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Field theories on conformally related space-times: Some global considerations

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The nature of the vacua appearing in the relation between the vacuum expectation value of stress tensors in conformally flat spaces is clarified. The simple but essential point is that the relevant spaces should have conformally related global Cauchy surfaces. Some commonly occurring conformally flat space-times are divided into two families according to whether they are conformally equivalent to Minkowski space or to the Rindler wedge. Expressions, some new, are obtained for the vacuum expectation value of the stress tensor for a number of illustrative cases. It is noted that thermalization relates the Green's functions of these two families.

I. INTRODUCTION

The renormalized vacuum expectation values of stress tensors for conformally invariant field theories in conformal space-times are related by the trace anomaly. This relation, specialized to conformally flat spaces, is[1]

$$\langle T_\mu{}^\nu\rangle = g^{-1/2} g_\flat{}^{1/2}\langle T_\mu{}^\nu\rangle_\flat + \frac{1}{2880\pi^2}\{a(s)[2(R_{;\mu}{}^\nu - RR_\mu{}^\nu)+\tfrac{1}{2}g_\mu{}^\nu(R^2-4\Box R)] + b(s)[\tfrac{2}{3}RR_\mu{}^\nu - R_{\mu\alpha}R^{\nu\alpha} - \tfrac{1}{4}g_\mu{}^\nu(R^2-2R_{\alpha\beta}R^{\alpha\beta})]\}, \qquad (1)$$

where the constants $a(s)$ and $b(s)$ depend only on the spin s of the field under consideration and take the values exhibited in Table I.[2] In (1) and in subsequent equations we use $\langle T_\mu{}^\nu\rangle$ to denote the *renormalized* vacuum expectation value and the symbol "$\flat$" to denote that the corresponding quantity refers to a flat manifold.

As has been noted, more or less explicitly, by several authors[1,3] relation (1) carries with it an assumption about boundary conditions, i.e., about the vacuum states implicit in the equation. In this paper we wish to examine the nature of these vacua through some illustrative examples. Before displaying these some general comments are perhaps in order.

From the consideration that relation (1) is derived by the integration of a functional differential equation governing the behavior of $\langle T_\mu{}^\nu\rangle$ under conformal deformations of the manifold it follows that it connects a flat space, $M_\flat$, with a curved space, M, each supposed globally hyperbolic and endowed with a conformal vacuum,[4] if the following criterion is satisfied:

There exist subspaces $V_\flat \subset M_\flat$ and $V \subset M$ and a conformal transformation ω such that (i) ω is a diffeomorphism of $V_\flat$ onto V and (ii) the Cauchy developments of $V_\flat$ and V are $M_\flat$ and M, respectively.

The essential point of this criterion is that $V_\flat$ and V should have a common global Cauchy surface under the conformal mapping. This ensures that the mapping preserves the definition of a positive frequency in the following sense: Where there exists on M a global conformal timelike Killing field distinct from the one inherited from $M_\flat$ under ω both Killing fields define the same vacuum.

Thus it is the Green's functions corresponding to these vacua that are conformally related in the usual way, i.e.,

$$G(x,x') = \Omega^{-1}(x)G_\flat(x,x')\Omega^{-1}(x'),$$

when $g_{\mu\nu}=\Omega^2 g^\flat_{\mu\nu}$. This has been detailed explicitly by Unruh in a number of cases.[5]

TABLE I. The values of the coefficients appearing in Eq. (1) for different values of the spin s.

s	$a(s)$	$b(s)$
0	$-\frac{1}{6}$	1
$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{11}{2}$
1	3	62

