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Evaluation of the effective potential in quantum electrodynamics

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Starting with Schwinger's proper-time integral in QED, we here present the analytical and numerical evaluation of the one-loop effective potential in the presence of a constant prescribed magnetic (electric) field.

I. INTRODUCTION

Up till now a great deal of research has been invested in studying the field-strength asymptotics of one-loop effective Lagrangians in various model field theories.

In two recent papers by one of the authors (W. D.),[1,2] the one-loop effective potential in QED was treated and, among others, an analytical expression was given for the one-loop effective Lagrangian in QED. Since there is growing interest in one-loop corrections to classical Lagrangians in Abelian as well as in non-Abelian gauge theories, we here present an explicit evaluation of the functional dependence of the effective Lagrangian in QED, which is valid in the entire range of the magnetic (electric) field strength. Graphical representations are also given, which, in the asymptotic limit, agree with the well-known approximations calculated earlier by Heisenberg and Euler,[3] Weisskopf,[4] and Schwinger.[5] The location of the relative minimum (maximum) for a constant magnetic (electric) field is found, the latter being of great interest in the theory of spontaneous symmetry breaking.[6]

II. EFFECTIVE LAGRANGIAN, ζ-FUNCTION REGULARIZATION

More than four decades ago Heisenberg and Euler[3] wrote down the quantum-mechanical correction to the classical Maxwell Lagrangian. Thereafter Weisskopf[4] and Schwinger[5] gave their own derivation of the so-called one-loop effective Lagrangian. Our treatment is based on Schwinger's source and proper-time technique as stated, e.g., in Ref. 5 or 7. The process which summarizes the effect that an external field can have on a single fermion loop is given by the vacuum persistence amplitude

$$\langle 0_+ | 0_- \rangle^A = \exp\{i\, W^{(1)}[A]\}\,, \tag{2.1}$$

where

$$i\, W^{(1)} = i \int d^4x\, \mathcal{L}^{(1)}(x) = -\tfrac{1}{2} \int_0^\infty \frac{ds}{s} \exp(-ism^2) \mathrm{Tr}\{\exp[is(\gamma\Pi)^2]\} \tag{2.2}$$

and

$$\Pi = \frac{1}{i}\partial - eA\,, \quad -(\gamma\Pi)^2 = \Pi^2 - \tfrac{1}{2} e\, \sigma F\,.$$

The trace Tr is to be taken in coordinate and spinor space. In Ref. 1. as well as in Ref. 8 one can find alternative ways of evaluating the integral in (2.2). The result is (for magnetic field only: $F_{12} = -F_{21} = H$)

$$\mathcal{L}^{(1)}[H] = \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s} \exp(-im^2 s) \times (z \cot z - 1 + \tfrac{1}{3} z^2)\,, \quad z = eHs\,. \tag{2.3}$$

With the rotation of the contour $s \to -is$ in (2.3), we find

$$\mathcal{L}^{(1)}[H] = -\frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m^2 s} \times (z \coth z - 1 - \tfrac{1}{3} z^2)\,. \tag{2.4}$$

This expression can be easily evaluated by using the n-dimensional regularization scheme together with the formula[9]

$$\int_0^\infty z^{\mu-1}\exp(-\beta z)\coth(z)\,dz=\Gamma(\mu)[2^{1-\mu}\zeta(\mu,\beta/2)-\beta^{-\mu}],\quad \mathrm{Re}\mu>1,\quad \mathrm{Re}\beta>0\,.$$

