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Inconsistencies of Glass's equation for spin-3/2 particles

P. M. Mathews, M. Seetharaman, and T. R. Govindarajan
Department of Theoretical Physics, University of Madras, Madras-600 025, India

J. Prabhakaran
Deparment of Physics, Loyala College, Madras-600 034, India
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A systematic study is made of the Glass equation for spin-3/2 particles with minimal electromagnetic interaction. The study is motivated by the knowledge that this equation is just as satisfactory as the Rarita-Schwinger equation in the absence of interactions and that a variety of problems crop up in the Rarita-Schwinger theory when minimal electromagnetic interaction is introduced. The hope that the Glass equation might fare better is belied, however, Not only does it suffer from the various ills (e.g., noncausal propagation, modes of complex frequency) which beset the Rarita-Schwinger theory but it also exhibits further troubles such as an increase in the number of "spin" degrees of freedom (something not encountered earlier in any theory with $s < 2$), nonlocality of anticommutators of field components, etc., depending on the nature of the external field. Further, unlike in the symmetric tensor theory for spin 2, nonminimal interactions do not help to remove the anomaly of the abnormal number of degrees of freedom resulting from the minimal electromagnetic interaction. The bearing of the alebra of the β matrices on the difficulties of the interacting theories is briefly referred to.

I. INTRODUCTION

The various types of inconsistencies which afflict theories of higher-spin particles (with $s \geqslant \frac{3}{2}$) have been the subject of extensive investigation in recent years. At the quantized level, it was shown long ago by Johnson and Sudarshan[1] that in the case of the spin-$\frac{3}{2}$ particles [described by the familiar Rarita-Schwinger (RS) theory] minimal coupling to an external electromagnetic field makes the sign of the anticommutators dependent on the magnitude of the magnetic field, so that quantization with a positive-definite metric is not always possible. (The same difficulty has been shown to arise also with other types of couplings of the RS field.[2]) Even more disturbing was the demonstration by Velo and Zwanziger[3] that the classical RS wave equation with electromagnetic coupling exhibits noncausal modes of propagation. Shamaly and Capri[4] and others[5] have shown that the same problem afflicts a wide variety of equations, with various kinds of coupling to external fields. Another type of difficulty which has been discovered at the c-number level is the appearance of complex frequencies associated with "stationary" solutions of the RS equation including interaction with a static and homogeneous external magnetic field (hmf) at large field strengths. This was first demonstrated by Seetharaman, Prabhakaran, and Mathews[6] by explicit solution of the RS equation in the presence of an hmf. The same type of difficulty was shown to be present also in the case of the wave equation for spin-2 particles described by a symmetric second-rank tensor.[7]

While searching for higher-spin theories wherein superluminal velocities of propagation do not appear, it was noted by Prabhakaran, Seetharaman, and Mathews[8] that both the mixed spin-$\frac{3}{2}$–spin-$\frac{1}{2}$ theory of Bhabha[9] and Gupta[10] and the theory of Fisk and Tait[11] remain causal in the presence of minimal coupling to an external electromagnetic field as well as with trilinear coupling to a spinor field and a scalar field.[12] It was later shown by Prabhakaran, Govindarajan, and Seetharaman[13] that the Bhabha-Gupta (BG) equation retains its causal character also in the simultaneous presence of external electromagnetic and gravitational fields. They pointed out that the satisfactory behavior of the BG equation in this respect is due to the diagonalizability of the matrix β_0 which occurs as the coefficient of π_0 ($\equiv -i\partial_0 - eA_0$) in the equation of motion. These equations are free of other troubles also: For example, the frequencies associated with stationary modes remain real in the presence of an arbitrary hmf, and the anticommutators evaluated using the Schwinger procedure are independent of the external field.[14] This last result has been shown, in a recent work by Cox,[15] to be valid for any general equation with a diagonalizable β_0. His proof employs the Yang-Feldman procedure. While the theories with diagonalizable β_0 are appealing on account of their freedom from the various inconsistencies mentioned above, the price paid for this advantage is the indefiniteness of the total charge[16] and the resultant problems with quantization.[17]

It becomes of interest then to examine theories with nondiagonalizable β_0 other than the RS theory which has been seen to be wanting in several respects. One promising candidate is Glass's equation.[18] The expression for the total charge in Glass's theory is positive definite, as in the Rarita-Schwinger case, though the algebra of the β matrices is not the same in the two cases. [The minimal equation for β_0 is $\beta_0^3(\beta_0^2-1)=0$ in Glass's theory[19] and $\beta_0^2(\beta_0^2-1)=0$ in the RS theory.] Our aim in this paper is to make a detailed study of Glass's equation with minimal electromagnetic interaction. First, we solve the equation in the presence of an hmf to see whether frequencies associated with any of the modes become complex (Sec. II). We find that this does in fact happen. What is worse, the number of independent solutions turns out to be more than what is required for the correct description of a spin-$\frac{3}{2}$ particle. This latter fact prompts us to make an investigation of the number of constraints derivable from the equations of motion when the coupling to a general electromagnetic field is present. This analysis is presented in Sec. III, where we show that there is in fact a loss of constraints except when the external field happens to be purely electric. This difference between the case when the external field is purely electric ($\vec{\mathcal{H}}=0$) and the case when $\vec{\mathcal{H}}\neq 0$ has interesting repercussions: When one tries to quantize the field using Schwinger's action principle formalism, one finds that for $\vec{\mathcal{H}}\neq 0$ the anticommutators are mutually consistent and local (though they become of indefinite sign for large $\vec{\mathcal{H}}$) while if $\vec{\mathcal{H}}=0$ some of the anticommutators become nonlocal. The contribution of the secondary constraints when $\vec{\mathcal{H}}=0$ to the generator in the Schwinger formalism is responsible for this phenomenon, as will be seen in Sec. IV where the quantization problem is dealt with. In Sec. V we explore the possibility that the introduction of nonminimal interaction terms might help to remove the above difficulties. With a general interaction linear in the electromagnetic field components $F^{\mu\nu}$, we find that there is no possibility of avoiding loss of constraints when a nonvanishing magnetic field is present. The last section (Sec. VI) is devoted to a brief discussion of the results against the background of earlier work and with special reference to a recent work of Singh and Hagen.[20]

II. SOLUTION OF GLASS'S EQUATION WITH AN hmf

Glass's equation for spin-$\frac{3}{2}$ particles employs a 20-component wave function ψ, which is equivalent in its transformation properties to a vector-spinor together with a Dirac spinor. The equation (with minimal electromagnetic coupling), when written in the standard form

$$(\beta\cdot\pi - m)\psi = 0\,, \tag{1}$$

involves matrices β_0 and $\vec{\beta}$ of dimension 20. These matrices have been so chosen as to satisfy the minimal equation $\beta_0^3(\beta_0^2-1)=0$ violating the Umezawa-Visconti condition. The explicit representation used by Glass may be found in Ref. 18. For our purposes it will be convenient to use a different representation related to the specific forms $\beta_\mu^{(G)}$ used by Glass by the transformation

$$\beta_\mu = \alpha\beta_\mu^{(G)}\alpha^{-1}\,, \tag{2}$$

with

$$\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & A \end{bmatrix}, \tag{3}$$

where A is a 6×6 matrix and 1 stands for the unit matrix of dimension 4. The matrix A is composed of 2×2 blocks which are multiples of the unit matrix dimension 2. With this understanding, the partitioned form of A is

