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Quantum nondemolition and gravity-wave detection

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A variety of quantum nondemolition measurement schemes are defined and examples are given to demonstrate their theoretical and practical possibilities. They show that the quantum limit of measurement of weak forces does not exist.

In the attempts to detect gravitational radiation there has been concern that the fundamental quantum nature of the detection apparatus might set an ultimate limit on the strength of gravitational radiation detectable. The principal problem lies not in the weakness of the gravitational radiation pulse (which is of the order of ergs/cm^2 sec at kilohertz frequencies, far from any limit in which the gravitational quantum effects are expected to play a role) but in the weakness of any known coupling of the gravitational radiation to present detectors. In particular, for realistic sources, estimates[1] have suggested that only of the order of one quanta would be deposited into a cold detector of the Weber type.

Braginsky[2] has suggested that these difficulties may have a solution, which has come under the name of "quantum nondemolition detection." He realized that the cross section for the interchange of a quantum of energy with the gravity wave could be increased by somehow placing the bar in a highly excited state. The coherent interaction of the many quanta in the bar with the gravity wave increases the probability of emission or absorption of one quantum by a factor N, the number of quanta in the bar.

The difficulty arose as to how one could detect the change induced in the bar by the gravity wave. The standard techniques suggested that one could at best detect a change in the number of quanta in the bar to within $\pm\sqrt{N}$ quanta, which would precisely eliminate the gain achieved by the increase in cross section due to the coherent stimulated emission/absorption effect.

The original question posed by Braginsky was whether one could design a scheme for measuring the number of quanta in the bar to sufficient accuracy (say ±1 quanta) to take advantage of the increased interaction.

Until recently,[3] there has been confusion as to exactly what one wanted to achieve by "quantum nondemolition" measurement of gravity waves. In clarifying this measurement technique, let us first define some component parts to the system (see Fig. 1). The first is the strength of the gravity wave itself at the detector, which I will denote by $G(t)$. I will assume throughout that $G(t)$ is a given c-number field. For gravitational radiation, this is a very good approximation. Furthermore, as there exists no quantum theory of gravity, one could not treat the gravity wave quantum mechanically, even if one wanted to. The second part of the system will be called the detector. It is via the interaction between the dynamic degrees of freedom of the detector and the gravity wave that one can obtain information about the gravity wave. The detector will be assumed to be a quantum device. Finally there will be the readout system which interacts with the detector. It is via correlations induced in the readout system by the interaction with the detector that one ultimately obtains information about the gravity wave. The first stages of the readout system will also be treated quantum mechanically.

We will write down the Hamiltonian for the system as

$$\mathcal{H}=\mu TG+H_D+\epsilon QR+H_R\,, \tag{1}$$

where μ and ϵ are coupling constants, H_D is the free Hamiltonian for the detector, T and Q are functions which depend only on the dynamic variables of the detector, and R depends only on the dynamic variables of the readout system.

I will work throughout in the Heisenberg representation. An examination of the behavior of any readout variable shows that it depends on the detector only via Q. That is, if ϕ is a readout var-

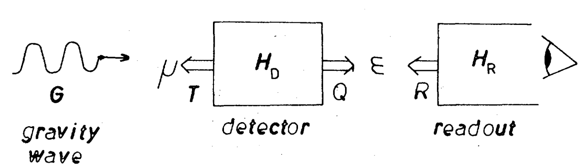

FIG. 1. Stages in detection apparatus for gravity wave.

iable, then

$$i\frac{d\phi}{dt}=[\phi,H_R]+\epsilon Q[\phi,R]. \tag{2}$$

As far as the readout system (and thus as far as the experimentalist) is concerned, we need only worry about the behavior of the variable Q of the detector. We find

$$i\frac{dQ}{dt}=[Q,H_D]+\mu G[Q,T]. \tag{3}$$

Following Thorne *et al.*[3] now define quantum nondemolition readout (QNDR) by demanding that the time development of Q does not depend on any property of the readout system if $G=0$ (or if $\mu=0$). This implies that in the absence of a gravity wave, Q will be determined only by the state of the detector itself, and will be completely independent of the state of the readout system. The readout system, therefore, cannot inject noise into the variable Q, no matter what the state of the readout system, and no matter what one does to the readout system.

The simplest requirement is that we demand

$$[Q,H_D]=0. \tag{4}$$

Another possibility is that

$$[Q,H_D]=\alpha Q \tag{5}$$

with α constant.

In general we want that

$$[P_i,Q]=0 \text{ for all } i \tag{6}$$

where

$$P_i=[\cdots[[Q,H_D],H_D]\cdots H_D]$$

with i commutators on the right-hand side.