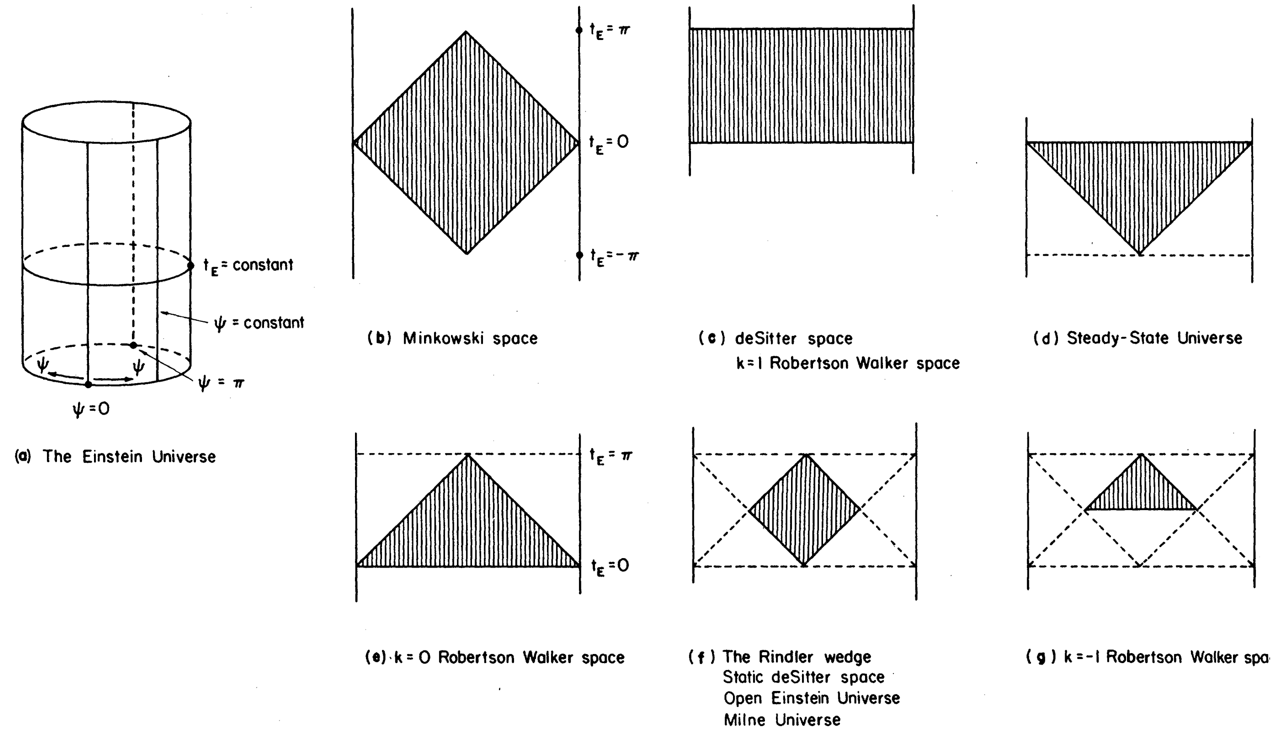

FIG. 1. The images under conformal transformation into the Einstein universe, of unit radius, of some commonly considered conformally flat space-times. Every point other than those for which $\psi = 0$, π represents a two-sphere. In (b)–(g) the Einstein universe is shown unwrapped, the two vertical lines are to be identified. The spaces in (a)–(e) have the surface $t_E = 0$, $0 \leq \psi < \pi$ in common as the image of global Cauchy surfaces, while those in (f) and (g) belong to a separate family sharing the surface $t_E = \pi/2$, $0 \leq \psi < \pi/2$ as the conformal image of global Cauchy surfaces.

II. EXAMPLES

We proceed now to the examples in order to clarify the above discussion. Figure 1 exhibits the conformal images in the Einstein universe[6] of several commonly occurring space-times. This representation is convenient because it allows one to appreciate easily the various conformal mappings. An example is provided by Fig. 2 where we have superimposed Figs. 1(b) and 1(c) in such a way that the surface $t_E = 0$, $0 \leq \psi < \pi$ is the common conformal image of a global Cauchy surface for each space. In this case the shaded region will represent the conformal image of both $V_\flat$ and V. Clearly the Cauchy developments of this region are the entire Minkowski and de Sitter spaces, respectively.