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}. \tag{4}$$

The explicit form of β_0 which we obtain from Eq. (2) is

$$\beta_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\tilde{B} \end{bmatrix}, \tag{5}$$

where the partitioning is of the same type as in Eq. (3), and[21]

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{6}$$

with each element standing for a 2×2 unit matrix. Using this and the corresponding forms of the β_i, we write down Glass's equation (1) as the set of Eqs. (7) wherein the 20-component ψ is broken up into two four-component parts, φ_1 and χ_1, and six two-component parts, $\varphi_2, \varphi_3, \varphi_4, \chi_2, \chi_3, \chi_4$:

$$(\pi_0 - m)\varphi_1 - \tfrac{2}{3}\vec{\Sigma}\cdot\vec{\pi}\chi_1 + \frac{1}{\sqrt{2}}\vec{u}\cdot\vec{\pi}(\chi_3+\chi_4) = 0\,, \tag{7a}$$

$$\pi_0(\varphi_3+\varphi_4) - m\varphi_2 + \tfrac{1}{3}\vec{\sigma}\cdot\vec{\pi}(\chi_2-\chi_3-2\chi_4) = 0\,, \tag{7b}$$

$$\pi_0\varphi_4 - m\varphi_3 + \frac{1}{\sqrt{2}}\vec{u}^{\dagger}\cdot\vec{\pi}\chi_1 - \tfrac{1}{3}\vec{\sigma}\cdot\vec{\pi}(\chi_2+2\chi_4) = 0\,, \tag{7c}$$

$$m\varphi_4 - \frac{1}{\sqrt{2}}\vec{u}^{\dagger}\cdot\vec{\pi}\chi_1 + \tfrac{2}{3}\vec{\sigma}\cdot\vec{\pi}(\chi_2+\chi_3) = 0\,, \tag{7d}$$

$$(\pi_0+m)\chi_1 - \tfrac{2}{3}\vec{\Sigma}\cdot\vec{\pi}\varphi_1 - \frac{1}{\sqrt{2}}\vec{u}\cdot\vec{\pi}(\varphi_2+\varphi_3) = 0\,, \tag{7e}$$

$$m\chi_2 - \frac{1}{\sqrt{2}}\vec{u}^{\dagger}\cdot\vec{\pi}\varphi_1 - \tfrac{2}{3}\vec{\sigma}\cdot\vec{\pi}(\varphi_3+\varphi_4) = 0\,, \tag{7f}$$

$$\pi_0\chi_2 + m\chi_3 - \frac{1}{\sqrt{2}}\vec{u}^{\dagger}\cdot\vec{\pi}\varphi_1 - \tfrac{1}{3}\vec{\sigma}\cdot\vec{\pi}(2\varphi_2+\varphi_4) = 0\,, \tag{7g}$$

$$\pi_0(\chi_2+\chi_3) + m\chi_4 - \tfrac{1}{3}\vec{\sigma}\cdot\vec{\pi}(2\varphi_2+\varphi_3-\varphi_4) = 0\,. \tag{7h}$$

Here Σ_i are the spin-$\frac{3}{2}$ representation of angular momentum matrices while the σ_i are the Pauli matrices. The u_i are rectangular matrices given by

$$u_1 = \frac{i\sqrt{2}}{3}\begin{bmatrix}\sqrt{3} & 0\\ 0 & 1\\ -1 & 0\\ 0 & -\sqrt{3}\end{bmatrix},\quad u_2 = \frac{\sqrt{2}}{3}\begin{bmatrix}\sqrt{3} & 0\\ 0 & 1\\ 1 & 0\\ 0 & \sqrt{3}\end{bmatrix},$$
$$u_3 = \frac{-2i\sqrt{2}}{3}\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}. \tag{8}$$

It is the set of Eqs. (7) which we now seek to solve, taking the external field to be a constant homogeneous magnetic field.

To obtain the solution we use the general method developed by one of us in Ref. 6, and applied later to various relativistic wave equations of higher-spin particles in an hmf. Choosing the direction of the magnetic field $\mathcal{H}$ to be the z axis, we take $A_3 = A_0 = 0$ and seek solutions having the time dependence e^{-iEt} and belonging to the eigenvalue p_3 $(-\infty < p_3 < \infty)$ for $\pi_3 \equiv -i\partial_3$. This means that π_0 and π_3 will be replaced by the numerical parameters $-E$ and p_3 in the equations of motion. Further, we form the combinations of π_1 and π_2, namely,

$$a = (2e\mathcal{H})^{-1/2}\pi_+ \quad\text{and}\quad a^{\dagger} = (2e\mathcal{H})^{-1/2}\pi_- \tag{9}$$

$$(\pi_\pm = \pi_1 \pm i\pi_2)\,,$$

which are easily seen to obey the algebra of the annihilation and creation operators of a harmonic oscillator.

We now define the four-component column states $|n, s_3\rangle$ which are simultaneous eigenstates of the "number operator"

$$N = a^{\dagger}a \tag{10}$$

and the third component of the spin operator $\vec{\Sigma}$

$$N|n, s_3\rangle = n|n, s_3\rangle\,, \tag{11a}$$

$$\Sigma_3|n, s_3\rangle = s_3|n, s_3\rangle\,, \tag{11b}$$

where $s_3 = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$. The operators $\Sigma_\pm = \frac{1}{2}(\Sigma_1 \pm i\Sigma_2)$ act as raising and lowering operators for the quantum number s_3 in the well-known manner,

$$\Sigma_\pm|n, s_3\rangle = \tfrac{1}{2}[(\tfrac{3}{2} \mp s_3)(\tfrac{3}{2} \pm s_3 + 1)]^{1/2}|n, s_3 \pm 1\rangle\,. \tag{12}$$

Let us also define two-component column states $|n, \pm\rangle$ which are simultaneous eigenstates of N and σ_3:

$$N|n, \pm\rangle = n|n, \pm\rangle\,, \tag{13a}$$

$$\sigma_3|n, \pm\rangle = \pm|n, \pm\rangle\,. \tag{13b}$$

The effect of $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ on these is given by

$$\sigma_+|n,+\rangle = \sigma_-|n,-\rangle = 0\,,$$
$$\sigma_+|n,-\rangle = |n,+\rangle\,,\quad \sigma_-|n,+\rangle = |n,-\rangle\,. \tag{14}$$