The result is[10]

$$-V_{\rm eff}^{(1)}[h]\equiv\mathcal{L}^{(1)}[h]=\frac{m^4}{32\pi^2}\frac{1}{h^2}[4\zeta'(-1,h)+h^2-\tfrac{1}{3}-(2h^2-2h+\tfrac{1}{3})\ln h]\,, \tag{2.5}$$

where

$$h=m^2/2eH=\tfrac{1}{2}H_{\rm cr}/H\,.$$

Here we meet Riemann's generalized ζ function, $\zeta(z,q)$; cf. also Ref. 9. Except for $z=1$, $\zeta(z,q)$ is holomorphic everywhere. For a purely electric field we obtain the following expression:

$$-V_{\rm eff}^{(1)}[\epsilon]\equiv\mathcal{L}^{(1)}[\epsilon]=-\frac{m^4}{32\pi^2}\frac{1}{\epsilon^2}[4\zeta'(-1,i\epsilon)-\epsilon^2-\tfrac{1}{3}+(2\epsilon^2+2i\epsilon-\tfrac{1}{3})\ln(i\epsilon)]\,, \tag{2.6}$$

where

$$\epsilon=m^2/2eE\,.$$

The corresponding effective potential for scalar QED has also been worked out and is given by (cf. also Ref. 11)

$$\text{spin 0:}\quad \mathcal{L}_0^{(1)}[h]=\frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s^3}e^{-m^2s}\left(\frac{z}{\sinh z}-1+\tfrac{1}{6}z^2\right)$$
$$=-\frac{m^4}{64\pi^2}\frac{1}{h^2}[4\zeta'(-1;\ h+\tfrac{1}{2})+h^2+\tfrac{1}{6}-(2h^2-\tfrac{1}{6})\ln h]. \tag{2.7}$$

Here we employed the formula[9]

$$\int_0^\infty dz\,z^{\mu-1}e^{-\beta z}\frac{1}{\sinh z}$$
$$=2^{1-\mu}\Gamma(\mu)\,\zeta(\mu;\tfrac{1}{2}(\beta+1))\,,\quad \mathrm{Re}\ \mu>1,\quad \mathrm{Re}\ \beta>-1.$$

The imaginary part in Eq. (2.6) is indicative of a nonvanishing probability for pair creation in an external electric field. That probability is expressed per unit time and volume by[5,7]

$$2\,\mathrm{Im}\mathcal{L}^{(1)}[E]=\frac{\alpha}{\pi^2}E^2\sum_{n=1}^\infty\frac{1}{n^2}\exp\left(-n\pi\frac{m^2}{eE}\right). \tag{2.8}$$

III. EFFECTIVE LAGRANGIAN, Γ_1-FUNCTION REGULARIZATION

Before we continue with a more detailed discussion of the generalized ζ function in Eq. (2.5), we present still another evaluation of Schwinger's proper-time integral, paralleling some parts of Ref. 12. That is, without resort to the n-dimensional regularization method, we will attempt to evaluate (2.3) directly in the four-dimensional space-time.

Again, we may start with Eq. (2.3) which we rewrite in the form

$$\mathcal{L}^{(1)}=\frac{m^4}{32\pi^2}\frac{1}{h^2}I(h),\quad h=\frac{m^2}{2eH}=\frac{H_{\rm cr}}{2H}, \tag{3.1}$$

where

$$I(h)=\int_0^\infty\frac{dz}{z^3}\exp(-i2hz)$$
$$\times(z\cot z-1+\tfrac{1}{3}z^2)\,. \tag{3.2}$$

In order to evaluate (3.2), we consider the derivative

$$\frac{d}{dh}I(h)=-\frac{1}{3h}+J(h)\,, \tag{3.3}$$

where

$$J(h)=-2i\int_0^\infty\frac{dz}{z^2}\exp(-i2hz)(z\cot z-1)\,. \tag{3.4}$$

$J(h)$ can be found by forming the derivative first and then employing Eqs. (21) and (25) of Ref. 12. This yields

$$\frac{d}{dh}J(h)=-4\int_0^\infty\frac{dz}{z}\exp(-i2hz)(z\cot z-1)$$
$$=-4\left[-\psi(1+h)+\ln h+\frac{1}{2h}\right]. \tag{3.5}$$

