Radiating fluid spheres in general relativity

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We study the time-dependent field equations for radiating fluid spheres, where we take the energy-momentum tensor as the sum of the energy-momentum tensor for a perfect fluid and a radially expanding radiation. We present four new exact analytic solutions with the corresponding pressure, density, and luminosity distributions. We also present a case where two of the three metric coefficients are given as a function of the other, thus reducing the problem to choosing one of the metric elements such that distribution of physical quantities is reasonable. The solutions we have considered correspond to fluids at rest and have radius independent of time. Hence they represent fluid spheres in equilibrium.

I. INTRODUCTION

Einstein's field equations were first published in 1916. Since then much work has been done to obtain exact analytic solutions and to classify them. Even today, in the age of fast computers, searching for analytic solutions remains valuable due to the fact that once such a solution is found one can immediately study all of its physical properties. Besides, because of the nonlinearity of the equations, even if one has a general solution there is always the possibility of the existence of singular solutions, with entirely new properties. In this paper we study radiating fluid spheres in general relativity and present four new exact analytic solutions for their interior.

During most stages of stellar evolution the variation of physical parameters with time is so slow that a quasistatic approximation becomes sufficient. That is, one sets all the time derivatives in the structure equations to zero and mimics the evolution of the star by a series of static models, with varying chemical composition for post-main-sequence stars and varying central density for cold stars. On the other hand this method fails for stars evolving very rapidly from one energy state to another. In such cases full time-dependent equations have to be solved to get realistic models.

In general relativity finding the field of radiating fluid spheres becomes a much more difficult problem, basically due to the fact that coupled nonlinear ordinary differential equations of the static case are now coupled nonlinear partial differential equations.[1] The first attempt to solve time-dependent equations outside the field of cosmology was done by Oppenheimer and Snyder.[2] The only analytic solution they found was for nonradiating spherically symmetric distributions in a state of free fall with zero pressure. Later their solution was generalized to include pressure gradient terms and radiation.[3] Several authors have also studied time-dependent field equations for perfect fluids.[4,5] Since a nonstatic system in general would be radiating energy their solutions could only apply to special cases. Besides, the discovery of extragalactic strong radio sources and their huge energy requirements motivated Hoyle and Fowler[6,7] to develop a theory of hot, convective supermassive stars where general-relativistic effects are important. Also Thorne and Zytkow[8] have numerically analyzed the structure of red supergiant stars with degenerate neutron cores. So far the only analytic solutions to time-dependent field equations with nontrivial pressure distribution and radiation are given by Vaidya.[9] Here we present four new exact analytic solutions, one of which is physically reasonable everywhere, and the other three could be used to represent portions of relativistic radiating stars.

II. FIELD EQUATIONS AND METHOD OF OBTAINING ANALYTIC SOLUTIONS

A nonstatic system will be radiating energy. Hence, the energy-momentum tensor can be taken as[9]

$$T^{\mu\nu}=(T^{\mu\nu})_{\text{mech}}+(T^{\mu\nu})_{\text{em}}. \tag{2.1}$$

Here $(T^{\mu\nu})_{\text{mech}}$ corresponds to the mechanical part of the energy-momentum tensor due to matter and can be taken as the energy-momentum tensor for a perfect fluid:

$$(T^{\mu\nu})_{\text{mech}}=(P+\rho)\,v^{\mu}v^{\nu}-Pg^{\mu\nu}, \tag{2.2}$$

$$v^{\mu}v_{\mu}=1. \tag{2.3}$$

$(T^{\mu\nu})_{\text{em}}$ corresponds to the energy-momentum tensor for spherically symmetric and radially expanding radiation and can be given as[10]

$$(T^{\mu\nu})_{\text{em}}=\sigma w^{\mu}w^{\nu}, \tag{2.4}$$

$$w^{\mu}w_{\mu}=0,\quad w^{\mu}{}_{;\nu}w^{\nu}=0, \tag{2.5}$$

where σ is the density of the flowing radiation and $w^{\mu}{}_{;\nu}$ means a covariant derivative of w^{μ}. Hence the final $T^{\mu\nu}$ can be written as

$$T^{\mu\nu}=(P+\rho)\,v^{\mu}v^{\nu}-Pg^{\mu\nu}+\sigma w^{\mu}w^{\nu}. \tag{2.6}$$

Next we will consider the most general form of line element that has spherical symmetry as

$$ds^2=-e^{\alpha}dr^2-r^2e^{\beta}(d\theta^2+\sin^2\theta\, d\phi^2)+e^{\gamma}dt^2, \tag{2.7}$$

$\alpha=\alpha(r,t)$, $\beta=\beta(r,t)$, $\gamma=\gamma(t,r)$, and $c=G=1$.