The rectangular matrices u_i and their adjoints $u_i^{\dagger}$ connect the states (11) and (13):

$$u_+|n,+\rangle = \frac{i\sqrt{2}}{\sqrt{3}}|n, \tfrac{3}{2}\rangle\,,\quad u_+|n,-\rangle = \frac{i\sqrt{2}}{3}|n, \tfrac{1}{2}\rangle\,,$$
$$u_-|n,+\rangle = -\frac{i\sqrt{2}}{3}|n, -\tfrac{1}{2}\rangle\,,\quad u_-|n,-\rangle = \frac{-i\sqrt{2}}{\sqrt{3}}|n, -\tfrac{3}{2}\rangle\,,$$
$$(u^{\dagger})_+|n, \tfrac{3}{2}\rangle = (u^{\dagger})_+|n, \tfrac{1}{2}\rangle = (u^{\dagger})_-|n, -\tfrac{1}{2}\rangle = (u^{\dagger})_-|n, -\tfrac{3}{2}\rangle = 0\,, \tag{15}$$
$$(u^{\dagger})_+|n, -\tfrac{1}{2}\rangle = \frac{i\sqrt{2}}{3}|n,+\rangle\,,\quad (u^{\dagger})_+|n, -\tfrac{3}{2}\rangle = \frac{i\sqrt{2}}{\sqrt{3}}|n,-\rangle\,,$$
$$(u^{\dagger})_-|n, \tfrac{3}{2}\rangle = \frac{-i\sqrt{2}}{\sqrt{3}}|n,+\rangle\,,\quad (u^{\dagger})_-|n, -\tfrac{1}{2}\rangle = \frac{-i\sqrt{2}}{3}|n,-\rangle\,.$$

Here,

$$u_\pm = \tfrac{1}{2}(u_1 \pm iu_2)\,.$$

In view of Eqs. (12), (14), and (15) as well as the well-known effect of a and $a^{\dagger}$ on the eigenstates of N, one can easily conclude by inspection of Eqs. (7) that their general solution must have the following form:

$$\varphi_1 = f_1|n, \tfrac{3}{2}\rangle + f_2|n-2, -\tfrac{1}{2}\rangle + g_1|n-3, -\tfrac{3}{2}\rangle + g_2|n-1, \tfrac{1}{2}\rangle\,, \tag{16a}$$

$$\varphi_2 = f_3|n-2, -\rangle + g_3|n-1, +\rangle\,, \tag{16b}$$

$$\varphi_3 = f_4|n-2,-\rangle + g_4|n-1,+\rangle, \tag{16c}$$

$$\varphi_4 = f_5|n-2,-\rangle + g_5|n-1,+\rangle, \tag{16d}$$

$$\chi_1 = f_6|n-3,-\tfrac{3}{2}\rangle + f_7|n-1,\tfrac{1}{2}\rangle + g_6|n,\tfrac{3}{2}\rangle + g_7|n-2,-\tfrac{1}{2}\rangle, \tag{16e}$$

$$\chi_2 = f_8|n-1,+\rangle + g_8|n-2,-\rangle, \tag{16f}$$

$$\chi_3 = f_9|n-1,+\rangle + g_9|n-2,-\rangle, \tag{16g}$$

$$\chi_4 = f_{10}|n-1,+\rangle + g_{10}|n-2,-\rangle \tag{16h}$$

$$(n = 3, 4, 5, \ldots).$$

The coefficients $f_1, f_2, \ldots, f_{10}$ and $g_1, g_2, \ldots, g_{10}$ here are parameters which are as yet undetermined. For any $n<3$ those terms in Eqs. (16) wherein the eigenvalue becomes negative [e.g., the third term in the right-hand side of (16a)] are to be dropped. For example, with $n=1$, Eqs. (16) reduce to

$$\begin{aligned}
&\varphi_1 = f_1|1,\tfrac{3}{2}\rangle + g_2|0,\tfrac{1}{2}\rangle,\\
&\varphi_2 = g_3|0,+\rangle, \quad \varphi_3 = g_4|0,+\rangle, \quad \varphi_4 = g_5|0,+\rangle,\\
&\chi_1 = f_7|0,\tfrac{1}{2}\rangle + g_6|1,\tfrac{3}{2}\rangle,\\
&\chi_2 = f_8|0,+\rangle, \quad \chi_3 = f_9|0,+\rangle, \quad \chi_4 = f_{10}|0,+\rangle.
\end{aligned} \tag{17}$$

Returning to the solution of Eqs. (7) we observe now that the substitution of the forms (16a)–(16h) in (7a)–(7h) followed by the use of Eqs. (12), (14), and (15) yields a set of linear equations for the f_i and g_i when the coefficients of linearly independent states on the left-hand side are equated to zero. With $p_3 = 0$, the f_i get completely decoupled from the g_i in the equations. The equations for the f_i are, in matrix form,

$$Df = 0, \tag{18}$$

where the matrix D is given by

$$\begin{bmatrix}
\epsilon-1 & 0 & 0 & 0 & 0 & -\frac{\rho_n}{\sqrt{3}} & 0 & 0 & \frac{i\rho_n}{\sqrt{3}} & \frac{i\rho_n}{\sqrt{3}}\\
0 & \epsilon-1 & 0 & 0 & 0 & -\frac{2\rho_{n-1}}{3} & -\frac{\rho_{n-2}}{\sqrt{3}} & 0 & -\frac{i\rho_{n-1}}{3} & -\frac{i\rho_{n-1}}{3}\\
0 & 0 & -1 & \epsilon & \epsilon & 0 & 0 & \frac{\rho_{n-1}}{3} & -\frac{\rho_{n-1}}{3} & -\frac{2\rho_{n-1}}{3}\\
0 & 0 & 0 & -1 & \epsilon+1 & 0 & 0 & \frac{\rho_{n-1}}{3} & \frac{2\rho_{n-1}}{3} & -\frac{2\rho_{n-1}}{3}\\
0 & 0 & 0 & 0 & 1 & \frac{i\rho_{n-1}}{3} & -\frac{i\rho_{n-2}}{\sqrt{3}} & \frac{2\rho_{n-1}}{3} & \frac{2\rho_{n-1}}{3} & 0\\
-\frac{\rho_n}{\sqrt{3}} & -\frac{2\rho_{n-1}}{3} & -\frac{i\rho_{n-1}}{3} & -\frac{i\rho_{n-1}}{3} & 0 & \epsilon+1 & 0 & 0 & 0 & 0\\
0 & -\frac{\rho_{n-2}}{\sqrt{3}} & \frac{i\rho_{n-2}}{\sqrt{3}} & \frac{i\rho_{n-2}}{\sqrt{3}} & 0 & 0 & \epsilon+1 & 0 & 0 & 0\\
0 & 0 & -\frac{2\rho_{n-1}}{3} & -\frac{\rho_{n-1}}{3} & \frac{\rho_{n-1}}{3} & 0 & 0 & \epsilon & \epsilon & 1\\
0 & 0 & -\frac{2\rho_{n-1}}{3} & \frac{2\rho_{n-1}}{3} & \frac{\rho_{n-1}}{3} & 0 & 0 & \epsilon-1 & 1 & 0\\
\frac{i\rho_n}{\sqrt{3}} & -\frac{i\rho_{n-1}}{3} & 0 & -\frac{2\rho_{n-1}}{3} & -\frac{2\rho_{n-1}}{3} & 0 & 0 & 1 & 0 & 0
\end{bmatrix} \tag{19}$$

and $f = \mathrm{col}(f_1, f_2, \ldots, f_{10})$. In (19),

$$\rho_n = (2ne\mathcal{H}/m^2)^{1/2} \text{ and } \epsilon = (E/m). \tag{20}$$

When there is no external field ($\mathcal{H}=0$), $\det D$ reduces to $(\epsilon^2-1)^2$, and it follows that there are four independent solutions for (18), corresponding to the four roots of $|D(\epsilon)|=0$. Four more solutions emerge from the equations for the g's, so that the total number of solutions does tally with the expected number $2(2s+1)$ with $s=\frac{3}{2}$.