ψ denotes the logarithmic derivative of the Γ function,

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)\,.$$

After integrating from 1 to h, Eq. (3.5) becomes

$$J(h)-J(1)=-4[\ln(1+h)+h\ln h-h+1+\tfrac{1}{2}\ln h]\,. \tag{3.6}$$

A subsequent integration from 1 to h of expression (3.3) then produces

$$\begin{aligned}I(h)=&I(1)-J(1)+3-\tfrac{1}{3}\ln h-2\ln 2\pi\\&+h[J(1)+(4/h)\ln\Gamma_1(1+h)-4\\&+2\ln 2\pi+h-2(1+h)\ln h]\,,\end{aligned} \tag{3.7}$$

where it is important to note that the following integrals are independent of h:

$$\begin{aligned}J(1)&=-2i\int_0^\infty\frac{dz}{z^2}\exp(-2iz)(z\cot z-1)\,,\\I(1)&=\int_0^\infty\frac{dz}{z^3}\exp(-2iz)(z\cot z-1+\tfrac{1}{3}z^2)\,.\end{aligned} \tag{3.8}$$

The generalized Γ function[13] $\Gamma_1(x)$, which is defined by

$$\ln\Gamma_1(x)=\int_0^x dt\,\ln\Gamma(t)+\tfrac{1}{2}x(x-1)-\tfrac{1}{2}x\ln 2\pi\,, \tag{3.9}$$

comes in by virtue of the integral of (3.6):

$$\begin{aligned}\int_1^h dt\,\ln\Gamma(1+t)&=\int_0^{1+h}dt\,\ln\Gamma(t)-\int_0^2 dt\,\ln\Gamma(t)\\&=\ln\Gamma_1(1+h)+\tfrac{1}{2}(1+h)(\ln 2\pi-h)\\&\quad+1-\ln 2\pi\,.\end{aligned} \tag{3.10}$$

We also needed the relations $\Gamma_1(2)=\Gamma_1(1)=1$.

Finally, in order to determine $J(1)$ and $I(1)$, which are h independent, we observe that in the weak-field limit ($h\gg 1$) the main contribution to $I(h)$ comes from $z\ll 1$. Then, with the aid of the expansion

$$z\cot z-1+\tfrac{1}{3}z^2\cong-\tfrac{1}{45}z^4\,,$$

we find

$$\begin{aligned}I(h)\underset{h\gg 1}{\longrightarrow}&-\tfrac{1}{45}\int_0^\infty z\,dz\exp(-2ihz)\\&=\frac{1}{45}\frac{1}{4h^2}\,.\end{aligned} \tag{3.11}$$

The formula stated in Eq. (3.7) should reproduce this result in the weak-field limit:

$$\begin{aligned}I(h)\underset{h\gg 1}{\longrightarrow}&[I(1)-J(1)+3-2\ln 2\pi+4L_1]\\&+h[J(1)-4+2\ln 2\pi]\,.\end{aligned} \tag{3.12}$$

Hence, by comparing coefficients in (3.11) and (3.12), we obtain

$$\begin{aligned}J(1)&=4-2\ln(2\pi)\,,\\I(1)&=J(1)-3+2\ln(2\pi)-4L_1=1-4L_1\,.\end{aligned} \tag{3.13}$$

The constant L_1 is known to be

$$L_1=\tfrac{1}{3}+\int_0^1 dx\,\ln\Gamma_1(1+x)\cong 0.2487\,. \tag{3.14}$$

Here then is the other version of the exact closed-form expression of Schwinger's proper-time integral

$$\mathcal{L}^{(1)}[h]=\frac{m^4}{32\pi^2}\frac{1}{h^2}I(h)\,, \tag{3.15}$$

with $I(h)$ given by

$$\begin{aligned}I(h)=&-(\tfrac{1}{3}+2h+2h^2)\ln h+h^2\\&-4L_1+4\ln\Gamma_1(1+h)\,.\end{aligned} \tag{3.16}$$

Similar procedures can be used to evaluate the spin-0 effective Lagrangian (2.7).

