Rotating universe with successive causal and noncausal regions

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We present a new exact solution of Einstein's equations. The source of the curvature is a shear-free nonexpanding but rotating fluid. There are successive concentric causal and noncausal regions, which generalize a typical behavior of Gödel's space-time.

Some years ago Gödel[1] discussed a cosmological solution of Einstein's modified equations of gravity (with non-null cosmological constant Λ) of the type

$$ds^2 = dt^2 - dr^2 - dz^2 + g(r)d\phi^2 + 2h(r)d\phi dt . \qquad (1)$$

Gödel has shown that if we set $g(r) = \sinh^4 r - \sinh^2 r$ and $h(r) = \sqrt{2}\sinh^2 r$, the geometry given by (1) satisfies Einstein's equations with the stress-energy tensor of a fluid with density of energy ρ without pressure. The congruence of curves comoving with the fluid is shear-free, has no expansion, but has a constant non-null rotation.

The existence of such non-null vorticity is responsible for many unusual properties of the Gödel space-time, among which we may cite the absence of a global Gaussian system of coordinates and the existence of closed timelike curves. This last fact can be easily seen by considering the behavior of the function $g(r)$. Let us define a critical radius R_c such that $\sinh R_c = 1$. Then the curves with constant $r = R$ and $t = z = 0$ such that $R > R_c$ have $g(R)$ positive definite and, consequently, are closed timelike curves.

The existence of such curves poses a difficult problem related to the possibility of violation of the well-established causality principle. One can argue, however, that as long as one remains inside the region bounded by the critical radius, nothing exceptional occurs, i.e., the causality principle is not violated.

Particularly curious should be the case of a geometry of the type described by the same form (1) in which $g(r)$ should have not only one but many roots for different values of r, that is, $g(r_1) = g(r_2) = \ldots = g(r_n) = 0$ for $r_1 < r_2 < \ldots < r_n$. Such a geometry would present a succession of concentric regions such that a causal (noncausal) region is encircled internally and externally by two noncausal (causal) regions. This would bring some additional difficulties besides those encountered in Gödel's model. In this paper we exhibit precisely a solution of Einstein's equations in which such a property is present. Although such a solution may not conform to any period of the real universe, it can help us in understanding the properties of Einstein's theory of gravity.

We start by defining the one-forms θ^A.

$$\begin{aligned} \theta^0 &= dt + h(r)d\phi , \\ \theta^1 &= dr , \\ \theta^2 &= \Delta(r)d\Phi , \\ \theta^3 &= dz , \end{aligned} \qquad (2)$$

in which we have set $\Delta(r) \equiv (h^2 - g)^{1/2}$. The geometry (1) is obtained from (2) by the expression

$$ds^2 = \theta^A\theta^B\eta_{AB} = (\theta^0)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 , \qquad (3)$$

where η_{AB} is the Minkowski tensor diag(+ - - -).

Then the tetrad frame defined by $\theta^A = e^A(\alpha)\,dx^\alpha$ is given by

$$e^0_{(0)} = 1, \quad e^0_{(2)} = h, \quad e^1_{(1)} = 1, \quad e^2_{(2)} = \Delta, \quad e^3_{(3)} = 1 .$$

A straightforward calculation gives the value of the curvature tensor in this tetrad frame. The unique non-null components of the contracted curvature tensor are

$$R_{00} = -\frac{1}{2}\frac{h'^2}{\Delta 2}, \qquad (4a)$$

$$R_{11} = \frac{\Delta''}{\Delta} - \frac{1}{2}\frac{h'^2}{\Delta^2}, \qquad (4b)$$

$$R_{22} = R_{11}, \qquad (4c)$$

$$R_{02} = \frac{1}{2}\frac{h'\Delta'}{\Delta^2} - \frac{1}{2}\frac{h''}{\Delta}, \qquad (4d)$$

where $h' \equiv dh/dr$.

Einstein's equations are given by

$$R_{AB} = -kT_{AB} + kT\eta_{AB} + \Lambda\eta_{AB} . \qquad (5)$$

We remark that in the tetrad system the tensor indices are lowered and raised by the Minkowski metric.

