Thin spherical shells

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The equation of motion of a thin spherical timelike shell is obtained directly from the Lanczos equations for all spherically symmetric embedding four-geometries. No explicit form for the intrinsic surface energy three-tensor of the shell is used. It is shown that for static embedding geometries the condition of positive-definite total proper shell mass guarantees the impossibility of the coalescence of inner and black-hole horizons but allows the coalescence of black-hole and cosmological horizons.

I. INTRODUCTION

The equation of motion of a thin spherical timelike shell with spherically symmetric, embedding four-geometries is obtained directly from the Lanczos equations without use of an explicit form for the intrinsic surface energy three-tensor of the shell. The procedure used here has the advantage that it streamlines the calculations and does not introduce a spurious constant of integration. With static embedding four-geometries it is shown that the impossibility of the coalescence of inner and black-hole horizons is an immediate consequence of the condition that the shell have positive-definite total proper mass. The possibility of the coalescence of black-hole and cosmological horizons, discussed recently,[1] can be traced directly to the behavior of null geodesics in the asymptotic regions of asymptotically de Sitter spacetimes.

II. EQUATION OF MOTION

A timelike hypersurface Σ, which divides spacetime into two distinct four-dimensional manifolds V^+ and V^-, represents the history of a thin shell if its extrinsic curvature three-tensor K_{ij} suffers a discontinuity when Σ is crossed, $[K_{ij}]\equiv K_{ij}|^+ - K_{ij}|^- \neq 0$. The intrinsic surface energy three-tensor S_{ij} is given by the Lanczos equations which can be written in the form[2]

$$-8\pi S_{ij}=\gamma_{ij}-g_{ij}\gamma\,, \tag{1}$$

where g_{ij} is the intrinsic metric of Σ, $\gamma\equiv g^{ab}\gamma_{ab}$, and $\gamma_{ij}\equiv[K_{ij}]$. In terms of the shell tangent u^i, the proper surface density σ is defined by the eigenvalue equation

$$S_i{}^j u^i=-\sigma u^j\,,\quad u^i u_i=-1. \tag{2}$$

From Eqs. (1) and (2) it follows that

$$-8\pi\sigma=\gamma_{ij}u^i u^j+\gamma\,. \tag{3}$$

For a spherical shell the intrinsic metric can be given in terms of "comoving" coordinates as

$$ds_\Sigma{}^2=R^2(\tau)(d\theta^2+\sin^2\theta d\phi^2)-d\tau^2\,, \tag{4}$$

where

$$u^i\equiv(\dot\theta,\dot\phi,\dot\tau)=(0,0,1)\,,$$

and the overdot denotes $d/d\tau$. From Eqs. (3) and (4) it follows that

$$\gamma_{\theta\theta}=-4\pi R^2\sigma\equiv -M(\tau)\,, \tag{5}$$

from which we obtain

$$K_{\theta\theta}{}^2|^+=\frac{1}{4M^2(\tau)}(K_{\theta\theta}{}^2|^- - K_{\theta\theta}{}^2|^+ - M^2(\tau))^2\,. \tag{6}$$

Write the metric in the four-geometries V^+ and V^- as[3]

$$ds_\pm{}^2=(2cdvdr-c^2fdv^2+r^2(d\theta^2+\sin^2\theta d\phi^2))_\pm\,, \tag{7}$$

where $c=c(v,r)$ and $f=f(v,r)$. For a simultaneous embedding of Σ in V^+ and V^- it follows from Eqs. (4) and (7) that $r_\Sigma=R(\tau)$, $\theta^+=\theta^-$, $\phi^+=\phi^-$, and

$$(c^2f\dot v^2-2c\dot R\dot v)|^\pm=1\,. \tag{8}$$

With $K_{ij}\equiv -n_\alpha\delta(\partial x^\alpha/\partial\xi^j)/\delta\xi^i$, ξ^i the intrinsic coordinates of Σ, and $\vec n$ the unit normals to Σ, it follows from Eq. (7) that

$$K_{\theta\theta}=\pm R(\tau)(cf\dot v-\dot R)\,, \tag{9}$$

where $(n_v,n_r,n_\theta,n_\phi)=\pm c(-\dot R,\dot v,0,0)$. From Eq. (8) we have

$$cf\dot v-\dot R=\pm(\dot R^2+f)^{1/2}\,. \tag{10}$$

Inserting Eq. (9) with Eq. (10) into Eq. (6) then yields the equation of motion

$$\dot R^2=\left(\frac{R}{2M}\right)^2(f_+-f_-)^2-\tfrac12(f_++f_-)+\left(\frac{M}{2R}\right)^2. \tag{11}$$