These conditions ensure that with $\mu=0$, Q will depend only on the dynamics of the detector and not of the readout systems. In the following discussion, I will choose the condition $[Q,H_D]=0$ as the definition of quantum nondemolition readout.

The second idea one can include under quantum nondemolition is the idea of full quantum nondemolition detection (QNDD). This assumes that QNDR obtains, and adds the condition that the *response* of Q to a gravity wave is also to be independent of the state of the readout system. This ensures that, if the readout accurately measures Q, one can uniquely determine the shape of the gravity wave which caused the given change in Q. It implies that the change in Q depends only on the gravity wave and the state of the detector. If this condition is not satisfied, but only QNDR is satisfied, then a change in Q will imply that a gravity wave has interacted with the system, but that the change in Q cannot be used to (uniquely) determine the properties of the wave.

In terms of a classical amplifier, QNDR would imply a noise free amplifier—i.e., no signal—no output, while QNDD would imply that the given input signal produces a unique output (non QNDD could be likened to a noise free amplifier with a partially random gain). (Note that noise free in this context is in terms of a classical input signal. The quantum noise, due to the amplifier's spontaneous emission of "gravitons" is neglected.[4])

In formal terms, the simplest way to implement QNDD is to place an additional demand on the forms of Q and T. In particular, we demand the conditions

$$\begin{aligned}&[Q,H_D]=0,\\&[Q,T]\neq 0,\\&[[Q,T],H_D]=0,\\&[[Q,T],Q]=0.\end{aligned} \tag{7}$$

The first ensures that we have QNDR. The second ensures that Q will be affected by the presence of a gravity wave. The third and fourth conditions together ensure that the response of Q to the presence of a gravity wave will be independent of the state of the readout system. The third could be weakened in a similar manner to the QNDR case [Eq. (6)], but I will maintain the conditions in this form.

A fully QNDD system which is physically realizable has yet to be found. The closest to such a system is a model for the Thorne *et al.*[3] system with an ideal clock to be described later.

The final condition one could demand of one's detector-readout system, is that the response of the detector be tunable by an appropriate choice of the initial state of the detector. It is this idea, which I will call detector-dependent response, DDR, which essentially motivated Braginsky in his original attempts at quantum nondemolition measurement. In formal terms, we can define DDR as a system which obeys the condition

$$[Q,T]\neq c \text{ number}. \tag{8}$$

As the above descriptions are rather formal, it will probably be of interest to examine a variety of systems to discover exactly how they fit into the above scheme. Most of the examples will be cases in which part of the detector is a simple harmonic oscillator. The reason is that the conventional Weber-bar-type detector can be regarded as a simple harmonic oscillator where the oscillator is the lowest longitudinal mode of the bar. I will also discuss another scheme of QNDR using a rotating sphere.

In all of these discussions I will completely ignore the possibility of other interactions with the detector besides those with the gravity wave and the readout system. In particular all sources of external noise besides those concerned with the readout system will be ignored for the present. Ultimately, of course, the response of the various systems to extraneous noise sources will also be of importance.

I. LINEAR COUPLINGS–HARMONIC-OSCILLATOR DETECTOR

The usual readout schemes (capacitive, squids, piezoelectric, etc.) are linear in one of the fundamental dynamic variables of the detector. In particular, we can model the detector as a simple harmonic oscillator with

$$H_D = \tfrac{1}{2}(p_1{}^2 + \omega^2 q_1{}^2)\,, \tag{9}$$

where p_1 and q_1 are the normalized momentum and coordinate of the detector (for the bar q_1 would be related to the amplitude of vibration).

The coupling with the gravity wave is invariably linear in p_1 and q_1, and can, by a canonical transformation, always be written such that

$$T = q_1\,, \tag{10}$$

i.e., the gravity wave acts as a generalized force on the system. Furthermore, the coupling with the readout system in these schemes is also linear—i.e., the readout system "measures" (couples to) usually q_1 the amplitude of vibration. We therefore have

$$Q = q_1\,. \tag{11}$$

We now have that Q obeys the equation

$$\frac{dQ}{dt} = \frac{dq_1}{dt} = p_1\,. \tag{12}$$

Although Q does not directly depend on the readout system, it does through p_1, for we have

$$\frac{dp_1}{dt} = -\omega^2 q - \mu G(t) - \epsilon R\,. \tag{13}$$

Because R is in general unknown and variable (partially because of the large number of in general unknown and unknowable interactions the readout system has with the outside world) the dependence of p_1 on R will produce a random and uncalculable effect on Q, and thus back on the readout system. Furthermore, because the dependence of p_1 and thus of Q on G is independent of the state of the detector, the change in Q produced by a given G will also be independent of the state of the detector. Analysis of this situation by Braginsky led him to the conclusion that $G(t)$ would have to be larger than some specified minimum to be detectable. This was what came to be known as the quantum limit on measurability of the gravitational wave.