We take for the stress-energy tensor the expression

$$T_{AB} = \rho\delta^0_A\delta^0_B - p h_{AB} + \pi_{AB} , \qquad (6)$$

where h_{AB} is the three-dimensional metric defined by $h_{AB} \equiv \eta_{AB} - \delta_A^0 \delta_B^0$. The anisotropic pressure π_{AB} is symmetric, trace-free, and orthogonal to the comoving observer ($\pi_{0A} = 0$).

Let us evaluate now the kinematical quantities associated with the congruence of the fluid.

In the tetrad frame the shear σ_{AB}, the expansion θ, and the vorticity ω_{AB} are given by

$$2\sigma_{AB} = -\gamma^0_{AB} - \gamma^0_{BA} + \eta_{B0}\gamma^0_{A0} + \gamma^0_{B0}\eta_{A0} - \tfrac{2}{3}\theta h_{AB}\,, \quad (7a)$$

$$\theta = -\gamma^0_{MN}\eta^{MN}\,, \quad (7b)$$

$$2\omega_{AB} = -\gamma^0_{AB} + \gamma^0_{BA} + \gamma^0_{A0}\delta^0_B - \gamma^0_{B0}\delta^0_A\,, \quad (7c)$$

$$\dot{V}_A = -\gamma^0_{A0}\,, \quad (7d)$$

in which γ^A_{BC} are the Ricci coefficients.

From (2) we easily obtain the nonvanishing coefficients

$$\gamma_{012} = -\gamma_{102} = \gamma_{201} = -\gamma_{021} = -\gamma_{120} = \gamma_{210} = \frac{1}{2}\frac{h'}{\Delta}\,, \qquad \gamma_{122} = -\gamma_{212} = \frac{\Delta'}{\Delta}\,. \quad (8)$$

Thus, the fluid has no expansion, no shear but has a non-null vorticity given by

$$\omega_{12} = -\frac{1}{2}\frac{h'}{\Delta}\,. \quad (9)$$

The vorticity vector $\omega^A = \frac{1}{2}\epsilon^{ABC}\omega_{BC}$ is given by $\omega^A = (0, 0, 0, -\frac{1}{2}h'/\Delta)$. This vector can be used in order to construct the vortex matrix $\Omega^A{}_B$ by means of the expression

$$\Omega^A{}_B = \omega^A\omega_B + \tfrac{1}{3}\Omega^2\delta^A{}_B\,, \quad (10)$$

where $\Omega^2 = -\omega^A\omega_A$. Thus we have

$$\Omega^1_1 = \Omega^2_2 = -\tfrac{1}{2}\Omega^3_3 = \tfrac{1}{3}\Omega^2\,.$$

Einstein's equations are

$$R_{00} = -\tfrac{1}{2}k\rho + \Lambda - \tfrac{3}{2}kp\,, \quad (11a)$$

$$R_{11} = -\tfrac{1}{2}k\rho + \tfrac{1}{2}kp - k\alpha - \Lambda\,, \quad (11b)$$

$$R_{22} = -\tfrac{1}{2}k\rho + \tfrac{1}{2}p - k\beta - \Lambda\,, \quad (11c)$$

$$R_{33} = -\tfrac{1}{2}k\rho + \tfrac{1}{2}kp + k(\alpha+\beta) - \Lambda\,, \quad (11d)$$

$$R_{02} = 0\,, \quad (11e)$$

in which we have written the anisotropic pressure as $\pi^1_1 = -\alpha$, $\pi^2_2 = -\beta$, $\pi^3_3 = \alpha + \beta$.

From Eq. (4c) we obtain $\alpha = \beta$. Thus the anisotropic and the vortex matrices are proportional. We set

$$\pi_{AB} = -\gamma^2\Omega_{AB}\,. \quad (12)$$

We remark that expression (12) is a particular case of the so-called principle of generalized viscosity[2] which tells us that we can set up phenomenological equations relating the fluid dynamical quantities (such as anisotropic pressure, heat flux) to the kinematical ones (such as shear, vorticity).