But when a magnetic field is present, the situation turns out to be different: it is easy to verify from (19) that for $\mathcal{H}\neq 0$, $\det D$ is a *cubic* in ϵ^2. Consequently one gets six solutions from the f equations (and six more from the g equations), so that the number of independent solutions *exceeds* the number of spin degrees of freedom of a spin-$\frac{3}{2}$ particle.

In addition to this anomaly, one encounters the further problem that complex values of ϵ arise for some range of values of $\mathcal{H}$ (the range being dependent on the particular state considered). This fact can be seen readily by examining the special solutions of the type (17). The equation for ϵ in this case separates into

$$\eta^2\epsilon^3 + (1-2\eta^2)\epsilon^2 - 2\eta^2\epsilon - 1 = 0 \tag{21}$$

(where $\eta^2 = 2e\mathcal{H}/3m^2$) and another equation differing from the above by the replacement of ϵ by $-\epsilon$. It is a simple exercise to verify that (21) does have complex solutions for $0<\eta\leq 1$.

III. DERIVATION OF THE CONSTRAINTS IN THE PRESENCE OF MINIMAL COUPLING TO THE EXTERNAL ELECTROMAGNETIC FIELD

We return now to the question of the number of degrees of freedom. Since the basic Eqs. (1) are first-order equations for the twenty independent components of the wave function ψ, it would appear *a priori* that there are 20 initial data (the initial values of ψ) to be specified. However, for a particle with spin $s=\frac{3}{2}$, one has only $2(2s+1)=8$ independent degrees of freedom available, and therefore 12 constraints are needed. These constraints should follow from the basic Eqs. (1), and in fact they do as long as there are no external fields. However, the results of the last section indicate that the number of degrees of freedom is more than the requisite when a magnetic field is present (however weak it may be). This would imply that there is a deficiency in the number of constraints. An examination of the constraints following from Eq. (1) seems therefore to be called for, and we proceed to do this now.[22]

Considering the set of Eqs. (7a)–(7h) we observe the evident fact that not all of them are equations of motion. In fact, (7d) and (7f) are constraints, since they do not involve the time derivatives of any of the components of the wave function. Since these constraints follow from the singular nature of β_0, we shall call them "primary" constraints, following Johnson and Sudarshan.[1] The number of such constraints is four, since each of (7d) and (7f) involves a two-component entity.

To obtain the further constraints in the theory, we have to differentiate the primary constraints with respect to time, and see if the time derivatives of the various components can be eliminated using the equations of motion. We note that the primary constraints involve only those components for which we have equations of motion. The secondary constraints which follow on eliminating the time derivatives of these after differentiating (7d) and (7f) are

$$\begin{aligned}&-\frac{\sqrt{2}}{3}(\vec{\mathrm{u}}^{\dagger}\cdot\vec{\pi})(\vec{\Sigma}\cdot\vec{\pi})\varphi_1 - \tfrac{2}{3}e(\vec{\sigma}\cdot\vec{\mathcal{H}})\varphi_2\\ &\quad-\tfrac{2}{9}(\vec{\pi}^2+2e\vec{\sigma}\cdot\vec{\mathcal{H}}-\tfrac{9}{2}m^2)\varphi_3-\tfrac{2}{9}(\vec{\sigma}\cdot\vec{\pi})^2\varphi_4\\ &\quad-\frac{ie}{\sqrt{2}}\vec{\mathrm{u}}^{\dagger}\cdot\vec{\mathcal{E}}\chi_1+\tfrac{1}{3}(m\vec{\sigma}\cdot\vec{\pi}+2ie\vec{\sigma}\cdot\vec{\mathcal{E}})\chi_2+\tfrac{2}{3}ie\vec{\sigma}\cdot\vec{\mathcal{E}}\chi_3=0\,,\end{aligned} \tag{22}$$

$$\begin{aligned}&-\frac{ie}{\sqrt{2}}\vec{\mathrm{u}}^{\dagger}\cdot\vec{\mathcal{E}}\varphi_1-\tfrac{2}{3}ie\vec{\sigma}\cdot\vec{\mathcal{E}}\varphi_3+\tfrac{1}{3}(m\vec{\sigma}\cdot\vec{\pi}-2ie\vec{\sigma}\cdot\vec{\mathcal{E}})\varphi_4\\ &\quad-\frac{\sqrt{2}}{3}(\vec{\mathrm{u}}^{\dagger}\cdot\vec{\pi})(\vec{\Sigma}\cdot\vec{\pi})\chi_1+\tfrac{2}{9}(\vec{\sigma}\cdot\vec{\pi})^2\chi_2\\ &\quad+\tfrac{2}{9}(\vec{\pi}^2+2e\vec{\sigma}\cdot\vec{\mathcal{H}}-\tfrac{9}{2}m^2)\chi_3+\tfrac{2}{3}e\vec{\sigma}\cdot\vec{\mathcal{H}}\chi_4=0\,.\end{aligned} \tag{23}$$

(Here $\vec{\mathcal{E}}$ is the electric field.) In arriving at these forms we have made use of the relations[18]

$$(\vec{\mathrm{u}}^{\dagger}\cdot\vec{\pi})(\vec{\mathrm{u}}\cdot\vec{\pi})=\tfrac{8}{9}\vec{\pi}^2+\tfrac{4}{9}e\vec{\sigma}\cdot\vec{\mathcal{H}}\,, \tag{24a}$$

$$(\vec{\sigma}\cdot\vec{\pi})^2=\vec{\pi}^2-e\vec{\sigma}\cdot\vec{\mathcal{H}}. \tag{24b}$$

Each of the set of Eqs. (22) and (23) is a set of two equations, so that we have obtained four secondary constraints. For the correct description of a spin-$\frac{3}{2}$ particle, we need four more constraints which ought to be obtained from the secondary constraints by time differentiation. However, this cannot be done. The reason is simply that the two quantities φ_2 and χ_4, *for which we have no equations of motion*, appear in (22) and (23) and therefore $\pi_0\varphi_2$ and $\pi_0\chi_4$ cannot be eliminated after time differentiation of (22) and (23). It is intersting to note that the terms $\pi_0\varphi_2$ and $\pi_0\chi_4$ contain the factor $(\vec{\sigma}\cdot\vec{\mathcal{H}})$, so the "loss" of constraints arises only in the presence of a *magnetic field*. In the free field case, or in the case when *only an electric field is present*, these drop out causing these equations to become constraints. In such circumstances the total number of constraints is just what is needed for the theory to describe a spin-$\frac{3}{2}$ particle.

IV. QUANTIZATION USING THE ACTION PRINCIPLE

In the last section we showed explicitly that the tertiary constraints are lost the moment an interaction with an external magnetic field is introduced. This prompts one to ask: In what manner will the loss of constraints be reflected in the nature of anticommutators if one attempts to quantize the theory? This question assumes enormous significance in the context of a recent work of

Singh and Hagen[20] who claim that in the light-front coordinate formalism, the RS equation with minimal electromagnetic interaction exhibits a "loss of constraints" and what is more, that with the reduced number of constraints, the anticommutators between the various field components (at equal values of light-front coordinates) turn out to be mutually inconsistent when the quantization is performed using Schwinger's action principle procedure. It becomes relevant therefore to quantize Glass's theory also by the same procedure, to see whether the loss of constraints (at fixed time in the present case) leads to mutual inconsistencies between equal-time anticommutators of various field components.