II. HARMONIC OSCILLATOR QNDR

In this example I will retain the model of the detector as a harmonic oscillator with

$$H_D = \tfrac{1}{2}(p_1{}^2 + \omega^2 q_1{}^2)\,. \tag{14}$$

Furthermore, I will retain the linear coupling with the gravitational field, $T = q_1$.

In order to obtain QNDR, we must have $[Q, H_D] = 0$. It can be shown that the only possible Q's which obey this are Q's which are functions of H_D and of the identity operator. The simplest such function is

$$Q = H_D/\omega\,, \tag{15}$$

where the division by ω is for later convenience. Q now corresponds to the operator giving the number of quanta in the oscillator.

An example of such a system is given in Fig. 2. Here the harmonic oscillator is an L-C circuit, where the first stage in the readout system is a pivoted bar connected to the inductor and the capacitor in such a way that L/C is independent of the angle of the bar. (Since the capacitance depends inversely on the plate separation, and the inductance depends inversely on the length of the inductor, the bar is arranged so as to alter the capacitor separation and the inductor length by the same ratio.)

If we define ϕ as the angle of the bar from the equilibrium position, J as the conjugate angular momentum of the bar, I as the moment of inertia of the bar, and L and C as the inductance and capacitance with L_0 and C_0 the values when $\phi = 0$, we obtain the total Hamiltonian

$$\mathcal{H} = \tilde{\mu} eG + \frac{e^2}{2C} + \frac{\pi^2}{2L} + \frac{J^2}{2I} + H_I\,, \tag{16}$$

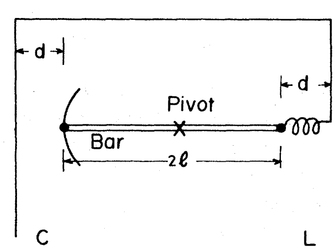

FIG. 2 Bar readout system of quanta in L-C circuit.

where e is the charge on the capacitor, π is the conjugate momentum to e and is proportional to the flux in the inductor, and H_I is the interaction of the bar with the rest of the readout system. I have also assumed that the "gravity wave" interacts only with the charge on the capacitor.

As both $1/L$ and $1/C$ are directly proportional to the inductor length and plate separation, respectively, and since both of these are of the form $d+\frac{1}{2}l\phi^2$ where d is the minimum inductor length and plate separation, and $2l$ is the length of the bar, we have

$$\mathcal{H}=\tilde{\mu}eG=\frac{e^2}{2C_0}+\frac{\pi^2}{2L_0}+\frac{l\phi^2}{2d}\left(\frac{e^2}{2C_0}+\frac{\pi^2}{2L_0}\right)+\frac{J^2}{2I}+H_I\,. \tag{17}$$

Defining

$$\begin{aligned} q_1&=\sqrt{L_0}e\,,\\ p_1&=\sqrt{L_0}\pi\,,\\ \omega&=(L_0C_0)^{-1/2}\,,\\ \epsilon&=\frac{l\omega}{d}\,,\\ R&=\frac{\phi^2}{2}\,,\\ \mu&=\tilde{\mu}/\sqrt{L_0}\,, \end{aligned} \tag{18}$$

we obtain the required form for $\mathcal{H}$. For $H_I=0$ and for $G=0$, this system is exactly solvable. The eigenfunctions are

$$\begin{aligned} \psi_{n,m}&=e^{-iE_{n,m}t}(\omega\Omega_n I)^{1/4}H_n(\sqrt{\omega}q)H_m(\sqrt{\Omega_n I}\,\phi)\,,\\ E_{n,m}&=n\omega+m\Omega_n+(\omega+\Omega_n)/2\,,\\ \Omega_n&=[(n+\tfrac{1}{2})\epsilon/I]^{1/2}\,, \end{aligned} \tag{19}$$

where H_n are the normalized oscillator eigenfunctions

$$\begin{aligned} &\frac{1}{2}\left(\frac{d^2}{dx^2}-x^2\right)H_n(x)=(n+\tfrac{1}{2})H_n(x)\,,\\ &\int H_n(x)H_{n'}(x)dx=\delta_{n,n'}\,. \end{aligned} \tag{20}$$

The pivoted bar therefore acts as a harmonic oscillator with frequency directly proportional to $(n+\frac{1}{2})^{1/2}$, where n is the number of quanta in the L-C circuit. By measuring the frequency of the bar oscillator, one can obtain a direct reading of the number of quanta in the circuit. This frequency can be measured, for example, by driving the bar by an external force to discover the resonance frequency. We have

$$\langle\psi_{n,m}H_I\psi_{n'm'}\rangle=0\,,$$

for $n\neq n'$ as H_I is a function only of the readout variables. Therefore the coupling to the bar cannot change the number of quanta in the L-C circuit. This is a demonstration of the QNDR feature of this system.