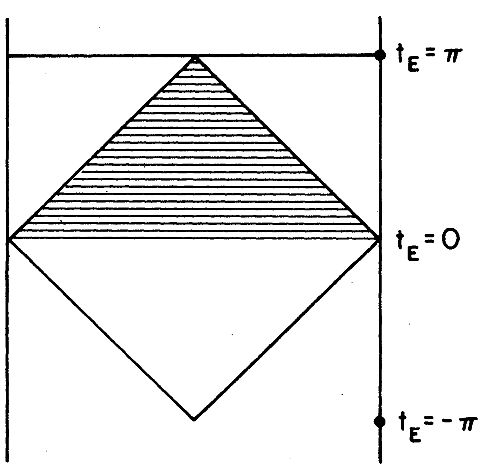

FIG. 2. The conformal images of Minkowski space and de Sitter space shown superimposed. The shaded region is the image of both $V_\flat$ and V.

It is now clear from Fig. 1 that the spaces in 1(a)–1(e) constitute a family which has in common the surface $t_E = 0$, $0 \leq \psi < \pi$ as the conformal image of global Cauchy surfaces. Relation (1) should therefore be applied to these spaces with $M_\flat$ Minkowski space, and hence $\langle T_\mu{}^\nu \rangle_\flat = 0$. The spaces in 1(f) and 1(g), however, constitute a separate family which has in common the surface $t_E = \pi/2$, $0 \leq \psi < \pi/2$ as the conformal images of global Cauchy surfaces. Relation (1) should therefore be applied to these spaces with $M_\flat$ the Rindler wedge and hence $\langle T_\mu{}^\nu \rangle_\flat \neq 0$.

To fix conventions we remind the reader that the Rindler wedge is the region $x > |t|$, $-\infty < y < \infty$, $-\infty < z < \infty$ of Minkowski space which is covered by the coordinates (τ, ξ, y, z) with

$$t = \xi \sinh\tau, \quad x = \xi\cosh\tau ,$$

in terms of which

$$ds^2 = -\xi^2 d\tau^2 + d\xi^2 + dy^2 + dz^2 \quad (0 < \xi < \infty) . \tag{2}$$

The vacuum stress for the Rindler wedge has been calculated to be[7]

$$\langle T_\mu{}^\nu\rangle_{\text{Rindler}} = -\frac{h(s)}{2\pi^2\xi^4}\int_0^\infty \frac{d\nu\,\nu(\nu^2+s^2)}{e^{2\pi\nu}-(-1)^{2s}} \times \text{diag}(-1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}) , \tag{3}$$

where $h(s)$ is the number of helicity states for the massless spin $s(=0,\frac{1}{2},1)$ field.

A few examples of the application of (1) follow.

First let M be the Einstein universe of radius a with metric

$$ds^2 = -dt_E{}^2 + a^2 d\psi^2 + a^2\sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2) . \tag{4}$$

Then by (1),

$$\langle T_\mu{}^\nu\rangle_{\text{Einstein}} = \frac{C(s)}{2\pi^2 a^4}\,\text{diag}(-1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}) , \tag{5}$$

where the coefficients $C(s)$ take the values

$$C(0) = \tfrac{1}{240}, \quad C(\tfrac{1}{2}) = \tfrac{17}{960}, \quad C(1) = \tfrac{11}{120} .$$

Expression (5) agrees with the standard result.[8] It was noticed[3,7] that, numerically,

$$C(s) = h(s)\int_0^\infty \frac{d\nu\,\nu(\nu^2+s^2)}{e^{2\pi\nu}-(-1)^{2s}} , \tag{6}$$

a somewhat surprising coincidence which we are now able to explain.