We shall show in this section that mutual inconsistencies of the type reported by Singh and Hagen[20] do not arise in the case of Glass's equation even in a situation ($\mathcal{H}\neq 0$) where there are fewer constraints than needed. On the contrary, one gets the curious result that when a nonvanishing electric field alone is present (no magnetic field), which is a situation wherein the number of constraints is just right, some of the anticommutators of the field components become nonlocal.

A. Quantization with $\mathcal{H}\neq 0$

We recall that in Schwinger's action principle method the generator G of infinitesimal field transformations is obtained from the time-derivative terms in the Lagrangian[23]

$$G=\frac{i}{2}\int d^3x(\psi^\dagger\eta\beta_0\delta\psi+\text{H.c.})$$
$$=\frac{i}{2}\int d^3x\mathcal{G}(x) \tag{25}$$

where $\eta\equiv\alpha^\dagger\eta^{(G)}\alpha$, $\eta^{(G)}$ being the "Hermitianizing matrix" in the representation used by Glass[18]:

$$\eta^{(G)}=\begin{bmatrix}1&&&\\&\rho&&\\&&-1&\\&&&-\rho\end{bmatrix}\quad\text{with }\rho=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}. \tag{26}$$

The partitioning here is same as in Eqs. (3) and (4) which define α. On using the explicit forms of β_0 and ψ in Sec. II, the generator G reduces to

$$G=\frac{i}{2}\int d^3x[\varphi_1^\dagger\delta\varphi_1+\chi_1^\dagger\delta\chi_1-(\varphi_3^\dagger+\varphi_4^\dagger)\delta\varphi_4$$
$$-\varphi_4^\dagger\delta\varphi_3-(\chi_2^\dagger+\chi_3^\dagger)\delta\chi_2-\chi_2^\dagger\delta\chi_3+\text{H.c.}]. \tag{27}$$

The commutation relations of the theory are to be determined from

$$[\Omega(x),G]=\frac{i}{2}\delta\Omega(x), \tag{28}$$

where $\Omega(x)$ represents any one of the field operators $\varphi_1,\varphi_3,\varphi_4$ or χ_1,χ_2,χ_3. On account of the primary constraints (7d) and (7f), the variations which appear in the generator are not independent, but are subject to the conditions

$$Z_1\equiv m\delta\varphi_4-\frac{1}{\sqrt{2}}\vec{u}^\dagger\cdot\vec{\pi}\delta\chi_1+\tfrac{2}{3}\vec{\sigma}\cdot\vec{\pi}(\delta\chi_2+\delta\chi_3)=0 \tag{29}$$

and

$$Z_2\equiv m\delta\chi_2-\frac{1}{\sqrt{2}}\vec{u}^\dagger\cdot\vec{\pi}\delta\varphi_1-\tfrac{2}{3}\vec{\sigma}\cdot\vec{\pi}(\delta\varphi_3+\delta\varphi_4)=0. \tag{30}$$

The most convenient way of handling these constraints is by the technique of Lagrange multipliers.[24] Thus for the choice $\Omega(x)\equiv\varphi_1(x)$, one writes (28) as

$$\int d^3x'\{[\varphi_1(x),\mathcal{G}'(x')]-\delta\varphi_1(x')\delta(\vec{x}-\vec{x}')$$
$$-\lambda_1(x,x')Z_1-\lambda_2(x,x')Z_2\}=0 \tag{31}$$

and considers all variations to be independent. Here $\lambda_1(x,x')$ and $\lambda_2(x,x')$ are to be determined such that (7d) and (7f) are satisfied. The result of this calculation is that

$$\lambda_1(x,x')=0 \tag{32}$$

and

$$\lambda_2(x,x')=-\frac{3}{2\sqrt{2}e}\delta(\vec{x}-\vec{x}')\vec{u}\cdot\vec{\pi}'\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right), \tag{33}$$

where π'_μ is the operator $i(\partial/\partial x'^\mu)+eA_\mu(x')$, *defined* to act on all functions occurring to the left. Then the equal-time anticommutation relations are

$$\{\varphi_1(x),\varphi_1^\dagger(x')\}=\delta(\vec{x}-\vec{x}')\left[1+\frac{3}{4e}\vec{u}\cdot\vec{\pi}'\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right)\vec{u}^\dagger\cdot\vec{\pi}'\right], \tag{34}$$

$$\{\varphi_1(x),\chi_1^\dagger(x')\}=\{\varphi_1(x),\chi_2^\dagger(x')\}$$
$$=\{\varphi_1(x),\varphi_3^\dagger(x')\}=0, \tag{35}$$

$$\{\varphi_1(x),\varphi_4^\dagger(x')\}=-\frac{1}{e\sqrt{2}}\delta(\vec{x}-\vec{x}')\vec{u}\cdot\vec{\pi}'\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right)\vec{\sigma}\cdot\vec{\pi}', \tag{36}$$

$$\{\varphi_1(x),\chi_3^\dagger(x')\}=-\frac{3m}{2\sqrt{2}e}\delta(\vec{x}-\vec{x}')\vec{u}\cdot\vec{\pi}'\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right). \tag{37}$$

Similarly, with the choices $\Omega(x)=\chi_1(x)$, $\Omega(x)=\varphi_3(x)$, and $\Omega(x)=\varphi_4(x)$, the anticommutation relations may be found to be

$$\{\chi_1(x),\varphi_1^\dagger(x')\}=\{\chi_1(x),\varphi_4^\dagger(x')\}$$
$$=\{\chi_1(x),\chi_3(x')\}=0, \tag{38}$$

$$\{\chi_1(x), \chi_1^\dagger(x')\} = \delta(\vec{x}-\vec{x}')\left[1+\frac{3}{4e}\vec{u}\cdot\vec{\pi}'\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right)\vec{u}^\dagger\cdot\vec{\pi}'\right], \tag{39}$$

$$\{\chi_1(x), \varphi_3^\dagger(x')\} = -\frac{3m}{2\sqrt{2}e}\delta(\vec{x}-\vec{x}')\vec{u}\cdot\vec{\pi}'\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right), \tag{40}$$

$$\{\chi_1(x), \chi_2^\dagger(x')\} = \frac{1}{e\sqrt{2}}\delta(\vec{x}-\vec{x}')\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right)\vec{\sigma}\cdot\vec{\pi}', \tag{41}$$

$$\{\varphi_3(x), \varphi_1^\dagger(x')\} = \{\varphi_3(x), \chi_3^\dagger(x')\} = 0, \tag{42}$$

$$\{\varphi_3(x), \chi_1^\dagger(x')\} = -\frac{3m}{2\sqrt{2}e}\delta(\vec{x}-\vec{x}')\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right)\vec{u}^\dagger\cdot\vec{\pi}', \tag{43}$$

$$\{\varphi_3(x), \varphi_4^\dagger(x')\} = -\delta(x-x'), \tag{44}$$

$$\{\varphi_3(x), \varphi_3^\dagger(x')\} = \delta(x-x')\left[1+\frac{3m^2}{2e}\left(\frac{\sigma\cdot\mathcal{H}}{\mathcal{H}^2}\right)\right], \tag{45}$$

