partial_2 - A_1) & (\partial_1 + A_2) & -\phi_1 & \phi_2 \\ \phi_1 & \phi_2 & -\partial_2 & \partial_1 \\ 0 & 0 & \partial_1 & \partial_2 \end{bmatrix}. \tag{3.3}$$

The index of the elliptic differential operator $\mathfrak{D}$ is defined by

$$\mathcal{I}(\mathfrak{D}) = \dim(\text{kernel }\mathfrak{D}) - \dim(\text{kernel }\mathfrak{D}^*)\,, \tag{3.4}$$

where

$$\mathfrak{D}^* = \begin{bmatrix} (-\partial_1 + A_2) & -(\partial_2 + A_1) & \phi_1 & 0 \\ (\partial_2 + A_1) & (-\partial_1 + A_2) & \phi_2 & 0 \\ \phi_2 & -\phi_1 & \partial_2 & -\partial_1 \\ \phi_1 & \phi_2 & -\partial_1 & -\partial_2 \end{bmatrix} \tag{3.5}$$

is the adjoint of $\mathfrak{D}$. We will first obtain a formula for $\mathcal{I}(\mathfrak{D})$ and then show that the kernel of $\mathfrak{D}^*$ vanishes, so that $\mathcal{I}(\mathfrak{D})$ is in fact the desired number of modes.

We begin by noting that the kernels of $\mathfrak{D}$ and $\mathfrak{D}^*$ are identical to those of $\mathfrak{D}^*\mathfrak{D}$ and $\mathfrak{D}\mathfrak{D}^*$, respectively. Furthermore, if ψ is an eigenfunction of $\mathfrak{D}^*\mathfrak{D}$ with nonzero eigenvalue, then $\mathfrak{D}\psi$ is an eigenfunction of $\mathfrak{D}\mathfrak{D}^*$ with the same eigenvalue. Assuming that the eigenfunctions form a complete basis, it follows that[10]

$$\mathcal{I}(\mathfrak{D}) = \mathrm{Tr}\left(\frac{M^2}{\mathfrak{D}^*\mathfrak{D}+M^2}\right) - \mathrm{Tr}\left(\frac{M^2}{\mathfrak{D}\mathfrak{D}^*+M^2}\right), \quad (3.6)$$

where M^2 is an arbitrary parameter. It will be most convenient to evaluate this expression in the limit $M^2 \to \infty$.

A short calculation shows that

$$\begin{aligned} \mathfrak{D}^*\mathfrak{D} &= \Delta - L_1\,, \\ \mathfrak{D}\mathfrak{D}^* &= \Delta - L_2\,, \end{aligned} \quad (3.7)$$

where

$$\Delta = -I(\partial_1{}^2 + \partial_2{}^2) \quad (3.8)$$

and L_1 and L_2 are first-order differential operators. We may write

$$\begin{aligned} \frac{M^2}{\mathfrak{D}^*\mathfrak{D}+M^2} &= M^2[(\Delta+M^2)^{-1} \\ &\quad + (\Delta+M^2)^{-1}L_1(\Delta+M^2)^{-1} + \cdots]\,, \\ \frac{M^2}{\mathfrak{D}\mathfrak{D}^*+M^2} &= M^2[(\Delta+M^2)^{-1} \\ &\quad + (\Delta+M^2)^{-1}L_2(\Delta+M^2)^{-1} + \cdots]\,. \end{aligned} \quad (3.9)$$

If these expressions are substituted into Eq. (3.6), all terms beyond those linear in the L_i will vanish in the limit $M^2 \to \infty$.[11] The L_i will thus only enter through their traces, which satisfy

$$\mathrm{tr}L_1 - \mathrm{tr}L_2 = 4(\partial_1 A_2 - \partial_2 A_1)\,. \quad (3.10)$$

Therefore,

$$\begin{aligned} \mathcal{I}(\mathfrak{D}) &= \lim_{M^2\to\infty} \int d^2x\, 4F_{12}(x) M^2 \langle x|(\Delta+M^2)^{-2}|x\rangle \\ &= \lim_{M^2\to\infty} \int d^2x\, 4F_{12}(x) \int \frac{d^2k}{(2\pi)^2}\frac{M^2}{(k^2+M^2)^2} = 2n\,. \end{aligned} \quad (3.11)$$

We now consider the kernel of $\mathfrak{D}^*$. Any solution of

$$0 = \mathfrak{D}^*\psi$$

must also be a solution of

$$0 = D\mathfrak{D}^*\psi\,, \quad (3.12)$$

where

$$D = \begin{bmatrix} \phi_1 & \phi_2 & -\partial_2 & \partial_1 \\ \phi_2 & -\phi_1 & \partial_1 & \partial_2 \\ 0 & 0 & \phi_2 & \phi_1 \\ 0 & 0 & -\phi_1 & \phi_2 \end{bmatrix}. \quad (3.13)$$