This system is not QNDD, however, The response of the oscillator to a nonzero G depends on the state of the bar. The first-order transition probability for the system to go from ψ_{nm} to $\psi_{n'm'}$ is given by

$$P_{n,m;n'm'}=\mu^2\left|\int e^{i(E_{n'm'}-E_{nm})t}G(t)dt\int H_{n'}(\sqrt{\omega}q_1)H_n(\sqrt{\omega}q_1)\sqrt{\omega}q_1dq_1\int(\Omega_{n'}\Omega_n)^{1/4}\sqrt{I}H_{m'}(\sqrt{\Omega_{n'}I}\,\phi)H_m(\sqrt{\Omega_n I}\,\phi)d\phi\right|^2. \tag{21}$$

Since

$$E_{n'm'}-E_{nm}=m'\Omega_{n'}-m\Omega+(n'-n)\omega$$

depends on m and on m', and since the final integral is not 0 for $m\neq m'$ the response of the system to the gravity wave will depend not only on the state of the detector (i.e., on n) but also on the initial and final state of the readout (i.e., on m and m'). This means that in addition to exciting the detector (changing n) the wave also has a nonzero probability of changing the number of quanta in the readout ($m'\neq m$). Furthermore, as $\Omega_{n'}\neq\Omega_n$, even if the number of quanta in the readout does not change, the change in energy of each such quantum creates an m dependence in the transition probability through the first term.

If (as would usually be the case if one wanted to keep track of n on a continuous basis) one had the ϕ bar interacting with further links in the readout, the response of the system to $G(t)$ would in general depend on the state of the whole readout chain. As it is practically impossible to take the dynamic evolution of the whole readout chain into account, this introduces a random unknown factor into the response of the system to the gravity wave. This reflects the fact that this system is not QNDD since

$$[[T,Q],Q]=\left(\frac{ip_1}{\omega},\frac{p_1^2+\omega^2q_1^2}{\omega}\right)=q_1\neq 0\,. \qquad (22)$$

Effectively, the readout system makes the frequency response of the detector uncertain. Referring back to Fig. 1, this is understandable, as monitoring the energy of the bar will jiggle the bar, which will alter both L and C and thus alter the frequency $\sqrt{LC}$. Furthermore, the more the bar is monitored (in order to keep closer watch on the number of quanta in the oscillator) the more unknown this frequency becomes.

III. QNDR-TWO-OSCILLATOR DETECTOR

If we introduce a second oscillator into our detector, such that we have

$$H_D=\tfrac{1}{2}(p_1^2+\omega^2q_1^2)+\tfrac{1}{2}(p_2^2+\Omega^2q_2^2)\,, \qquad (23)$$

then there are a few more possibilities for a QNDR system. Let us assume that again we have $T=q_1$ (i.e., the gravity wave interacts with only one of the oscillators). In addition to choosing Q to be a function of H_D, it can be an arbitrary function of

$$H_1=\tfrac{1}{2}(p_1^2+\omega^2q_1^2)$$

and (24)

$$H_2=\tfrac{1}{2}(p_2^2+\Omega^2q_2^2)\,.$$

The dependence of Q on H_2 is not of much direct use as H_2 cannot be affected by a gravity wave ($[H_2,T]=0$). It can, however, be of use in canceling out some of the background effect of H_1 on the readout system. In particular, if we choose $Q=H_1/\omega-H_2/\Omega$ and choose the initial number of quanta in each oscillator to be the same, then the initial force of the detector on the readout system is zero. The readout system, would, however, still be sensitive to changes in Q caused by the gravity wave. Furthermore, since the interaction with the gravity wave depends only on the state of the first oscillator ($[Q,T]=ip_1/\omega$) we can have the advantage of increased coupling with the gravity wave, without also increasing the interaction with the readout system in the absence of gravity waves. (For example, if in the previous L-C circuit example the interaction of the L-C circuit with the bar deviated from that assumed for large ϕ, we could rig up a second L-C circuit to cancel the static force of the first on the bar and keep the system out of the regime where the simple analysis fails to apply.)