To this end we take M in (1) to be the open Einstein universe, $T\times H^3$, and $M_\flat$ the Rindler wedge. It is implicit in (1) that the coordinates of the two spaces should be conformally related. Thus we take the metric for the open Einstein universe in the form

$$ds^2 = a^2\left(-d\tau^2 + \frac{d\xi^2+dy^2+dz^2}{\xi^2}\right), \tag{7}$$

where the coordinates have the same ranges as in (2). The relation between the coordinates in (7) and the more usual hyperbolic set can be found in the Appendix together with the transformations that justify Fig. 1.

Now, it can be shown independently[3] that $\langle T_\mu{}^\nu\rangle = 0$ on the open Einstein universe. The reason for this is that the WKB approximation is exact on $T\times H^3$ and since there is only one geodesic connecting any two points on H^3 (in contradistinction to the situation on S^3) the renormalization removes the entire expression. Then, since the Ricci tensor terms in (1) are the same for both the open and closed Einstein universe, we find that

$$\langle T_\mu{}^\nu\rangle_{\text{Rindler}} = -\frac{a^4}{\xi^4}\langle T_\mu{}^\nu\rangle_{\text{Einstein}} , \tag{8}$$

which is just the relation (6). This last equality is all the more striking when account is taken of the fact that the anomalous trace takes the value zero in both flat space and the Einstein universe. Nevertheless this trace is not zero "in between" and its residual influence manifests itself in the end-point relation (8).

It is of interest to record that dimensional regularization, with its different value for $a(1)$, alters the values of $\langle T_{\mu\nu}\rangle$ in both the Einstein and open Einstein universes for spin one. In particular it is nonzero for the latter space. This modifies equation (8) such that the difference of $\langle T_\mu{}^\nu\rangle$ in these two spaces enters on the right-hand side.

In this approach we can also see the origin of the "nongeometrical" term that Bunch finds[3] in $\langle T_\mu{}^\nu\rangle$ for the $k=-1$ Robertson-Walker space and the Milne universe. It is just the first term on the right-hand side of (1), viz.,

$$g^{-1/2} g^{1/2}{}_{\text{Rindler}}\langle T_\mu{}^\nu\rangle_{\text{Rindler}} .$$

For completeness we next treat de Sitter space which has been discussed already from this point of view.[9] Taking $M_\flat$ to be Minkowski space we have

$$\langle T_\mu{}^\nu\rangle_{\text{deSitter}} = -\frac{b(s)K^2}{960\pi^2}\, g_\mu{}^\nu . \tag{9}$$

where $K^{-1/2}$ is the radius of the de Sitter hyperboloid and $b(s)$ is as in Table I.

We now proceed to static de Sitter space with metric

$$ds^2 = \frac{(1-Kr^2)}{K}\left[-d\tau^2 + \frac{K dr^2}{(1-Kr^2)^2} + \frac{Kr^2}{(1-Kr^2)}(d\theta^2 + \sin^2\theta d\phi^2)\right] , \tag{10}$$

where we shall obtain an expression for $\langle T_\mu{}^\nu\rangle$ in the "observer dependent" vacuum of Gibbons and Hawking.[10] Figure 1(f) reveals that $M_\flat$ is again the Rindler wedge [for the relation that connects the coordinates in (2) and (10) see the Appendix]. Relation (1) then yields

$$\langle T_\mu{}^\nu \rangle = -\frac{C(s)}{2\pi^2}\frac{K^2}{(1-Kr^2)^2}\,\mathrm{diag}(-1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}) - \frac{b(s)K^2}{960\pi^2}\, g_\mu{}^\nu \,.$$

We observe that the stress tensor diverges (in an orthonormal frame) at the horizons $Kr^2 = 1$ which bound the static region.

The final example is the steady-state universe. From Fig. 1(d) and (1) it is immediately apparent that the vacuum stress takes the same value as in de Sitter space (9) since the Ricci tensor terms in (1) are the same in each case.