Using comoving coordinates where

$$v^1=v^2=v^3=0,\quad v^4=e^{-\gamma/2}, \tag{2.8}$$

$$w^{\mu}{}_{;\nu}w^{\nu}=0,\quad w^{\mu}w_{\mu}=0,\quad w^2=w^3=0, \tag{2.9}$$

and $T_{\mu}{}^{\nu}{}_{;\nu}=0$, we can derive an equation that replaces the Tolman-Oppenheimer-Volkoff (TOV) equation for radiating fluid spheres as[9]

$$P'=-\tfrac{1}{2}(P+\rho)\gamma'+e^{(\alpha-\gamma)/2}\left[\dot{\rho}+(P+\rho)\left(\dot{\beta}+\frac{\dot{\alpha}}{2}\right)\right], \tag{2.10}$$

where prime and dot denote partial derivatives with respect to r and t, respectively.

It can be shown that for $P/\rho\to 0$ and quasistatic processes, $T^{1\nu}{}_{;\nu}=0$ reduces to the classical hydrostatic equilibrium equation with radiation pressure,[9]

$$\frac{d}{dr}(P_g+P_r)=-\frac{M(r)}{r^2}\rho. \tag{2.11}$$

Now we can write down Einstein's field equations for the metric (2.7) as

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R=-8\pi T^{\mu\nu}, \tag{2.12}$$

$$8\pi T_1^1=-e^{-\alpha}\left(\frac{\beta'^2}{4}+\frac{\beta'\gamma'}{2}+\frac{\beta'+\gamma'}{r}+\frac{1}{r^2}\right)+\frac{e^{-\beta}}{r^2}+e^{-\gamma}\left(\ddot{\beta}+\tfrac{3}{4}\dot{\beta}^2-\frac{\dot{\beta}\dot{\gamma}}{2}\right), \tag{2.13}$$

$$8\pi T_2^2=8\pi T_3^3=-e^{-\alpha}\left(\frac{\beta''}{2}+\frac{\gamma''}{2}+\frac{\beta'^2}{4}+\frac{\gamma'^2}{4}-\frac{\alpha'\beta'}{4}+\frac{\beta'\gamma'}{4}-\frac{\alpha'\gamma'}{4}-\frac{\alpha'}{2r}+\frac{\beta'}{r}+\frac{\gamma'}{2r}\right)$$
$$+e^{-\gamma}\left(\frac{\ddot{\alpha}}{2}+\frac{\ddot{\beta}}{2}+\frac{\dot{\alpha}^2}{4}+\frac{\dot{\beta}^2}{4}+\frac{\dot{\alpha}\dot{\beta}}{4}-\frac{\dot{\alpha}\dot{\gamma}}{4}-\frac{\dot{\beta}\dot{\gamma}}{4}\right), \tag{2.14}$$

$$8\pi T_4^4=-e^{-\alpha}\left(\beta''+\tfrac{3}{4}\beta'^2-\frac{\alpha'\beta'}{2}+\frac{3\beta'}{r}-\frac{\alpha'}{r}+\frac{1}{r^2}\right)+\frac{e^{-\beta}}{r^2}+e^{-\gamma}\left(\frac{\dot{\alpha}\dot{\beta}}{2}+\frac{\dot{\beta}^2}{4}\right), \tag{2.15}$$

$$8\pi T_1^4=-e^{-\gamma}\left(\dot{\beta}'-\frac{\dot{\beta}\gamma'}{2}+(\dot{\beta}-\dot{\alpha})\frac{\beta'}{2}+(\dot{\beta}-\dot{\alpha})\frac{1}{r}\right). \tag{2.16}$$