If $\omega=\Omega$, it would seem that there are more possibilities for Q. In particular, we then have the quantities

$$X_1=\frac{1}{2\omega}(p_1p_2+\omega^2q_1q_2),$$

$$X_2=p_1q_2-p_2q_1, \qquad (25)$$

$$X_3=\frac{1}{2\omega}[(p_1^2+\omega^2q_1^2)-(p_2^2+\omega^2q_2^2)],$$

which all commute with H_D and are thus all possible candidates for Q. Furthermore, X_1 and X_2 are linear in the variables of the first oscillator.

However, all three of these quantities are essentially equivalent as far as their use as possible candidates for Q. Each can be obtained from any of the others by a simple canonical transformation. Furthermore, these quantities obey the angular momentum commutation relations, and thus have identical spectra. As X_3 simply corresponds to the previous choice of Q for two oscillators, we can find that the only advantage to using any of these is that one can choose the initial state so that the effect of the detector on the readout is minimal while the coupling with the gravity wave is large. For example, choosing $Q=X_2$ we find that this system is DDR:

$$[Q,T]=[X_2,q_1]=-iq_2\,. \qquad (26)$$

One might imagine that one could choose the second oscillator to be in a highly excited state (for example a highly excited semiclassical state for which $p_2\approx A\cos\omega t$). However, this assumption leads to difficulties. In particular, during the measurement of X_1 the value of P_2 will become unknown. If we determine X_1 to have the exact value λ (i.e., we are in an eigenstate $|\lambda\rangle$ of X_1) then we find

$$\begin{aligned}\langle\lambda|p_2|\lambda\rangle&=2\omega i\langle\lambda|[X_1,q_1]|\lambda\rangle\\&=2\omega i\langle\lambda|(\lambda q_1-q_1\lambda)|\lambda\rangle\\&=0\neq A\cos\omega t\,.\end{aligned} \qquad (27)$$

This arises because of the effect of the readout system on the second oscillator. We have

$$\frac{dp_2}{dt}=-\frac{\epsilon\omega q_1R}{2}-\omega^2q_2\,. \qquad (28)$$

Although q_1 may initially be very small (the 1st oscillator in a low state of excitation), the process of determining X_1 will quickly drive the first oscillator to a high level of excitation and will therefore also affect the second oscillator.

The above discussion indicates that one must use great care in analyzing QND systems. Thorne *et al.* proposed a quantity

$$\tilde{X} = \cos\omega t q_1 + \frac{\sin\omega t}{\omega} p_1 \tag{29}$$

as a suitable candidate for Q in a QNDD system. However, time-dependent functions can be realized in nature only by a suitable choice of dynamic variables and states. (That is, one must design a clock or oscillator to produce the requisite function $\cos\omega t$.) The above arguments indicate that a second harmonic oscillator is not a suitable candidate for such a clock to realize the Thorne *et al.* system as an exact QNDD system. It is of course, QNDR, just as the energy measuring system (example II) is.

IV. QNDD SYSTEM

It is, however, possible to realize, at least mathematically, the Thorne *et al.* kind of QNDD system. The detector will consist of a harmonic oscillator with Hamiltonian $\frac{1}{2}(p^2 + \omega^2 q^2)$. It will also have as a component part a clock with Hamiltonian H_c. We now demand that

$$Q = \Pi q - \Psi p \tag{30}$$

where Π and Ψ are functions of the clock's dynamic variables. In order that this system be QNDR, we find that we must have

$$\frac{d\Pi}{dt} = -\omega^2\Psi, \quad \frac{d\Psi}{dt} = \Pi, \tag{31a}$$

or

$$[H_c, \Pi] = +\omega^2\Psi, \quad [H_c, \Psi] = -\Pi. \tag{31b}$$

The interaction with the gravity wave is assumed to be via linear interaction with $T = q$.

We now obtain QNDD only if the variable

$$\Psi = i[Q, T] \tag{32}$$

is independent of the readout system (in particular, only if Ψ commutes with Q). This therefore gives

$$[\Pi, \Psi] = 0. \tag{33}$$

The ideal clock system therefore obeys

$$\frac{d\Pi}{dt} = -\omega^2\Psi, \quad \frac{d\Psi}{dt} = \Pi, \quad [\Pi, \Psi] = 0. \tag{34}$$

The first two relations are the simple harmonic-oscillator relations, but the last relation ensures that the clock cannot be realized as a simple harmonic oscillator.