$$\{\varphi_3(x), \chi_2^\dagger(x')\} = -\frac{m}{e}\delta(x-x')\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right)\vec{\sigma}\cdot\vec{\pi}', \tag{46}$$

$$\{\varphi_4(x), \chi_1^\dagger(x')\} = \{\varphi_4(x), \chi_2^\dagger(x')\} = 0, \tag{47}$$

$$\{\varphi_4(x), \varphi_1^\dagger(x')\} = -\frac{1}{e\sqrt{2}}\delta(x-x')\vec{\sigma}\cdot\vec{\pi}'\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right)\vec{u}^\dagger\cdot\vec{\pi}', \tag{48}$$

$$\{\varphi_4(x), \varphi_3^\dagger(x')\} = -\delta(\vec{x}-\vec{x}'), \tag{49}$$

$$\{\varphi_4(x), \varphi_4^\dagger(x')\} = \frac{2}{3e}\delta(\vec{x}-\vec{x}')\vec{\sigma}\cdot\vec{\pi}'\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right)\vec{\sigma}\cdot\vec{\pi}', \tag{50}$$

$$\{\varphi_4(x), \chi_3^\dagger(x')\} = \frac{m}{e}\delta(\vec{x}-\vec{x}')\vec{\sigma}\cdot\vec{\pi}'\left(\frac{\vec{\sigma}\cdot\vec{\mathcal{H}}}{\mathcal{H}^2}\right). \tag{51}$$

It is readily verified that the above anticommutation relations form a mutually consistent set. This is in contrast to what has been found by Singh and Hagen in the case of RS theory, quantized in light-front coordinates, which shares with Glass's theory (with nonzero magnetic field) the property of being deficient in the number of constraints. In fact, it is in a situation with the correct number of constraints, curiously enough, that new troubles arise in Glass's theory. This is the case for $\vec{\mathcal{H}}=0$; we show in the next subsection that when $\vec{\mathcal{H}}=0$ but $\vec{\mathcal{E}}\neq 0$, the expressions for the anticommutators become nonlocal. It may be noted, incidentally, that expressions (34) to (51) for the anticommutators become singular in the limit $\vec{\mathcal{H}}\to 0$. However, to get the correct anticommutators when $\vec{\mathcal{H}}=0$, one has to perform the quantization afresh, taking into account the extra constraints (compared to $\vec{\mathcal{H}}\neq 0$) which exist in this case. This is done in the next subsection.

Incidentally, it may be observed, from Eq. (45) for example, that the anticommutators suffer also from the Johnson-Sudarshan problem of indefinite sign.

B. Quantization in the presence of a pure electric field

We have stated in Sec. III that if the external field is purely electric, the number of constraints is the same as in the free case. This means that in working out the anticommutators in this case, the secondary constraints (which did not play any role in the case of $\mathcal{H}\neq 0$) should also be taken into account. This is because with setting of $\mathcal{H}=0$, the variations $\delta\varphi_2$ and $\delta\chi_4$ which do not appear in G drop out from these constraint equations. The additional constraints on the variations which result thus are

$$\begin{aligned} Z_3 \equiv & -\frac{\sqrt{2}}{3}\vec{u}^\dagger\cdot\vec{\pi}\vec{\Sigma}\cdot\vec{\pi}\delta\varphi_1 - \left(\tfrac{2}{9}\vec{\pi}^2 - m^2\right)\delta\varphi_3 \\ & -\tfrac{2}{9}\vec{\pi}^2\delta\varphi_4 - \frac{1}{\sqrt{2}}ie\vec{u}^\dagger\cdot\vec{\mathcal{E}}\delta\chi_1 \\ & +\tfrac{1}{3}(m\vec{\sigma}\cdot\vec{\pi} + 2ie\vec{\sigma}\cdot\vec{\mathcal{E}})\delta\chi_2 + \tfrac{2}{3}ie\vec{\sigma}\cdot\vec{\mathcal{E}}\delta\chi_3 = 0 \end{aligned} \tag{52}$$

and

$$\begin{aligned} Z_4 \equiv & -\frac{ie}{\sqrt{2}}\vec{u}^\dagger\cdot\vec{\mathcal{E}}\delta\varphi_1 - \tfrac{2}{3}ie\vec{\sigma}\cdot\vec{\mathcal{E}}\delta\varphi_3 \\ & +\tfrac{1}{3}(m\vec{\sigma}\cdot\vec{\pi} - 2ie\vec{\sigma}\cdot\vec{\mathcal{E}})\delta\varphi_4 + \tfrac{2}{9}\vec{\pi}^2\delta\chi_2 \\ & +\left(\tfrac{2}{9}\vec{\pi}^2 - m^2\right)\delta\chi_3 - \frac{\sqrt{2}}{3}\vec{u}^\dagger\cdot\vec{\pi}\vec{\Sigma}\cdot\vec{\pi}\delta\chi_1. \end{aligned} \tag{53}$$

The determination of anticommutators proceeds along the same lines as before, except that one has to take into account Eqs. (52) and (53) also with the aid of two more Lagrange multipliers $\lambda_3(x,x')$ and $\lambda_4(x,x')$. One gets thus, for example,

$$\begin{aligned} \{\varphi_1(x), \varphi_1^\dagger(x')\} = & \delta(x-x') - \frac{\lambda_2(x,x')}{\sqrt{2}}\vec{u}^\dagger\cdot\vec{\pi}' \\ & +\frac{\sqrt{2}}{3}\lambda_3(x,x')\vec{u}^\dagger\cdot\vec{\pi}'\vec{\Sigma}\cdot\vec{\pi}' \\ & +\frac{1}{\sqrt{2}}ie\lambda_4(x,x')\vec{u}^\dagger\cdot\vec{\mathcal{E}}. \end{aligned} \tag{54}$$

The Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ appearing in this and other anticommutators are to be determined, so that the anticommutators of the left-hand side of the constraint Eqs. (7d), (7f), (22), and (23)—taken with $\vec{\mathcal{H}}=0$—with all the $\varphi_i^\dagger$ and $\chi_i^\dagger$ vanish. This requirement leads to a set of equations for the λ's which can be solved:

$$\begin{aligned} \lambda_3(x,x') = & \frac{3}{4\sqrt{2}em}\delta(\vec{x}-\vec{x}')\vec{u}\cdot\vec{\pi}' \\ & \times\left(-\frac{i}{2}\vec{\sigma}\cdot\vec{\mathcal{E}} + \frac{9}{32em^2}\zeta\vec{\sigma}\cdot\vec{\mathcal{E}}\zeta\right)^{-1}, \end{aligned} \tag{55}$$

$$\lambda_4(x,x')=\frac{3i}{8em}\lambda_3(x,x')\left(\frac{\zeta\vec{\sigma}\cdot\vec{\mathcal{E}}}{\mathcal{E}^2}\right), \tag{56}$$

where

$$\zeta=m^3+\tfrac{2}{3}e\vec{\sigma}\cdot\vec{\mathcal{E}}\times\vec{\pi}'. \tag{57}$$

It is clear that on account of the nature of the operator in the large parentheses in (55), these expressions are nonlocal.