III. THERMAL CONSIDERATIONS

It is well known that the Minkowski vacuum is a thermal state of temperature $(2\pi)^{-1}$ relative to the Rindler-Fock space.[11] This is the origin of the thermal character of expression (3) for $\langle T_\mu{}^\nu \rangle_{\text{Rindler}}$. Despite the non-Planckian form of the numerator in (3) the spectrum is precisely thermal. The unconventional form of this factor is due to the fact that the important frequencies in the integral are of order ξ^{-1} corresponding to wavelengths of the same order as the distance from the boundary. Normally the Planck spectrum is derived under the assumption that the important wavelengths are very much smaller than the size of the cavity. The surprise, if any, then is that the deviation from the Planckian form is so simple. This is explained by noting that the space relevant for thermal considerations is not so much the physical space as the optical one [12,13] which in this case is just the open Einstein universe $T \times H^3$ of unit radius [cf. metric (7)]. As we have previously noted the WKB approximation is exact on this space so that all the Hamidew coefficients, a_n, vanish bar a_0 and, for $s = \frac{1}{2}$, 1, a_1. Furthermore there are no "exponentially small" terms, as there would be in the Einstein universe $T \times S^3$ for which the WKB approximation is exact as well. From this consideration it follows that the asymptotic distribution of eigenvalues is also exact being just $h(s)(\nu^2 + s^2)$. As a corollary we note that the thermal distribution in $T \times H^3$, at a temperature T, will be

$$\frac{h(s)}{2\pi^2}\int_0^\infty \frac{d\nu\,\nu(\nu^2+s^2)}{e^{\nu/T}-(-1)^{2s}} = \frac{h(s)}{96}\left[\left(\frac{15+(-1)^{2s}}{5}\right)\pi^2 T^4 + s^2\left(\frac{9+(-1)^{2s}7}{2}\right)T^2\right], \tag{11}$$

i.e., a combination of T^4 and T^2 for *all* T.

That a_1 should be nonzero for $s = \frac{1}{2}$, 1 is because the relevant *second*-order equations are not conformally invariant. This has the effect of producing a tail in the propagation inside the light cone.

We have remarked previously that the Green's function for the spaces within one family are conformally related. For those static spaces in the Rindler family the factor Ω is time independent so that the corresponding finite-temperature Green's functions, G_β $(\beta = T^{-1})$, obtained by (anti) periodizing in imaginary time are similarly conformally related. For the nonstatic spaces, the Milne universe and the $k = -1$ Robertson-Walker space, the thermal theory can be defined by conformal transformation from any one of the static spaces.

The statement that the Minkowski vacuum is a thermal Rindler-Fulling state of temperature $(2\pi)^{-1}$ is the statement that

$$G_{2\pi}^{\text{Rindler}} = G^{\text{Minkowski}} \,.$$

We then see that the following diagram must commute:

the Rindler wedge	$\xrightarrow{(2\pi)^{-1}}$	Minkowski space
$\updownarrow$		$\updownarrow$
static de Sitter space	$\xrightarrow{(2\pi K^{-1/2})^{-1}}$	de Sitter space
$\updownarrow$		$\updownarrow$
open Einstein universe	$\xrightarrow{(2\pi i a)^{-1}}$	Einstein universe
$\updownarrow$		$\updownarrow$
$k = -1$ Robertson-Walker		$k = 1$ Robertson-Walker
$\updownarrow$		$\updownarrow$
the Milne universe		$k = 0$ Robertson-Walker
		$\updownarrow$
		steady-state universe,

where the vertical arrows denote conformal transformation of the Green's functions and the horizontal arrows signify thermalization to the indicated temperature. The appearance of an imaginary temperature connecting the two types of Einstein universe is dictated by the fact that the process links the dual spaces H^3 and S^3. To be more precise (for spin zero, with a similar equation obtaining for other spins)

$$G_{T\times S^3}(t_E - t'_E, \gamma, a) = \sum_{n=-\infty}^{\infty} G_{T\times H^3}(t_E - t'_E + n2\pi a, \gamma, ia), \quad (12)$$

where γ is the spatial geodesic distance. Note that the Green's functions on the right-hand side of this equation refer to a hyperboloid of imaginary radius. Thus the Casimir (zero temperature) energy in the Einstein universe can be regarded, in a sense justified by (12), as having a thermal character. This sheds further light on (6).