The components of $T_{\mu\nu}$ in comoving coordinates become

$$T_1^1=-P+\sigma w_1w^1,\quad T_2^2=T_3^3=-P, \tag{2.17}$$

$$T_4^4=\rho+\sigma w_4w^4,\quad T_1^4=\sigma w_1w^4, \tag{2.18}$$

where

$$T_{\mu}^{\nu}=(P+\rho)\,v_{\mu}v^{\nu}-Pg_{\mu}^{\nu}+\sigma w_{\mu}w^{\nu}, \tag{2.19}$$

with

$$v^1=v^2=v^3=0,\quad v^4=e^{-\gamma/2} \tag{2.20}$$

and

$$w^2=w^3=0,\quad w_{\mu}w^{\mu}=0,\quad w^{\mu}{}_{;\nu}w^{\nu}=0. \tag{2.21}$$

From (2.17), (2.18) and (2.13)–(2.16) one can obtain the physical variables in terms of the metric coefficients:

$$P=-T_2^2,\quad \rho=T_1^1+T_4^4-T_2^2,\quad \sigma=-e^{-(3\alpha-\gamma)/2}(w^1)^{-2}T_1^4, \tag{2.22}$$

$$L(\text{luminosity})=-4\pi r^2e^{\beta}\sigma w^1, \tag{2.23}$$

$$w^4=e^{(\alpha-\gamma)/2}w^1,\quad w^{\mu}{}_{;\nu}w^{\nu}=0 \text{ (see Appendix A)}. \tag{2.24}$$

From (2.17) and (2.18) one can obtain the analogous equation to Eq. (2.5) in Bayin[11]:

$$T_1^1-T_2^2=e^{(\gamma-\alpha)/2}T_1^4. \tag{2.25}$$

Using (2.25) and Eqs. (2.13)–(2.16) we obtain the following differential equation which is not altered by introducing the cosmological constant to the field equations:

$$\frac{\beta''}{2}+\frac{\gamma''}{2}+\frac{\gamma'^2}{4}-\frac{\alpha'\beta'}{4}-\frac{\beta'\gamma'}{4}-\frac{\gamma'\alpha'}{4}-\frac{\alpha'}{2r}-\frac{\gamma'}{2r}-\frac{1}{r^2}+\frac{e^{\alpha-\beta}}{r^2}$$
$$+e^{\alpha-\gamma}\left(\frac{\ddot\beta}{2}-\frac{\ddot\alpha}{2}-\frac{\dot\alpha^2}{4}+\frac{\dot\beta^2}{2}-\frac{\dot\alpha\dot\beta}{4}-\frac{\dot\beta\dot\gamma}{4}+\frac{\dot\alpha\dot\gamma}{4}\right)+e^{(\alpha-\gamma)/2}\left(\dot\beta'-\frac{\dot\beta\gamma'}{2}+(\dot\beta-\dot\alpha)\frac{\beta'}{2}+(\dot\beta-\dot\alpha)\frac{1}{r}\right)=0\,. \tag{2.26}$$

In order to solve this differential equation in α, β, and γ we need two more relations, one of them corresponding to an equation of state and the other to the law of energy transfer. But due to mathematical difficulties we will use the same method discussed by Bayin[11] and assume two relations among α, β, and γ and later check the solution for physical reasonableness. We take a solution to be physically reasonable if pressure and density are positive and monotonically decreasing functions of r throughout the star, and pressure vanishes at finite radius. Since we are using comoving coordinates the boundary of the star should be independent of time. This alone greatly limits the possible forms for time dependence of the metric coefficients. Also we should have positive σ and w^1 for radiating fluid spheres.