Using the equations satisfied by the unperturbed fields to reexpress $\partial_i\phi_j$, Eq. (3.12) becomes

$$\begin{aligned} 0 &= [-(\partial_1{}^2+\partial_2{}^2) + \phi_1{}^2 + \phi_2{}^2]\psi_3\,, \\ 0 &= -(\partial_1{}^2+\partial_2{}^2)\psi_4\,, \\ 0 &= (\phi_1{}^2+\phi_2{}^2)\psi_1 + (\phi_2\partial_2 - \phi_1\partial_1)\psi_3 \\ &\quad - (\phi_2\partial_1 + \phi_1\partial_2)\psi_4\,, \\ 0 &= (\phi_1{}^2+\phi_2{}^2)\psi_2 - (\phi_2\partial_1 + \phi_1\partial_2)\psi_3 \\ &\quad + (\phi_1\partial_1 - \phi_2\partial_2)\psi_4\,. \end{aligned} \quad (3.14)$$

Since there are no square-integrable solutions to the first two of these equations, ψ_3 and ψ_4 must vanish.[12] The last two equations then require that ψ_1 and ψ_2 also vanish, and consequently so must the kernel of $\mathfrak{D}^*$.

Thus, about an arbitrary n-vortex solution of Eqs. (2.11) there are precisely $2n$ modes of fluctuation (other than gauge transformations) under which the energy is stationary.

IV. AN EXAMPLE

It is perhaps wise to check the formal methods of Sec. III by an explicit example. This is easily done by considering n vortices superimposed at the origin.[13] De Vega and Schaposnik[6] have shown that in this case there is a solution of the form

$$\begin{aligned} \phi(r,\theta) &= e^{in\theta}f(r)\,, \\ A_j(r,\theta) &= -\epsilon_{jk}x_k\frac{n}{r^2}a(r)\,, \end{aligned} \quad (4.1)$$

where the real functions $f(r)$ and $a(r)$ satisfy

$$\begin{aligned} 0 &= r\frac{df}{dr} - n(1-a)f\,, \\ 0 &= \frac{2n}{r}\frac{da}{dr} + f^2 - 1\,, \end{aligned} \quad (4.2)$$

with $f(\infty) = a(\infty) = 1$ and $f(0) = a(0) = 0$. At the origin f vanishes like r^n.

We may write perturbations about these solutions in the form

$$\begin{aligned} \delta\phi(r,\theta) &= ne^{in\theta}f(r)h(r,\theta)\,, \\ \delta A_1(r,\theta) &= \frac{n}{r}[-\sin\theta\, b(r,\theta) + \cos\theta\, c(r,\theta)]\,, \\ \delta A_2(r,\theta) &= \frac{n}{r}[\cos\theta\, b(r,\theta) + \sin\theta\, c(r,\theta)]\,. \end{aligned} \quad (4.3)$$

Requiring that h be real is sufficient to fix the gauge. Substituting these expressions into Eqs. (2.11) and keeping terms linear in the fluctuations leads, with the aid of Eq. (4.2), to

$$0=r\frac{\partial h}{\partial r}+b\,,$$
$$0=\frac{\partial h}{\partial \theta}-c\,, \tag{4.4}$$
$$0=\frac{1}{r}\frac{\partial b}{\partial r}-\frac{1}{r^2}\frac{\partial c}{\partial \theta}+f^2h\,.$$

Substituting the first two of these into the third, we obtain[14]

$$0=-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right)-\frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2}+f^2h\,. \tag{4.5}$$

If we now write

$$h(r,\theta)=\sum_{k=0}^{\infty}\left[h_k^{(1)}(r)\cos k\theta+h_k^{(2)}(r)\sin k\theta\right], \tag{4.6}$$

we obtain

$$0=-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}h_k^{(i)}\right)+\left(f^2+\frac{k^2}{r^2}\right)h_k^{(i)}\,. \tag{4.7}$$

Solutions of this equation will behave like $C_1r^{-k}+C_2r^k$ at the origin and like $C_3e^{-r}+C_4e^r$ as $r\to\infty$. We must require that $\delta\phi$ be nonsingular; since $f(r)$ has an nth-order zero at the origin, $h(r)$ may be as singular as r^{-n}. Thus for $k\leq n$ we can always obtain an acceptable solution to Eq. (4.7) by choosing the proper behavior as $r\to\infty$. For $k>n$ we must require that the solution be regular both at the origin and as $r\to\infty$. However, by multiplying Eq. (4.7) by $rh_k^{(i)}$ and integrating we obtain

$$0=\int_0^{\infty}dr\,r\left[\left(\frac{dh_k^{(i)}}{dr}\right)^2+\left(f^2+\frac{k^2}{r^2}\right)(h_k^{(i)})^2\right]-rh_k^{(i)}\frac{dh_k^{(i)}}{dr}\Bigg|_{r=0}^{r=\infty}. \tag{4.8}$$

If $h_k^{(i)}$ is nonsingular, the second term must vanish; since the integral is positive-semidefinite, $h_k^{(i)}$ must vanish. (Note that this also excludes the case $k=0$.) There are thus $2n$ acceptable solutions, as predicted by the arguments of Sec. III.