From (11e) we have

$$\frac{h'}{\Delta} = 2\Omega = \text{constant}\,. \quad (13)$$

The remaining set of Einstein's equations are

$$k(p-\rho) = 2\Lambda - k\tfrac{4}{3}\gamma^2\Omega^2\,, \quad (14a)$$

$$k(3p+\rho) = 2\Lambda + 4\Omega^2\,, \quad (14b)$$

$$\frac{\Delta''}{\Delta} - 2\Omega^2 = -\tfrac{1}{2}k\rho + \tfrac{1}{2}kp - \tfrac{1}{3}k\gamma^2\Omega^2 - \Lambda\,. \quad (14c)$$

A solution of this set of equations is obtained if we set

$$\Delta = \sin(mr)\,. \quad (15)$$

Then we have

$$k\rho = -\Lambda + (1 + k\gamma^2)\Omega^2\,, \quad (16a)$$

$$kp = \Lambda + (1 - \tfrac{1}{3}k\gamma^2)\Omega^2\,, \quad (16b)$$

$$m^2 = (k\gamma^2 - 2)\Omega^2\,. \quad (16c)$$

The positivity of energy and pressure are guaranteed if we have a negative (or null) cosmological constant and if the coefficient of viscosity γ satisfies the relation

$$2 < k\gamma^2 < 3\,. \quad (17)$$

Finally, the functions $h(r)$ and $g(r)$ can be evaluated from (15) and (13). We obtain

$$g(r) = \left(\frac{k\gamma^2 + 2}{k\gamma^2 - 2}\right)\cos^2(mr) - 1\,, \quad (18a)$$

$$h(r) = -\frac{2}{(k\gamma^2 - 2)^{1/2}}\cos(mr)\,. \quad (18b)$$

It is now straightforward to see that our geometry does indeed contain the property which we enunciated at the beginning of this paper. The roots of $g(r_n)$ are given by the values

$$mr_n = \text{arc}\cos\left(\frac{k\gamma^2 - 2}{k\gamma^2 + 2}\right) + n\pi\,,$$

where n is a natural number.

In the region $0 < r < r_0$, $g(r)$ is positive definite. In this region the circle $z = t = 0$, $r = \text{constant}$ is timelike and consequently this is a noncausal region. In the subsequent region, $r_0 < r < r_1$, the function $g(r)$ is everywhere negative and thus the closed curves are spacelike. This is a causal region. The next region is noncausal, and so on. This proves our previous assertion.

As a final property of our geometry let us observe that it is homogeneous like Gödel's solutions. The Killing vectors which generates the isometries can be evaluated and we find

$$k_{\rm I} = \frac{\partial}{\partial t},$$

$$k_{\rm II} = \frac{\partial}{\partial z},$$

$$k_{\rm III} = \frac{\partial}{\partial \phi},$$

$$k_{\rm IV} = \cos(m\phi)\left(\Delta - \frac{h}{4\Omega}\frac{\Delta'}{\Delta}\right)\frac{\partial}{\partial t} + \frac{m}{4\Omega}\sin(m\phi)\frac{\partial}{\partial r} + \frac{1}{4\Omega}\frac{\Delta'}{\Delta}\cos(m\phi)\frac{\partial}{\partial \phi},$$

$$k_{\rm V} = -\sin(m\phi)\left(\Delta - \frac{h}{4\Omega}\frac{\Delta'}{\Delta}\right)\frac{\partial}{\partial t} + \frac{m}{4\Omega}\cos(m\phi)\frac{\partial}{\partial r} - \frac{1}{4\Omega}\frac{\Delta'}{\Delta}\sin(m\phi)\frac{\partial}{\partial \phi}.$$

The algebra and subalgebras of these Killing vectors are similar to the case discussed by Gödel.[3]