III. BOUNDARY CONDITIONS

A nonstatic distribution would be radiating energy, and so it would be surrounded by an ever-expanding zone of radiation. Hence the system will have two boundaries. The first boundary is at $r=R_1(t)$, where the matter pressure vanishes and the second boundary is at $R_2=R_2(t)$ where the radiation zone ends. For $r\geq R_2$ the line element is simply the Schwarzschild exterior solution,

$$ds^2=-\left(1-\frac{2M}{r}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2)+\left(1-\frac{2M}{r}\right)dt^2\,. \tag{3.1}$$

For $R_1\leq r\leq R_2$ the line element is found by solving the field equations for radially expanding radiation with energy-momentum tensor given in Eq. (2.4). This solution is obtained by Vaidya[10] as

$$ds^2=-\left(1-\frac{2m}{r}\right)dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2)+\frac{\dot m^2}{F^2}\left(1-\frac{2m}{r}\right)dt^2\,, \tag{3.2}$$

where $F(m)$ is an arbitrary function of m given by

$$m'\left(1-\frac{2m}{r}\right)=F(m)\,. \tag{3.3}$$

Continuity of the metric at $r=R_2$ determines the arbitrary function $\Phi(t)$ which appears when (3.3) is solved for m. $F(m)$ is determined by the conditions at $r=R_1$, then Eq. (3.3) is solved to obtain $m=m(r,t)$. Taking the classical limit of (3.2) one sees that classically $F(m)$ corresponds to luminosity.

IV. SOME ANALYTIC SOLUTIONS

First we assume $\alpha=\beta$ which reduces the metric to isotropic coordinates: Equation (2.26) becomes

$$\frac{A''}{A}+\frac{B''}{B}-\frac{2B'^2}{B^2}-\frac{2A'B'}{AB}-\frac{1}{r}\left(\frac{A'}{A}+\frac{B'}{B}\right)=-\frac{B}{A}\left(\frac{2\dot B'}{B}-\frac{2B'\dot B}{B^2}-\frac{2\dot BA'}{BA}\right), \tag{4.1}$$

where

$$A^2(r,t)=e^{\gamma(r,t)},\quad B^2(r,t)=e^{\beta(r,t)}\,.$$

Equation (4.1) is separable in space and time variables. Hence we make the substitution

$$A(r,t)=f(r)g(t)\,,\qquad B(r,t)=h(r)k(t)\,, \tag{4.2}$$

which gives the following differential equations:

$$f''-\left(\frac{2h'}{h}+\frac{1}{r}\right)f'+\left(\frac{h''}{h}-\frac{2h'^2}{h^2}-\frac{1}{r}\frac{h'}{h}\right)f=(2sh)\frac{f'}{f}\,, \tag{4.3}$$

$$\dot k=sg\,, \tag{4.4}$$

where s is an arbitrary constant. Note that this equation reduces to a second-order homogeneous differential equation for $f(r)$ once $h(r)$ is known for a static case.[12]

Next we assume

$$e^{\beta}=B^2=B_0\Phi^b(r)k^2(t)\,, \tag{4.5}$$

$$e^{\gamma}=A^2=A_0\Phi^{-a}(r)g^2(t)\,, \tag{4.6}$$

where B_0, A_0, b, a are constants. Substituting these into (4.3) gives us the differential equation to be solved for $\phi(r)$:

$$\Phi'' - \frac{\Phi'}{r} - c\,\frac{\Phi'^2}{\Phi} = -2\left(\frac{B_0}{A_0}\right)^{1/2} \Phi^{(b+a)/2}\left(\frac{a}{b-a}\right) s\Phi', \tag{4.7}$$

where

$$c = \frac{\frac{1}{2}b^2 - \frac{1}{2}a^2 - ab + b - a}{b-a}. \tag{4.8}$$

We also have $\dot{k} = sg$ to be solved for the time-dependent part of the metric. The only solution we have found for general a and b is $\phi = \text{constant}$. It is also possible to solve (4.7) for $a=0$ and $a=-b$. Now we study these solutions.