V. NONMINIMAL INTERACTION

Is it possible that by introducing nonminimal terms in the Lagrangian we can overcome the deficiencies of the minimally coupled theory? This question arises naturally in view of the example of the spin-2 theory,[25] wherein the constraints lost on introduction of minimal electromagnetic interaction were restored by the addition of the "Federbush term." We therefore examine now the effect of adding a nonminimal term

$$\frac{e}{m}\bar{\psi}S_{\mu\nu}F^{\mu\nu}\psi \tag{58}$$

to the Lagrangian with minimal coupling. There are ten second-rank independent antisymmetric tensors Hermitian with respect to the metric η of Eq. (26), which can be constructed from the β_μ and $S_{\mu\nu}$ in (58) will be taken to be an arbitrary linear combination of these tensors, which are given by

$$\begin{aligned}
T_1^{\mu\nu}&=i[\beta^\mu,\beta^\nu],\\
T_2^{\mu\nu}&=[\beta^\rho\beta_\rho,[\beta^\mu,\beta^\nu]],\\
T_3^{\mu\nu}&=i\{\beta^\rho\beta_\rho,[\beta^\mu,\beta^\nu]\},\\
T_4^{\mu\nu}&=i(\beta^\mu\beta^\rho\beta_\rho\beta^\nu-\beta^\nu\beta^\rho\beta_\rho\beta^\mu),\\
T_5^{\mu\nu}&=\{\beta^\mu,\beta_\rho\beta^\nu\beta^\rho\}-\{\beta^\nu,\beta_\rho\beta^\mu\beta^\rho\},\\
T_6^{\mu\nu}&=i\beta_\rho[\beta^\mu,\beta^\nu]\beta^\rho,\\
T_7^{\mu\nu}&=i[\beta^\mu,\beta_\rho\beta^\nu\beta^\rho]-i[\beta^\nu,\beta_\rho\beta^\mu\beta^\rho],\\
T_8^{\mu\nu}&=i[[\beta_\rho\beta^\rho,\beta^\mu],[\beta_\rho\beta^\rho,\beta^\nu]],\\
T_9^{\mu\nu}&=i[\{\beta_\rho\beta^\rho,\beta^\mu\},\{\beta_\rho\beta^\rho,\beta^\nu\}],\\
T_{10}^{\mu\nu}&=[[\beta^\rho\beta_\rho,\beta^\mu],\{\beta_\rho\beta^\rho,\beta^\nu\}]\\
&\quad+[\{\beta_\rho\beta^\rho,\beta^\mu\},[\beta_\rho\beta^\rho,\beta^\nu]].
\end{aligned} \tag{59}$$

It will suffice for our purpose to consider $F_{\mu\nu}$ in (58) to be a pure magnetic field since only the magnetic field enters into the terms in (22) and (23) which are responsible for the loss of constraints. On taking the direction of the magnetic field to be the z axis, $S_{\mu\nu}F^{\mu\nu}$ reduces to $2S_{12}F^{12}=2\mathcal{H}S_{12}$. Now it may be shown from the forms of the β matrices that a general linear combination of the tensors $T_i^{\mu\nu}$ leads to the following structure for S^{12}:

$$S_{12}=\begin{pmatrix} X & 0\\ 0 & X^\dagger\end{pmatrix}, \tag{60}$$

with

$$X=\begin{bmatrix}
\alpha_1\Sigma_3 & \sqrt{2}\alpha_2u_3 & \sqrt{2}\alpha_3u_3 & \sqrt{2}\alpha_4u_3\\
-\sqrt{2}\alpha_4^*u_3^\dagger & \frac{1}{3}\alpha_5\sigma_3 & \frac{1}{3}\alpha_6\sigma_3 & \frac{1}{3}\alpha_7\sigma_3\\
-\sqrt{2}\alpha_3^*u_3^\dagger & \frac{1}{3}\alpha_8\sigma_3 & \frac{1}{3}\alpha_9\sigma_3 & \frac{1}{3}\alpha_6^*\sigma_3\\
-\sqrt{2}\alpha_2^*u_3^\dagger & \frac{1}{3}\alpha_{10}\sigma_3 & \frac{1}{3}\alpha_8^*\sigma_3 & \frac{1}{3}\alpha_5^*\sigma_3
\end{bmatrix}. \tag{61}$$

The α_i $(i=1,2,3,\ldots,10)$ in (61) are certain complex linear combinations of the 10 arbitrary real parameters which are the coefficients of the various $T_i^{\mu\nu}$ in $S^{\mu\nu}$. The modified equations of motion including the nonminimal interaction terms may now be readily written down. Considering in particular Eq. (7d) which provided two of the primary constraints in the minimal case, we find that it is modified to

$$m\varphi_4-\frac{1}{\sqrt{2}}\vec{u}^\dagger\cdot\vec{\pi}\chi_1+\tfrac{2}{3}\vec{\sigma}\cdot\vec{\pi}(\chi_2+\chi_3)+\frac{\sqrt{2}e\mathcal{H}}{m}\alpha_2^*u_3^\dagger\varphi_1+\frac{e\mathcal{H}}{3m}\alpha_{10}\sigma_3\varphi_2+\frac{e\mathcal{H}}{3m}\alpha_8^*\sigma_3\varphi_3+\frac{e\mathcal{H}}{3m}\alpha_5^*\sigma_3\varphi_4=0. \tag{62}$$

While it still does not involve any time derivatives, it contains, unlike (7d), a term in the φ_2 for which no equation of motion exists at the primary stage. Therefore the time differentiation of (62) would lead to an equation of motion for φ_2 instead of yielding secondary constraints as before. There would then be loss of constraints even at the secondary stage unless

$$\alpha_{10}=0. \tag{63}$$

Requiring that this be so, we obtain on differentiation of (62) and use of the (modified) equations of motion for φ_1, φ_3, χ_1, χ_4, and $(\chi_2+\chi_3)$ the following secondary constraint equation:

$$\frac{e\mathcal{H}}{3}(\alpha_8+\alpha_8^*+2)\sigma_3\varphi_2+\frac{e^2\mathcal{H}^2}{m^2}[2\alpha_2^*\alpha_2u_3^\dagger u_3+\tfrac{1}{9}\alpha_8^*(\alpha_8-\alpha_5)\sigma_3{}^2-\tfrac{1}{9}\alpha_8\alpha_5^*\sigma_3{}^2]\varphi_2-\frac{e\mathcal{H}}{m}(\alpha_2u^\dagger\cdot\pi u_3-\alpha_2^*u_3^\dagger u\cdot\pi+\tfrac{2}{9}\alpha_5\sigma\cdot\pi\sigma_3-\tfrac{2}{9}\alpha_5^*\sigma_3\sigma\cdot\pi)\chi_4+\cdots=0. \tag{64}$$

Equations (24) have been used to arrive at this form. The dots in (64) are to indicate that a number of terms which are directly irrelevant to the present analysis have not been shown. [Equation (64) is the counterpart of Eq. (22) of the minimally coupled interacting case.] We have to see now whether, by a suitable choice of the parameters α_i, we can ensure that this equation—unlike Eq. (22)—can be made to yield the denied tertiary constraints by operating on it with π_0. This operation will result in a constraint equation only if (64) is free of terms in φ_2 and χ_4 (these being quantities for which we have no equations of motion). The condition for this to happen may readily be seen to be

$$1+\mathrm{Re}\alpha_8=0\,, \tag{65a}$$

$$\mathrm{Im}(4\alpha_2+\alpha_5)=0\,, \tag{65b}$$

$$\mathrm{Re}(2\alpha_2-\alpha_5)=0\,, \tag{65c}$$

$$16\alpha_2^*\alpha_2+\alpha_8^*\alpha_8-(\alpha_8^*\alpha_5+\alpha_5^*\alpha_8)=0\,. \tag{65d}$$