V. CONCLUSIONS

We have seen that for the special value $\lambda=1$, static n-vortex solutions of the Ginzburg-Landau equations may be obtained by solving a set of first-order equations. Solutions of these equations will have energy equal to n times that of a single vortex. About any such solution there are $2n$ physical modes of fluctuation under which the energy is stationary; thus, rather than being discrete, these solutions belong to $2n$-parameter families. It is tempting to conjecture that these parameters are just those needed to specify the positions of the n vortices; the results of Jacobs and Rebbi lend support to this view. Since solutions exist for n vortices superimposed at a point, and should certainly be expected to exist for n widely separated vortices, it is natural to make the further conjecture that these parameters may take on all real values. Stated somewhat differently, the vortices appear to behave very much like noninteracting point particles.

Many features of the $\lambda=1$ Ginzburg-Landau theory (e.g., the replacement of the second-order field equations by a set of first-order equations) are reminiscent of the problem of finding instanton[15] solutions to Yang-Mills theory in four-dimensional Euclidean space. In fact, the methods of Sec. III are just those which have been used to count parameters for multi-instanton solutions. Explicit expressions for some multi-instanton solutions have been obtained, and recently the problem of finding all multi-instanton solutions has been reduced to a purely algebraic one.[16] Whether the same methods can be fruitfully applied to the Ginzburg-Landau theory remains an interesting unanswered question.

ACKNOWLEDGMENTS

The author's work was supported by an Alfred P. Sloan Foundation Fellowship and in part by the U. S. Department of Energy.

[1]V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. 20, 1064 (1950); A. A. Abrikosov, *ibid*. 32, 1442 (1957) [Sov. Phys.—JETP 5, 1174 (1957)].
[2]H. B. Nielsen and P. Olesen, Nucl. Phys. B61, 45 (1973).
[3]E. B. Bogomol'nyi, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. 24, 449 (1976)].
[4]L. Jacobs and C. Rebbi, Phys. Rev. B (to be published).
[5]M. Atiyah and I. Singer, Ann. Math. 87, 484 (1968); M. Atiyah, R. Bott, and V. Patodi, Inventiones Math. 19, 279 (1973).
[6]H. J. de Vega and F. A. Schaposnik, Phys. Rev. D 14, 1100 (1976).
[7]The calculations for $n<0$ are the same except for some changes in the signs; the final result is that there are $2|n|$ modes of fluctuation.
[8]Methods similar to those of this section appear in

L. Brown, R. Carlitz, and C. Lee, Phys. Rev. D 16, 417 (1977); and C. Bernard, A. Guth, and E. Weinberg, *ibid*. 17, 1053 (1978).

[9]This condition does not rule out gauge transformations with a harmonic gauge function. It is, however, sufficient for our purposes since it eliminates those square-integrable fluctuations due to gauge transformations which might be confused with physically meaningful fluctuations.

[10]The fact that the spectra of $\mathfrak{D}^*\mathfrak{D}$ and $\mathfrak{D}\mathfrak{D}^*$ are not entirely discrete should not cause any essential difficulty. Since the Abelian Higgs model contains only massive particles, there will be a gap between the discrete zero eigenvalues, in which we are interested, and the physical part of the continuum spectrum. There will be a part of the continuum extending down to zero, but this corresponds to gauge fluctuations and is independent of the values of the fields; it will therefore have the same effect as it does for the vacuum case ($A_i=0$, $\phi=$constant), where its contributions to the two terms in Eq. (3.6) clearly cancel.

[11]There are terms quadratic in the L_i containing two derivatives which are potentially nonvanishing in the limit $M^2\rightarrow\infty$. However, these terms cancel after the traces are taken.

[12]There is of course the possibility of taking $\psi_3=0$ and ψ_4 constant; this is just the lower limit of that part of the continuum spectrum corresponding to gauge fluctuations. In Eq. (3.6) it is canceled by the corresponding part of the spectrum of $\mathfrak{D}^*\mathfrak{D}$.

[13]As in Sec. III, we are assuming $n>0$.

[14]This equation is also considered in Ref. 3.

[15]A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin, Phys. Lett. 59B, 85 (1975).

[16]V. Drinfeld and Yu. Manin, Funkts. Anal. Priloz. 12, 78 (1978); V. Drinfeld and Yu. Manin, Usp. Math. Nauk. (to be published); M. Atiyah, N. Hitchin, V. Drinfeld, and Yu. Manin, Phys. Lett. 65A, 185 (1978); N. Christ, E. Weinberg, and N. Stanton, Phys. Rev. D 18, 2013 (1978); E. Corrigan, D. Fairlie, S. Templeton, and P. Goddard, Nucl. Phys. B140, 31 (1978).