Since a hyperboloid of imaginary radius is a sphere, Eq. (12) is just as well expressed by saying that the usual, zero-temperature, Green's function on the Einstein universe is obtained by thermalizing to a temperature $T=(2\pi ai)^{-1}$ its own direct path contribution. This gives in general the high-temperature expansion[13] of, say, $\langle T_0{}^0\rangle$ which here has only the two terms proportional to T^4 and T^2 corresponding to the nonvanishing of a_0 and a_1. In this way we may directly relate the high- and low-temperature theories on the Einstein universe.[14]

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APPENDIX

We record here the conformal transformations that justify Fig. 1.

The steady-state metric

$$ds^2 = -dt^2 + \exp(2K^{1/2}t)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)],$$

$k=0$ Robertson-Walker metric

$$ds^2 = -dt^2 + R^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)],$$

and, trivially, the Minkowski space metric can all be brought to the form

$$ds^2 = \Omega_1{}^2(t)[-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)].$$

A further coordinate transformation with

$$t \pm r = \tan\left(\frac{t_E \pm \psi}{2}\right)$$

relates them to the Einstein universe (4) of unit radius:

$$ds^2 = \frac{\Omega_1{}^2}{4} \sec^2\left(\frac{t_E+\psi}{2}\right)\sec^2\left(\frac{t_E-\psi}{2}\right) \times [dt_E{}^2 + d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)].$$

The de Sitter space metric

$$ds^2 = -dt^2 + K^{-1}\cosh^2(K^{1/2}t) \times [d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

and the $k=1$ Robertson-Walker metric

$$ds^2 = -dt^2 + R^2(t)[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

are trivially written in the form

$$ds^2 = \Omega^2(t)[-dt^2 + d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)].$$

The metrics for the remaining spaces, which are those referred to in Figs. 1(f) and 1(g) are dealt with by first reducing them to "hyperbolic" form

$$ds^2 = \Omega_1{}^2[-d\tau^2 + d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)]. \quad \text{(A1)}$$

For the $k=-1$ Robertson-Walker space

$$ds^2 = -dt^2 + R^2(t)[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

and the Milne universe with metric

$$ds^2 = -dt^2 + t^2[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

this is trivially done. Static de Sitter space, (10), is brought to the form (A1) by the coordinate change $K^{1/2}r = \tanh\chi$. To bring the remaining two cases, the Rindler metric (2) and the open Einstein metric (7), to the form (A1) requires the coordinate transformation

$$\xi = \frac{1}{\cosh\chi - \sinh\chi\cos\theta},$$

$$y = \frac{\sinh\chi\sin\theta\cos\phi}{\cosh\chi - \sinh\chi\cos\theta},$$

$$z = \frac{\sinh\chi\sin\theta\sin\phi}{\cosh\chi - \sinh\chi\cos\theta}.$$

For these cases we will have Ω_1 equal to ξ and a, respectively. Finally we define

$$\tan\left(\frac{t_E \pm \psi}{2}\right) = \tanh\left(\frac{\tau \pm \chi}{2}\right)$$

so that (A1) becomes

$$ds^2 = \frac{\Omega_1^2}{\cos(t_E+\psi)\cos(t_E-\psi)} \times [-dt_E^2 + d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)].$$

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[2]Dimensional regularization gives the value -2 for $a(1)$ instead of 3, the value given by point-separation and ζ-function renormalization. All three renormalization techniques agree on the values of the remaining coefficients in Table I.

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