IV. NUMERICAL TREATMENT OF THE EFFECTIVE LAGRANGIAN

The two different integrated forms, (2.5) and (3.15) of the effective Lagrangian, (2.3), are actually identical to each other. This can be easily shown if we observe the following representation of Riemann's generalized ζ function,

$$\zeta'(-1,x)=\ln\Gamma_1(x)+\zeta'(-1)\,, \tag{4.1}$$

the recurrence relation of the generalized Γ function,[13]

$$\ln\Gamma_1(1+x)=x\ln x+\ln\Gamma_1(x) \tag{4.2}$$

and the numerical identity[14,15]

$$\zeta'(-1)=\tfrac{1}{12}-L_1\,. \tag{4.3}$$

However, the expression (3.16) is more suitable for numerical computation. If we use the explicit representation of $\ln\Gamma_1(1+x)$,

$$\ln\Gamma_1(1+x)=\int_1^{1+x}dt\,\ln\Gamma(t)+\tfrac{1}{2}x(x+1)-\tfrac{1}{2}x\ln 2\pi\,,$$

and introduce the notation that H stands for the magnetic field strength in units of the critical field H_{cr} [$H_{cr}=m^2/(4\pi\alpha)^{1/2}$], the effective Lagrangian can be expressed in the following form:

$$\mathcal{L}^{(1)}[H]=\frac{\alpha}{2\pi}\left[\tfrac{3}{4}-H(\ln 2\pi-1)-4H^2L_1+(\tfrac{1}{2}+H+\tfrac{1}{3}H^2)\ln 2H+(2H)^2\int_1^{1+1/2H}\ln\Gamma(t)dt\right], \tag{4.4}$$

where $\mathcal{L}^{(1)}[H]$ is given in units of $H_{cr}{}^2=m^4/4\pi\alpha$. This expression can be evaluated numerically. Figure 1 shows the effective potential in the H-field region (in units of H_{cr}): $0.1\leq H\leq 0.35$. The relative minimum of the effective Lagrangian (relative maximum of the effective potential) is located at $H\simeq 0.23\ H_{cr}$.

In the strong-field limit we can approximate the integral in (4.4) by

$$\begin{aligned}4H^2\int_1^{1+1/2H}\ln\Gamma(t)dt &\simeq 4H^2\int_1^{1+1/2H}\left[\ln\Gamma(1)+\frac{d}{dt}\ln\Gamma(t)\Big|_{t=1}(t-1)\right]dt\\ &=4H^2\int_1^{1+1/2H}[-C(t-1)]dt=-\tfrac{1}{2}C\,,\end{aligned} \tag{4.5}$$

where $C=0.572\,2$ denotes Euler's constant. Now it is easy to observe that the strong-field behavior of (2.5) or (4.4) is given by the expression

$$\mathcal{L}^{(1)}[H]\xrightarrow[H\gg H_{cr}]{}\frac{(eH)^2}{8\pi^2}\frac{1}{3}\left[\ln\frac{2eH}{m^2}-1+12\zeta'(-1)\right].$$

Figure 2 shows the result of the numerical evaluation of the effective Lagrangian for a constant H field in the range $0\leq H\leq 50H_{cr}$.

Returning to the effective Lagrangian for a constant electric field, Eq. (2.6), we obtain likewise

$$\begin{aligned}\mathcal{L}^{(1)}[E]=&\frac{\alpha}{2\pi}\left[\tfrac{3}{4}+\tfrac{1}{2}\ln(2E)-\frac{\pi}{2}E+E^2(\tfrac{1}{3}-4\zeta'(-1))-\tfrac{1}{3}E^2\ln(2E)+4E^2\int_0^{1/2E}\operatorname{Im}\ln\Gamma(1+iy))dy\right]\\ &+i\frac{\alpha}{2\pi}\left\{-\frac{\pi}{4}-E\ln(2E)+E(\ln(2\pi)-1)+\frac{\pi}{6}E^2-4E^2\int_0^{1/2E}\operatorname{Re}\ln[\Gamma(1+iy)]dy\right\},\end{aligned} \tag{4.6}$$