However, the Hamiltonian

$$H_c = P_x{}^2 + P_y{}^2 - \gamma(yP_x - xP_y), \tag{35}$$

which is that of a charged particle in a suitable constant magnetic field and quadrupole electric field (to cancel the quadratic terms in the magnetic Hamiltonian), and with

$$\Pi = \gamma P_x, \quad \Psi = P_y, \quad \gamma = \omega \tag{36}$$

gives the relations of, e.g., (34). Such a clock will be perfect in the sense that the readout system will not affect the clock. By setting up the initial state of the clock $|A\rangle$ such that

$$\Pi_{t=0}|A\rangle = |A\rangle, \quad \psi_{t=0}|A\rangle = 0, \tag{37}$$

we will have

$$\Pi|A\rangle = A\cos\omega t|A\rangle, \quad \psi|A\rangle = \frac{A}{\omega}\sin\omega t|A\rangle, \tag{38}$$

at all times, and

$$Q = qA\cos\omega t + \frac{pA}{\omega}\sin\omega t. \tag{39}$$

This therefore represents an ideal realization of the Thorne *et al.* prescription for QNDD. However, I have not yet discovered a physically realistic technique for coupling such a clock to a harmonic oscillator and a readout system in the required manner.

V. QNDR–ROTATING SPHERE

Most gravity wave detectors are harmonic oscillators to good approximation, and the above arguments have therefore concentrated on QND on harmonic oscillators. The technique is, however, more widely applicable. I will now sketch the application of QNDR to determining changes in the angular momentum of a spinning sphere. Although such a sphere is probably not a good gravity wave detector (although a gravity wave would be expected to interact with such a sphere, mainly by changing the moment of inertia of the sphere, the effect is probably too small to be of real benefit), the frame-dragging experiment of Everitt and Fairbank[5] does look for changes in the angular momentum direction.

As a simple demonstration of a QNDR system for measuring changes in one component of angular momentum let us place a uniformly charged spinning sphere within a set of superconducting Helmholtz coils. The coils are the first step in the readout system. The free Hamiltonian of the

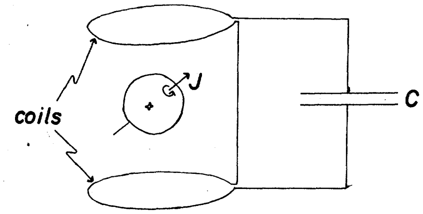

FIG. 3. L-C readout system for angular momentum of spinning sphere.

sphere will just be $J^2/2I$ where I is the moment of inertia of the sphere and J is its total angular momentum. The interaction of the sphere with the coils will be via the magnetic moment of the sphere which is proportional to the angular momentum.[5] In particular, for the arrangement shown in Fig. 3, the interaction is proportional to $J_z B_z$ where B_z is the magnetic field produced by the current in the coils. If the coils are now connected to a capacitor C, we can finally write the full Hamiltonian as

$$\mathcal{H}=\frac{J^2}{2I}+\frac{\alpha^2 J_z}{2L}+\alpha p J_z+\frac{p^2}{2L}+\frac{q^2}{2C}, \quad (40)$$

where q is the charge on the capacitor plates, p is essentially the flux through the coils, L is the coil inductance, and α is the coupling constant which is related to the charge distribution on the sphere, the dimensions of the coils, etc.

We therefore have

$$H_D=\frac{J^2}{2I}+\frac{\alpha^2 J_z^2}{2L},$$
$$Q=L_z, \quad (41)$$
$$R=p.$$

As $[Q,H_D]=0$, this system is QNDR.

From the equations of motion for the L-C circuit we find

$$\frac{d^2q}{dt^2}=-LCq-C\alpha\frac{dJ_z}{dt}. \quad (42)$$

If the frequency of the LC circuit is very high, the term $C\,\alpha\, dJ_z/dt$ will simply act as a displacement of the equilibrium charge on the capacitor. By monitoring the equilibrium position of the oscillator (i.e., the average charge on the capacitor plates or average field in the capacitor over a number of cycles) one can obtain a direct measurement of the changes in J_z.

The simplest coupling to such a sphere to change its z component of angular momentum would be via the angular momentum—e.g., the coupling would be of the form

$$\mu F(t) J_x. \quad (43)$$

[If, $F(t)$ were a magnetic field in the x direction, the coupling would be of just this form.] In this case we find

$$\frac{dJ_z}{dt}=\mu J_y F(t),$$
$$\frac{dJ_y}{dt}=-\mu Jz F(t)+\epsilon J_x R. \quad (44)$$

The system is not QNDD because of the dependence of J_z on R via J_y. It is, however, QNDR and DDR. The latter can be put to use by placing the sphere into a state of large J_y (i.e., the sphere spinning rapidly about an axis ⊥ to the z axis).