Solution I

For $a=-b$, $c=\frac{1}{2}b+1\neq 1$, $s\neq 0$, we find

$$\phi^{-c+1} = c_1 e^{zr}(zr-1) + c_2,$$

where

$$z = \left(\frac{B_0}{A_0}\right)^{1/2} s. \tag{4.9}$$

This gives the following metric coefficients:

$$e^{\beta} = B_0\left[c_1 e^{zr}(zr-1)+c_2\right]^{-2} k^2(t), \tag{4.10}$$

$$e^{\gamma} = A_0\left[c_1 e^{zr}(zr-1)+c_2\right]^{-2} g^2(t). \tag{4.11}$$

Now we determine $k(t)$ and $g(t)$ using (4.4) and the physical reasonableness conditions discussed in Sec. II. In order to have radius independent of time we may choose $k(t)=g(t)$ which leads to $k(t) = c_0 e^{st}$ from Eq. (4.4). Actually $k(t)=c_0 e^{st}$ is the only form that has the desired property (see Appendix B).

The pressure and density distributions now become

$$8\pi P = \left[c_1 e^{zr}(zr-1)+c_2\right]^2 \frac{e^{-2st}}{c_0^2}\left[\frac{-2c_1 z^2 e^{zr}}{B_0}\left(\frac{\left(-\frac{1}{2}c_1 z^2 r^2 - 2c_1 + zc_1 r\right)e^{zr} + 2c_2 + zc_2 r}{\left[c_1 e^{zr}(zr-1)+c_2\right]^2}\right) - \frac{s^2}{4A_0}\right], \tag{4.12}$$

$$8\pi\rho = \left[c_1 e^{zr}(zr-1)+c_2\right]^2 \frac{e^{-2st}}{c_0^2}\left[\frac{z^2 e^{zr}}{B_0}\left(\frac{\left(c_1 e^{zr} + c_2 + zrc_1 e^{zr}\right)6c_1 - 3c_1^2 z^2 r^2 e^{zr}}{\left[c_1 e^{zr}(zr-1)+c_2\right]^2}\right) + \frac{3s^2}{4A_0}\right], \tag{4.13}$$

$$L(r,t) = -\frac{c_1 s}{E} z^2 r^3 \left[c_1 e^{zr}(zr-1)+c_2\right]^{-1} e^{zr}, \tag{4.14}$$

$$8\pi P_c = (c_2-c_1)^2 \frac{e^{-2st}}{c_0^2}\left[-\frac{4c_1 z^2}{B_0(c_2-c_1)} - \frac{s^2}{4A_0}\right], \tag{4.15}$$

$$8\pi\rho_c = (c_2-c_1)^2 \frac{e^{-2st}}{c_0^2}\left[\frac{z^2}{B_0}\,\frac{(c_2+c_1)6c_1}{(c_2-c_1)^2} + \frac{3s^2}{4A_0}\right], \tag{4.16}$$

$$L_c = 0.$$

This solution could be physically reasonable throughout the star. The radius is defined by $P(R)=0$ and R is independent of time but it cannot be found analytically from Eq. (4.12).

Solution II

For $a=0$, $c=(\frac{1}{2}b+1)\neq 1$, $s\neq 0$, we have

$$\phi(r) = (c_0 r^2 + c_1)^{1/(1-c)}, \tag{4.17}$$

$$e^{\beta} = B_0(c_0 r^2 + c_1)^{-2} e^{2st}, \tag{4.18}$$

$$e^{\gamma} = A_0 e^{2st}, \tag{4.19}$$

with the properties

$$8\pi P = e^{-2st}\left(-\frac{4c_0 c_1}{B_0} - \frac{s^2}{4A_0}\right), \tag{4.20}$$

$$8\pi\rho = e^{-2st}\left(\frac{12C_0 C_1}{B_0} + \frac{3s^2}{4A_0}\right). \tag{4.21}$$

Note that $\rho = T^4_4$ since $T^4_1 = 0$. This solution has pressure and density as functions of time only; also, there is no radiation. Hence, even though it could be useful in cosmology with the addition of a cosmological constant to obtain positive pressure, it is not a suitable solution to represent interiors of radiating fluid spheres.