It seems worthwhile to make some additional comments on the source of our geometry, as the presentation of our stress-energy tensor may appear somewhat unusual. For the present geometry (and, to a certain extent, for more general ones) a realization of the non-Stokes-type character of the fluid which we used here is given by the electromagnetic field. Indeed, let us write the stress-energy tensor of the electromagnetic field, decomposing it for the observer V^A, under the form

$$T_{AB}^{(\rm em)} = \rho_{(\rm em)} V_A V_B - p_{(\rm em)} h_{AB} + \Pi_{AB},$$

in which $p_{(\rm em)} = \frac{1}{3}\rho_{(\rm em)}$, $\Pi_{AB} = T_{CD}^{(\rm em)} h_A^C h_B^D + \frac{1}{3}\rho_{(\rm em)} h_{AB}$. In the comoving frame, the electric and magnetic vectors are parallel to the vorticity. We set $E^\alpha = (0,0,0,A)$, $H^\alpha = (0,0,0,B)$. Maxwell's equations are satisfied if we set $A = e\sin(2\Omega z)$ and $B = e\cos(2\Omega z)$, in which e is a constant.

In the above frame the heat flux (Poynting vector) vanishes. Consequently, the anisotropic pressure of the electromagnetic field reduces to

$$\Pi_{AB}^{(\rm em)} = -E_A E_B + \tfrac{1}{3}E^2 h_{AB} - H_A H_B + \tfrac{1}{3}H^2 h_{AB},$$

in which

$$H^2 = H_A H^A = -B^2,$$

$$E^2 = E_A E^A = -A^2.$$

Thus, in the tetrad frame we have

$$\Pi_{11}^{(\rm em)} = \Pi_{22}^{(\rm em)} = -\tfrac{1}{2}\Pi_{33}^{(\rm em)} = \tfrac{1}{3}e^2.$$

Consequently, we obtain Eq. (12). Thus we can interpret the non-Stokes-type fluid in terms of a fluid with density of energy ρ_0 and pressure p_0 plus an electromagnetic field. Indeed, we can see this by the following. By setting $\Pi_{AB} = -\gamma^2\Omega_{AB}$ we obtain $e^2 = \gamma^2\Omega^2$. Thus, the set of equations (16) turns into

$$\rho_0 + \frac{\gamma^2\Omega^2}{2} = -\Lambda + (1+\gamma^2)\Omega^2,$$
$$p_0 + \tfrac{1}{6}\gamma^2\Omega^2 = \Lambda + (1 - \tfrac{1}{3}\gamma^2)\Omega^2, \qquad (16')$$
$$m^2 = (\gamma^2 - 2)\Omega^2,$$

in which we have set $k = 1$.

In this case, instead of the previous restriction (17) we have only the condition $\gamma^2 > 2$ and the positivity of the cosmological constant. The positivity of density ρ_0 and pressure p_0 is guaranteed if the value of the cosmological constant remains bounded by the limits

$$\Omega^2(\tfrac{1}{2}\gamma^2 - 1) < \Lambda < \Omega^2(\tfrac{1}{2}\gamma^2 + 1).$$

This ends the proof. The non-Stokes-type fluid generated by assuming Eq. (12) can be interpreted in terms of well-known sources, that is, in terms of a perfect fluid (ρ_0, p_0) plus an electromagnetic field.

Finally, it seems worthwhile to make the following remark. In the last few decades there have been published many exact solutions of the equations of motion of the gravitational field (Einstein's equations) which represent universes filled with an electromagnetic field or anisotropic fluids, etc., as the main source of curvature. What is the meaning of these solutions? What is the interest in them? Did we find magnetic fields in the universe? Did we find large anisotropy of the background radiation? No. The meaning of these solutions is not intimately related to direct observations. The meaning of them is this: They provide a set of geometries which satisfy Einstein's equations for a physical stress-energy tensor. This set will be (at least, we believe) of great help in the future analysis of the properties of the spectra of all possible real (physical) cosmological solutions, in the search of the unique geometry of the cosmos.

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[1]K. Gödel, Rev. Mod. Phys. 21, 447 (1949).
[2]M. Novello and J. B. d'Olival (unpublished).
[3]We remark that although in Gödel's cosmos the closed timelike curves are not geodesics, and thus should not be the paths of the free matter, in our present solution there are closed timelike geodesics.