Note that Eq. (11) follows directly from the Lanczos equations (1), the spherical symmetry, and the form (7) for the metric of the enveloping spacetimes. In particular, no explicit form of the intrinsic surface energy three-tensor S_{ij} has been used. To obtain $M(\tau)$ (and thus the history of Σ) we must, through use of the field equations, relate an explicit form of the three-tensor S_{ij} to the jump across Σ of the energy-momentum tensors of the enveloping spacetimes.[4] Equation (11) is usually derived from an explicit form of S_{ij} (e.g., an ideal fluid) via integration of $\ddot{R}$, a procedure which introduces a spurious constant of integration.

III. RESTRICTION

Whereas Eq. (11) is indifferent to the labeling of V^+ and V^-, the behavior of Σ is not. In particular, we must distinguish the "exterior" to Σ.[5] To do this, consider the history of Σ in the two-spaces of V^+ and V^- determined by constant angular coordinates θ and ϕ. At some τ in the history of Σ consider a spacelike geodesic g of an enveloping four-geometry to Σ, parametrized by λ such that $\lambda=0$ on Σ and $n^\alpha = dx^\alpha/d\lambda$ on Σ. If the area of two-spheres along g is increasing at Σ, that is, $dr/d\lambda|_\Sigma > 0$ [where $dr/d\lambda|_\Sigma = n^r = \pm(cf\dot{v} - \dot{R}) = \pm(\dot{R}^2+f)^{1/2}$], we say that $\vec{n}$ points to the exterior of Σ. As long as $\sigma \neq 0$, only one of the enveloping four-geometries can be exterior to Σ. Suppose we label V^+ as exterior to Σ at τ such that $\dot{R}^2 \neq -f_+$, then the exterior normal to Σ is determined, and V^+ remains exterior to Σ until τ_* such that $\dot{R}^2 = -f_+$ ($\Rightarrow f_+ < 0$ for finite $\dot{v}_+$). The "interior" or "exterior" character of V^+ subsequent to τ_* depends on the explicit form of S_{ij}.

For shells of positive-definite total proper mass $M(\tau)$, it follows from Eq. (5) that $K_{\theta\theta}|^+ < K_{\theta\theta}|^-$. Moreover, as long as V^+ is exterior to Σ, $K_{\theta\theta}|^+ > 0$, so that from Eqs. (9) and (10) it follows that

$$f_+ < f_- \; . \tag{12}$$

The coalescence of the inner and black-hole horizons of the Reissner-Nordström and Reissner–Nordström–de Sitter (or anti–de Sitter) spacetimes violates condition (12). Thus, with the reasonable initial conditions that $f_\pm > 0$ and $\dot{v}_\pm$ finite on some nonsingular spacelike initial surface, the third law of black-hole mechanics[6] (for *any* spherical timelike shell "perturbation") applied to inner and black-hole horizons can be viewed as an immediate consequence of the condition $M(\tau) > 0$.[7]

In asymptotically de Sitter spacetimes the null goedesics are asymptotically totally divergent near the future spacelike infinity (or totally convergent near the past spacelike infinity). The coalescence of the black-hole and cosmological horizons does not violate condition (12); rather it is the reverse transformation which does.[8]

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[1] K. Lake, Phys. Rev. D 19, 421 (1979), Sec. III.

[2] W. Israel, Nuovo Cimento 44B, 1 (1966); 48B, 463 (1967).

[3] This form of the most general spherically symmetric line element is due to H. Bondi. See, for example, Nature (Phys. Sci.) 238, 58 (1972). If $c=$ const, then we can arrange that $c^2=1$, $c=+1$ for "ingoing" null v and $c=-1$ for "outgoing" null v. For a discussion of the case $c^2=1$ see J. Plebanski and J. Stachel, J. Math. Phys. 9, 269 (1968).