Next we make the following substitution to Eq. (2.26):

$$A^2(r,t) = e^{\gamma}, \quad B^2(r,t) = e^{\alpha}, \quad C^2(r,t) = e^{\beta}, \tag{4.22}$$

and $A(r,t) = f(r)g(t)$, $B(r,t) = h(r)k(t)$, $C(r,t) = l(r)m(t)$, to obtain

$$\frac{l''}{l}-\frac{l'^2}{l^2}+\frac{f''}{f}-\frac{f'l'}{fl}-\frac{f'h'}{fh}-\frac{h'l'}{hl}-\frac{h'}{hr}-\frac{f'}{fr}-\frac{1}{r^2}+\frac{h^2k^2}{l^2m^2r^2}$$
$$+\frac{h^2k^2}{f^2g^2}\left(\frac{\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{\ddot{k}}{k}-\frac{\dot{k}\dot{m}}{km}-\frac{\dot{g}\dot{m}}{gm}+\frac{\dot{g}\dot{k}}{gk}\right)+\frac{hk}{fg}\left[2\frac{l'\dot{m}}{lm}-2\frac{\dot{m}}{m}\frac{f'}{f}-\frac{2\dot{k}l'}{kl}+\left(\frac{\dot{m}}{m}-\frac{\dot{k}}{k}\right)\frac{2}{r}\right]=0. \tag{4.23}$$

If we take

$$l(r)=1,\quad c(r,t)=m(t),\ \text{and}\ k(t)=m(t), \tag{4.24}$$

Eq. (4.23) becomes separable. Hence the differential equations to be solved for r and t, respectively, become

$$\frac{f^2}{hf'}\left(\frac{f''}{f}-\frac{f'h'}{fh}-\frac{h'}{hr}-\frac{f'}{fr}-\frac{1}{r^2}+\frac{h^2}{r^2}\right)=s \tag{4.25}$$

and

$$\frac{2\dot{m}}{g}=s, \tag{4.26}$$

where s is a separation constant which in general could take complex values:

$$s=s_0+is_1. \tag{4.27}$$

Of course to obtain physically reasonable answers at the end we have to take the real part of the physical quantities.

For the above choice (4.24) the line element can be given as

$$ds^2=-h^2(r)m^2(t)dr^2-m^2(t)r^2(d\theta^2+\sin^2\theta d\phi^2)$$
$$+f^2(r)g^2(t)dt^2, \tag{4.28}$$

which could be considered as the time-dependent generalization of the Schwarzschild metric in canonical coordinates.

Using $m=g$, Eq. (4.26) is immediately integrable giving (see Appendix B)

$$m(t)=c_0e^{(s/2)t}, \tag{4.29}$$

and (4.25) gives the following first-order differential equation for $h(r)$, once $f(r)$ is known:

$$h'\left(\frac{f'}{f}+\frac{1}{r}\right)=h\left(\frac{f''}{f}-\frac{f'}{fr}-\frac{1}{r^2}\right)$$
$$-h^2\left(s\frac{f'}{f^2}\right)+h^3\left(\frac{1}{r^2}\right). \tag{4.30}$$

This is Abel's equation of the first kind and is quite difficult to solve. Note that for static models (4.30) reduces to the Bernoulli equation, which can be reduced to quadratures immediately.[11]

Now we present various solutions to (4.30).

Solution III

For

$$f'/f^2=0, \tag{4.31}$$

Eq. (4.30) is immediately integrable leading to the following metric coefficients:

$$f(r)=C_1, \tag{4.32}$$

$$h(r)=(c_2r^2+1)^{-1}, \tag{4.33}$$

$$m(t)=c_0e^{(s/2)t}, \tag{4.34}$$

$$T_1^4=0. \tag{4.35}$$

This solution again is not suitable for representing interiors of radiating fluid spheres. Actually it can be transformed into the form of solution II by a new choice of radial marker. Hence we will not discuss it any further.

Equation (4.30), which can be written as

$$h'=f_1(r)h+f_2(r)h^2+f_3(r)h^3, \tag{4.36}$$

can be converted into one of the second order by the following transformation[13]:

$$h(r)=u(z)\,v(r),\quad z=\int vf_2(r)dr,\quad v=\exp\left[\int f_1dr\right]. \tag{4.37}$$

In the new independent variable,

$$u'(z)=u^2+g(z)u^3,\quad g(z)=v(r)\frac{f_3}{f_2}. \tag{4.38}$$

Now let

$$z'(\bar{r})+\frac{1}{\bar{r}u(z)}=0, \tag{4.39}$$

so that

$$\bar{r}^2z''(\bar{r})+g(z)=0. \tag{4.40}$$

If (4.40) can be solved, then (4.39) will give us

$u(z)$, and (4.37) gives the general solution of (4.30).