with $\mathcal{L}^{(1)}[E]$ expressed in units of $m^4/4\pi\alpha$ and the field strength E measured in units of m^2/e. The structure of the imaginary part can be made more transparent by noticing[9]

$$\operatorname{Re}\ln[\Gamma(1+iy)]=-\tfrac{1}{2}\ln\frac{\sinh(\pi y)}{\pi y}$$

and thereafter performing an integration by parts:

$$\int_0^{1/2E}1\ln\frac{\sinh(\pi y)}{\pi y}dy=\frac{1}{2E}\ln\left(2E\sinh\frac{\pi}{2E}\right)-\int_0^{1/2E}[\pi y\coth(\pi y)-1]dy\,.$$

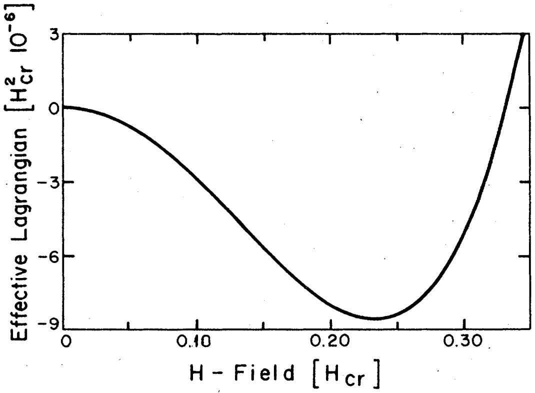

FIG. 1. Effective Lagrangian of a constant magnetic field, $0\leq H\leq 0.35H_{cr}$.

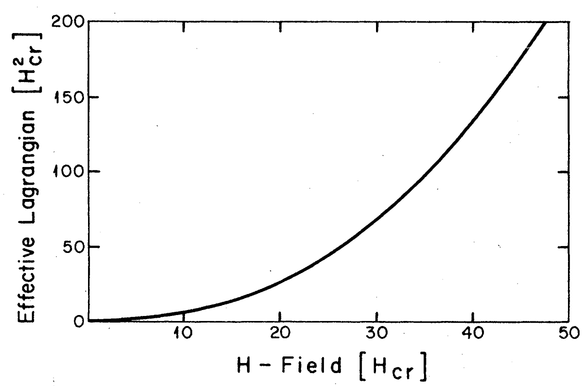

FIG. 2. Effective Lagrangian of a constant magnetic field, $0\leq H\leq 50H_{cr}$.

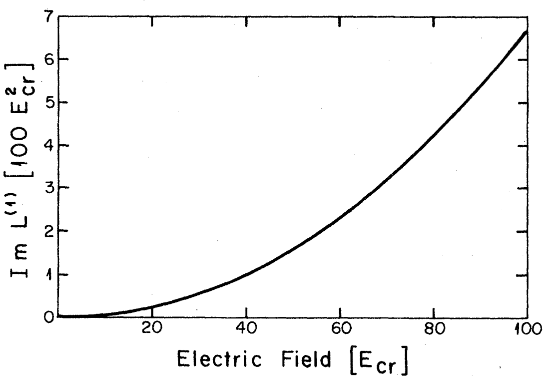

FIG. 3. Imaginary part of the effective Lagrangian in a constant electric field.

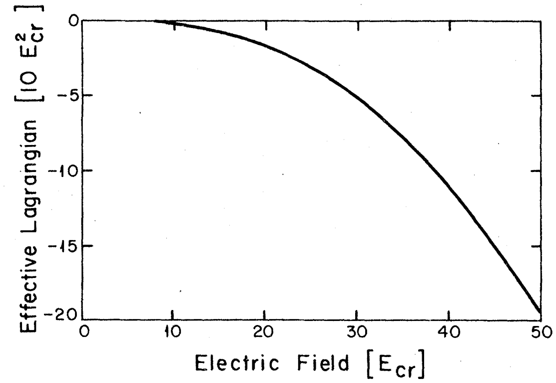

FIG. 4. Real part of the effective Lagrangian of a constant electric field, $0 \le E \le 50E_{cr}$.