VI. CONCLUSIONS

In the above, I have formally defined what is meant by a number of possible quantum nondemolition measurement schemes and have given a number of examples to illustrate that such schemes are in theory possible. Furthermore, I have demonstrated that at least QNDR is realistic in that one could hope to realize it physically as well.

One point which should be clarified is the division of the measuring apparatus into detector and readout system. This division is of course arbitrary, and the dividing line could be placed anywhere along the chain leading from the gravity wave to the experimentalist. The basic assumption is that the detector be part of the beginning of the chain which is simple enough to be completely analyzable. The readout system will then consist of the rest of the chain, which is in general far too complex to be analyzed in any way but a crude semiclassical manner. The purpose of quantum nondemolition measurement (as opposed to DDR) is to isolate the analyzable portion of the detection chain from the rest of the chain, so that one can accurately predict the response of the system to a gravity wave.

The above schemes are also what could be called strict quantum nondemolition schemes in which the detector is exactly isolated. One could, however, also approximate quantum nondemolition schemes by suitable couplings. This was the approach followed by myself[6] and by Braginsky[7] in some of the original attempts at quantum nondemolition. For example, instead of coupling to the energy of the free harmonic oscillator $p^2+\omega^2 q^2$, one could couple to the quantity q^2 alone. This is not QNDR. However, if the readout system can be set up so as to respond only to the low-frequency components of q^2, the system becomes essentially equivalent to a coupling to $p^2+\omega^2q^2$ as long as the change in $p^2+\omega^2q^2$ due to the gravity wave is not too rapid. For example in the L-C circuit oscillator, a device measuring the long-term averaged force on the capacitor plates will be approximately QNDR.

One can furthermore design systems which are QNDR, and which approximate a QNDD-type system. Essentially the problem with the QNDR systems is that the response of the system to gravity waves is made random by the dependence of the "amplifying factor" $[T,Q]$ on the readout system. This amplifying factor need not be exactly known, however, as it can cause only a fractional (not an absolute) error in the determination of $G(t)$, and thus by itself offers no limit to the weakness of gravity waves one can detect.

However, we have for a non-QNDD scheme that

$$[[Q,T],Q]\neq 0\,. \tag{45}$$

Thus we have an uncertainty relation

$$\Delta[Q,T]\Delta Q \geqslant \tfrac{1}{2}|\langle[[Q,T],Q]\rangle|\,. \tag{46}$$

In general, therefore, to maintain $\Delta[Q,T]/[Q,T]$, the fractional error in the "amplifying factor", small, we must have a nonzero ΔQ—i.e., we cannot measure the desired quantity Q to arbitrary accuracy and still keep the fractional error in the "amplifying factor" small.

To give an example of an approximate QNDD system, let us return to example II of a QNDR system in which the coupling was to the energy of the oscillator. The problem here is that in the process of measuring the energy exactly, the readout system's interaction with the oscillator will make the amplifying factor

$$i[Q,T]=i[H/\omega,q]=p/\omega \tag{47}$$

uncertain. Using a crude uncertainty-principle argument, we have that, if H/ω is known exactly, then

$$\begin{aligned}&\langle p/\omega\rangle=0\,,\\&\langle p^2/\omega^2\rangle=\langle H/\omega^2\rangle.\end{aligned} \tag{48}$$

We therefore have a very large uncertainty in p and thus a very large uncertainty in the reaction of the oscillator (and in particular of the relevant variable $Q=H/\omega$) to a gravity wave. This uncertainty in p/ω is essentially due to the "energy-phase uncertainty" for a harmonic oscillator. If we know the energy exactly, we do not know the phase of the oscillator at all, and thus do not know p/ω.

The time development of p and of q reinforces this view,

$$\begin{aligned}&\dot p=-(1+\epsilon R)\omega^2 q\,,\\&\dot q=(1+\epsilon R)p\,.\end{aligned} \tag{49}$$

An approximate solution is

$$p=p_0\cos\left[\omega\int(1+\epsilon R)dt\right] + q_0\omega\sin\left[\omega\int(1+\epsilon R)dt\right]. \tag{50}$$

The uncertainty in R is now reflected as an uncertainty in the phase of p.

To minimize the uncertainty in the phase, the probable values of R for the state of the readout system must be kept as small as possible. However, one can show that if the readout system is to be observably affected by changes in the number of quanta H/ω, the readout cannot be in the state of $R=0$, and R must have some finite uncertainty.

Let ϕ be some variable of the readout system which is to respond to changes in H/ω.

We have

$$\Delta\phi\Delta R\geqslant\tfrac{1}{2}|\langle[\phi,R]\rangle|\,. \tag{51}$$

But we also have

$$-i\frac{d\phi}{dt}=[\phi,H_R]+\epsilon[\phi,R]Q\,. \tag{52}$$

The change in ϕ caused by Q is just proportional to $[\phi,R]$ which enters the uncertainty relation.