[4] From the Gauss-Codazzi equations, the Einstein field equations, and the Lanczos equations it follows that

$$S^j_{i;j} = -\left[T_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} n^\beta\right]$$

[see Ref. 2 and V. de la Cruz and W. Israel, Nuovo Cimento 51, 744 (1967)]. For an ideal fluid of surface pressure P,

$$S_{ij} = (\sigma + P)u_i u_j + P g_{ij},$$

so that

$$\mathring{M} + 8\pi R\mathring{R}P = 4\pi R^2 [T_{\alpha\beta}u^\alpha n^\beta].$$

This expression is clearly invariant under the change $T_{\alpha\beta} \to T_{\alpha\beta} - \Lambda g_{\alpha\beta}$. If $[T_{\alpha\beta}u^\alpha n^\beta] = P = 0$, then (with given initial conditions) the information governing the history of the dust shell is completely determined by (11). More generally, from (1) and (4) it follows that

$$8\pi P = \tfrac{1}{2}(\gamma - \gamma_{ij}u^i u^j) = -\gamma_{\cdot\tau\tau} + 4\pi\sigma,$$

and from (7) we have

$$K_{\tau\tau} = \pm\left(-\frac{\ddot{v}}{\dot{v}} - \frac{\dot{v}c}{2}\frac{\partial f}{\partial r} - \frac{\mathring{c}}{c} - \left(\frac{\partial c}{\partial r}\right)\left(\frac{1 + c\mathring{R}\dot{v}}{c^2\dot{v}}\right)\right)$$

and

$$4\pi R^2[T_{\alpha\beta}u^\alpha n^\beta] = \pm\frac{R^2}{2}\left[-\frac{c\dot{v}^2}{R}\frac{\partial f}{\partial v} + \frac{\dot{v}^2}{c}A + \frac{\mathring{R}}{c}B\right],$$

where

$$A = -\frac{1}{c}\left(\frac{\partial c}{\partial v}\right)^2 + \frac{1}{2}\frac{\partial^2 c}{\partial v^2} + \frac{fc}{2}\frac{\partial^2 c}{\partial v \partial r} + \frac{c}{4}\left(\frac{\partial f}{\partial v}\frac{\partial c}{\partial r} - \frac{\partial c}{\partial v}\frac{\partial f}{\partial r}\right) - f\frac{\partial c}{\partial v}\frac{\partial c}{\partial r}$$

and

$$B=(cf\mathring{v}-\dot{R})\left(\frac{2}{R}\frac{\partial c}{\partial r}-\frac{1}{c}\left(\frac{\partial c}{\partial r}\right)^2+\frac{1}{2}\frac{\partial^2 c}{\partial r^2}\right)$$

for

$$(n^v,n^r,n^\theta,n^\phi)=\pm(\mathring{v},\ cf\mathring{v}-\dot{R},0,0).$$

[5]The following discussion generalizes and simplifies that of K. Lake and R. C. Roeder, Phys. Rev. D 17, 1935 (1978).

[6]J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. 31, 161 (1973).

[7]D. G. Boulware, Phys. Rev. D 8, 2363 (1973), has shown that with a flat interior a shell with $M>0$ cannot collapse to a naked Reissner-Nordström singularity. This is an immediate consequence of relation (12). Clearly one can arrange a sequence of "reverse" transformations with "naked" or "degenerate" interior and regular (nondegenerate) exterior embedding geometries.

[8]Condition (12) is easily visualized in terms of the convergence (or divergence) of null geodesics. Along null geodesics of the metric (7) (except the radial null geodesics with $v=\text{const}$ on which the metric is based),

$$c^2f=2c\ dr/dv+r^2\sin^2\theta_0(d\phi/dv)^2.$$

Suppose $\dot{c}=0$, then if an apparent horizon forms in V^- $[f^-(\tau_0)=0]$, it follows that either $f^+<0$, or that an apparent horizon has formed in $V^+[f^+(\tau)=0,\ \tau<\tau_0]$. Consider the static case $c^2=1$, $f=f(r)$. In terms of null geodesics condition (12) gives the relation

$$c\frac{dr}{dv}\bigg|^- <0\Rightarrow c\frac{dr}{dv}\bigg|^+<0.$$

With $c_+=c_-$, under the conditions governing relation (12), we have the following: (i) For $c=+1$, the convergence $(dr/dv<0)$ of null geodesics in the interior V^- demands the convergence of null geodesics in the exterior V^+. (ii) For $c=-1$, the divergence $(dr/dv>0)$ of null geodesics in the interior V^- demands the divergence of null geodesics in the exterior V^+.