Consequently we obtain

$$\mathrm{Im}\mathcal{L}^{(1)}[E]=\frac{\alpha}{2\pi}\left[-\frac{\pi}{4}+E\ln(2\pi)+\frac{\pi}{6}E^2+E\ln\sinh\frac{\pi}{2E}-2E^2\int_0^{1/2E}\pi y\coth(\pi y)dy\right]. \tag{4.7}$$

In the strong-field limit we may approximate the integrand in (4.7) by $\frac{1}{3}\pi^2y^2$, so that

$$\mathrm{Im}\mathcal{L}^{(1)}[E]=\frac{\alpha}{2\pi}\left[-\frac{\pi}{4}+E(\ln 2\pi-1)+\frac{\pi}{6}E^2+E\ln\sinh\frac{\pi}{2E}-\frac{\pi^2}{36}\frac{1}{E}\right]. \tag{4.8}$$

This formula is, incidentally, a good approximation of the infinite series (2.8). Figure 3 shows graphically the functional dependence of $\mathrm{Im}\mathcal{L}^{(1)}$ in terms of the electrical field strength.

Similarly one can readily approximate the real part of the effective Lagrangian for strong electric fields:

$$\mathrm{Re}\mathcal{L}^{(1)}[E]=\frac{\alpha}{2\pi}\left[\tfrac{3}{4}+\tfrac{1}{2}\ln(2E)-\frac{\pi}{2}E+E^2(\tfrac{1}{3}-4\zeta'(-1))-\tfrac{1}{3}E^2\ln(2E)-\tfrac{1}{2}C\right]. \tag{4.9}$$

The numerical evaluation of the real part of Eq. (4.6) is pictorially presented in Fig. 4. At last, Fig. 5 shows the real part in the region $0 \leq E \leq 5$. Here we plotted the negative value of the effective Lagrangian, the effective potential, which shows a local minimum at $E = 3E_{cr}$. It is exactly this relative minimum that makes the effective Lagrangian an important object in the theory of spontaneous symmetry breaking.

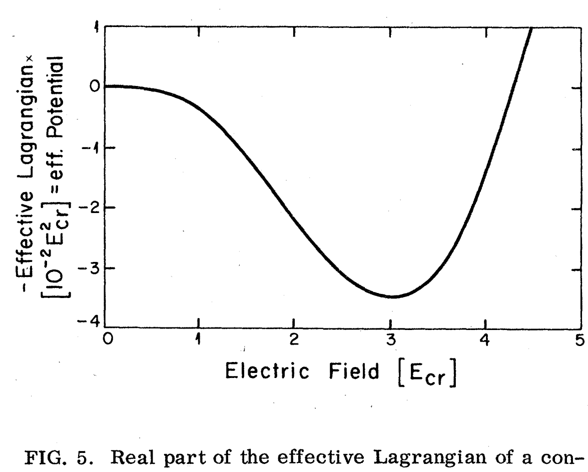

FIG. 5. Real part of the effective Lagrangian of a constant electric field, $0 \le E \le 5E_{cr}$.

V. CONCLUSION

Recently the Heisenberg-Euler Lagrangian enjoyed renewed interest in investigations concerning the theory of spontaneous symmetry breaking.

Starting with the one-loop approximation in QED, we presented two alternative calculations of Schwinger's proper-time integral. The connection between the two regularization schemes is given by a simple formula relating the two constants of integration. The dependence of the effective Lagrangian for constant magnetic (electric) fields was then numerically evaluated and plotted over the entire field-strength region. Finally, we exhibited graphically the location of the relative minimum of the effective potential in the presence of a constant electric field.

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