Unfortunately, these equations cannot be simultaneously satisfied. To see this we separate the α into their real and imaginary parts and write

$$\alpha_2=p+iq\,, \tag{66a}$$

$$\alpha_5=2p-4iq\,, \tag{66b}$$

$$\alpha_8=-1+it\,, \tag{66c}$$

using Eqs. (65a)–(65c). On substituting these in (65d) we obtain the relation

$$t^2+8qt+16(p^2+q^2)+4p+1=0\,. \tag{67}$$

Considered as a quadratic equation in t, this equation has the discriminant $D=-4(16p^2+4p+1)$, which is negative definite. Hence Eq. (67) possesses no real solutions for t. But t, by definition has to be real. Therefore the set of Eqs. (65) admits no solutions, and consequently one cannot obtain the tertiary constraints by the differentiation of the secondary constraint equation (64).

We find thus that it is impossible to avoid loss of constraints through the introduction of any nonminimal interaction terms linear in the $F_{\mu\nu}$. The behavior of Glass's theory for spin $\frac{3}{2}$ is in this respect worse than that of the spin-2 theory[25,26] where the Federbush term linear in $F_{\mu\nu}$ was sufficient to ensure preservation of the number of constraints.

VI. RESULTS AND DISCUSSION

We now recapitulate the results obtained. The solution of Glass's equation in interaction with an hmf shows that there is an increase in the number of independent degrees of freedom for Glass's particle when interaction with a magnetic field is introduced. The explicit analysis of Sec. III confirms that there is a deficiency in the number of constraints obtainable from the field equations, when $\vec{\mathcal{H}}\neq 0$. In addition to this difficulty which had so far not been encountered in theories for spin $s<2$, the appearance of complex energy modes in an hmf which was already seen to be a problem with the RS equation[6], persists in Glass's case also. Further, as in the RS theory with minimal electromagnetic coupling, the indefinite sign of anticommutators of field components makes its appearance in the Glass theory also when quantization is carried out (with the reduced number of constraints) using the Schwinger action principle approach. But the nonlocality of the anticommutators described in Sec. IV, seems to have no parallel in any of the relativistic theories studied so far.

It is pertinent here to remark on the recent work of Singh and Hagen[20] in the context of the above results. These authors find that for constant light-front coordinate (unlike for constant time) the RS equation has a deficiency of constraints and also suffers from mutual contradictions of the anticommutation rules among field components. The question whether these two types of problems are concomitants of each other is answered in the negative by our study which, in the case of Glass's equation, has revealed the anticommutators to be mutually consistent even in a situation ($\mathcal{H}\neq 0$) where there is a deficiency of constraints. It appears therefore that the use of light-front coordinates creates extra problems whose significance it is difficult to assess.

We have found in Sec. V that amelioration of the constraint problem is not possible through the introduction of any simple nonminimal interaction. The failure of this attempt in this case in contrast to the case of spin-2 can be understood in a heuristic fashion as follows. In the spin-2 theory of Ref. 26, the spin magnetic moment is 1 unit as shown by Hagen[26] and by Mathews, Seetharaman, and Prabhakaran.[7] This differs from the canonical value of $1/s$ (for a particle of spin s) conjectured by Belinfante.[27] We have noted in earlier work that the "optimal coupling[7]" is not in general the minimal but which leads to the canonical value $1/s$ for the spin magnetic moment. The Federbush term, in fact, does just this (bringing the magnetic moment to the value $\frac{1}{2}$). In Glass's theory for spin $\frac{3}{2}$, on the other hand, the minimal coupling is already optimal, in that the magnetic moment, as seen from the equation for the leading components of the wave function in the nonrelativistic limit, has the conjectured value $\frac{2}{3}$. One is therefore not surprised if the introduction of nonminimal terms does not help to ease the problem of an already optimal theory.

The dependence of the number of "spin" degrees of freedom on the nature of the external field (whether or not $\mathcal{H}=0$) seems to imply a breakdown of Lorentz invariance, since one can go by a suitable Lorentz transformation from a frame in which the external field is purely electric to one in which there is a magnetic field, and hence fewer constraints than in the original frame of reference. A similar observation has already been made in respect of the symmetric tensor theory[7] for spin 2 with minimal electromagnetic coupling.[28]

The results of the present study bring out clearly the fact that as the degree n of the minimal equation, $\beta_0^{n-2}(\beta_0^2-1)=0$, for β_0 increases, the types of inconsistencies arising on introducing interactions increase in number and variety. The Rarita-Schwinger equation which has $\beta_0^2(\beta_0^2-1)=0$ suffers from noncausal propagation at the classical level. Glass's equation, the β_0 of which obeys $\beta_0^3(\beta_0^2-1)=0$, is plagued by the further trouble of loss of constraints, nonlocal anticommutators, etc.

Recently it has been shown by Deser and Zumino[29] that in a supergravity theory of coupled massless fields of spin $\frac{3}{2}$ and spin 2 the problem of violation of causality does not arise. As for spin-$\frac{3}{2}$ fields minimally interacting with the electromagnetic field, the only equations known to us which are free of all troubles at the c-number level are the BG equation[8,13] and the Fisk-Tait equation[11,14] which are characterized by diagonalizable β_0 and do not have an irreducible mass/spin content. Quantization of such theories (which necessarily involve indefinite metric) has been recently attempted[17] and needs to be pursued further. On the other hand, the connection between the algebraic properties of the β matrices in unique-spin, unique-mass theories, and the various types of inconsistencies which arise in the presence of interactions in such theories need to be studied systematically. A study of this question has been undertaken, and we shall present some of the results in a sequel to this paper.

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[21]By a further similarity transformation one could reduce B to the Jordan from

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

but with this representation the final equations are not so symmetric as in the representation we have chosen.

[22]In what follows the external potential A_μ is taken to be an arbitrary function of x_μ.

[23]Schwinger's formalism is conventionally expressed in terms of real field variables (the number of components being doubled for this purpose if the basic field happens to be complex). We have preferred to work directly with complex variables.

[24]From the secondary constraints (22) and (23) one gets conditions which involve the variations $\delta\varphi_2$ and $\delta\chi_4$ which do not appear in the generator G. It can be shown as a consequence of this that if these constraints are included in the analysis the Lagrange multipliers associated with them would have to vanish.

[25]P. Federbush, Nuovo Cimento 19, 572 (1961).

[26]C. R. Hagen, Phys. Rev. D 6, 984 (1972).

[27]F. J. Belinfante, Phys. Rev. 92, 997 (1953).

[28]It is a curious fact that as compared to the spin-2 theory, the roles of the electric and magnetic fields in relation to the question of number of constraints are reversed in Glass's theory.

[29]S. Deser and B. Zumino, Phys. Rev. Lett. 38, 1433 (1977). See also D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, Phys. Rev. D 13, 3214 (1976).