Crudely, in a time τ, the change in ϕ caused by Q is

$$|\delta\phi|\approx\epsilon|\langle[\phi,R]\rangle|Q\tau\,. \tag{53}$$

Once we determine $\delta\phi$ of the readout, which we will know only with accuracy $\pm\Delta\phi$, we will know Q with accuracy ΔQ,

$$Q\pm\Delta Q\approx\frac{\delta\phi\pm\Delta\phi}{\epsilon\langle[\phi,R]\rangle\tau}\,, \tag{54}$$

or

$$\Delta Q\approx\frac{\Delta\phi}{\epsilon|\langle[\phi,R]\rangle|\tau}\geqslant\frac{\Delta\phi}{\epsilon\Delta\phi\Delta R\tau}\approx\frac{1}{\epsilon\Delta R\tau}\,. \tag{55}$$

(I have neglected all time dependence of $\langle[\phi,R]\rangle$, etc.) We therefore have

$$\epsilon\Delta Q(\Delta R\tau)\geqslant 1\,.$$

For $Q=H/\omega$, $\epsilon\Delta R\tau$ will be approximately the uncertainty in the phase of p as in Eq. (50).

This suggests a possible approximate QNDD scheme. The change in H/ω caused by a gravity wave $G(t)$ is given by

$$\frac{d(H/\omega)}{dt}=\frac{p}{\omega}G\,. \tag{56}$$

At any time t, let us assume that we know the magnitude $\bar p$ of p to an accuracy of $\pm\Delta\bar p$. Let us assume that over a short time period we measure H/ω to an accuracy $\Delta(H/\omega)$. The value of $G(t)$

during this time period is given by

$$G\delta t = \frac{\delta(H/\omega) \pm \Delta(H/\omega)}{\bar{p}/\omega \pm \Delta(\bar{p}/\omega)}$$
$$\approx \frac{\delta(H/\omega)}{\bar{p}/\omega}\left(1 \pm \frac{\Delta p}{\bar{p}}\right) \pm \frac{\Delta(H/\omega)}{\bar{p}/\omega}, \qquad (57)$$

where $\delta(H/\omega)$ is the measured change in H/ω in the time interval δt.

Now the error Δp in p is assumed to be principally due to the error in phase $\epsilon \Delta R \delta t$. The error introduced during the readout is at least

$$\Delta \bar{p} \approx \bar{p}\, \epsilon \Delta R \delta t\,.$$

By careful selection of the readout state we can at best have [by Eq. (55)]

$$\epsilon \Delta R \sim 1/\Delta(H/\omega)\delta t\,.$$

We therefore have an error in the measurement of G of

$$\Delta G \approx \frac{\Delta(H/\omega)}{\bar{p}/\omega\delta t} + \frac{G}{\Delta(H/\omega)}\,. \qquad (58)$$

The second term represents a fractional error in the measurement of G. For an accuracy of 1% in the determination of G, $\Delta H/\omega$ must be greater than or of the order of 100 quanta. The first term represents an absolute error in the determination of G, and sets the limit on the strength of the signal which is detectable. For any given accuracy of measurement of $\Delta H/\omega$, this term can be made arbitrarily small by making $\bar{p}$ sufficiently large. (i.e., by placing the detector into a highly excited state).

This QNDR system therefore can be used as an approximate QNDD system by making the interaction between the readout and the oscillator sufficiently weak (so that in the measuring time δt the number of quanta H/ω cannot be measured to absolute accuracy), by choosing the state of the readout system sufficiently carefully to minimize ΔR given the accuracy with which H/ω is to be measured, and by placing the oscillator into an initial state with a very large amplifying factor p. (As p is an oscillating function, we will have p large only during certain periods of oscillation of the oscillator, which implies that the oscillators sensitivity will vary depending on when in its cycle the gravity wave hits the oscillator.)

Furthermore, as the phase error caused by the uncertainty in R is cumulative, one will have to measure p occasionally to reduce the error in p to an acceptable fractional magnitude. This will of course upset the QNDR nature of the readout system. It is thus obvious that an ideal QNDD readout system would be the best possible form of readout, as one then does not have to concern oneself with the state of the readout system (required above to reduce ΔR) except insofar as one must measure changes in the readout to detect changes in the appropriate oscillator variable Q.

However, one can probably design an approximate QNDD system to sufficient accuracy to be of use in the ultimate detection of gravity waves. Such systems are at present undergoing intensive investigation by a group working with Thorne.

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