Now we will try the following form for $f(r)$:

$$f(r)=a_0 r^n,\quad n(\text{constant})\neq -1. \tag{4.41}$$

Evaluating the integrals (4.37) we get

$$v(r)=r^{(n^2-2n-1)/(n+1)}, \tag{4.42}$$

$$z=\frac{s(n+1)}{2a_0}r^{-2n/(n+1)},\qquad g(z)=-\frac{a_0}{sn}\left(\frac{2a_0 z}{s(n+1)}\right)^{-(n^2-n-1)/n}. \tag{4.43}$$

Hence (4.40) becomes

$$\bar r^2 z''(\bar r)+k_0 z^m=0, \tag{4.44}$$

where

$$k_0=-\frac{a_0}{sn}\left(\frac{2a_0}{s(n+1)}\right)^m,\quad m=-\frac{(n^2-n-1)}{n}. \tag{4.45}$$

We do not have a solution to (4.45) for general m, but for $m=1, n=1$ we have the following solutions:

$$z(\bar r)=\sqrt{\bar r}\,[c_1 z_1(\bar r)+c_2 z_2(\bar r)], \tag{4.46}$$

(i) $k_0-\frac14=r_0^2>0,\quad z_1(\bar r)=\cos r_0\ln\bar r,$

$$z_2(\bar r)=\sin r_0\ln\bar r, \tag{4.47}$$

(ii) $r_0^2<0,\quad z_1(\bar r)=(\bar r)^{r_0},\quad z_2(\bar r)=(\bar r)^{-r_0},$ (4.48)

(iii) $r_0^2=0,\quad z_1=1,\quad z_2(\bar r)=\ln\bar r,$ (4.49)

where $k_0=-a_0^2/s^2$. Now we will evaluate the physical parameters.

Solution IV

For $r_0^2>0$ we have found

$$z(\bar r)=F_0\sqrt{\bar r}\ln\bar r, \tag{4.50}$$

where $F_0=c_1\cos r_0+c_2\sin r_0$.

We will use $\bar r$ as our radial marker, where

$$r=\frac{s}{a_0F_0}(\sqrt{\bar r}\ln\bar r)^{-1}. \tag{4.51}$$

The physical parameters can be given as

$$f(\bar r)=\frac{s}{F_0}(\sqrt{\bar r}\ln\bar r)^{-1}, \tag{4.52}$$

$$h(\bar r)=-\frac{2a_0}{s}\frac{\ln\bar r}{\ln\bar r+2},\quad m(t)=c_0e^{(s/2)t}, \tag{4.53}$$

$$8\pi P(\bar r,t)=\frac{F_0^2e^{-st}}{c_0^2}\left[\frac{\bar r(\ln\bar r+2)^2}{4}\times\left(3-\frac{2\ln\bar r(\ln\bar r+4)}{(\ln\bar r+2)^2}\right)-\frac{\bar r}{4}(\ln\bar r)^2\right], \tag{4.54}$$

$$8\pi\rho(\bar r,t)=\frac{F_0^2e^{-st}}{c_0^2}\left[-\frac34\bar r(\ln\bar r+2)^2+\left(\frac{2a_0^2}{s^2}-\frac34\right)(\sqrt{\bar r}\ln\bar r)^2\right], \tag{4.55}$$

$$L(\bar r,t)=\frac{s^4}{8Ea_0^3F_0}\frac{(\ln\bar r+2)^2}{(\ln\bar r)^3\sqrt{\bar r}}. \tag{4.56}$$

The radius is defined by $P(\bar R)=0$, and is given by

$$\ln\bar R=\tfrac13(11\mp\sqrt{139}). \tag{4.57}$$

Note that we have the condition

$$r_0^2>0,\quad -\frac{a_0^2}{s^2}-\frac14>0, \tag{4.57$'$}$$

with the above equations, which could only be satisfied by complex s. Hence the real parts of the physical variables have to be taken after substituting $s=s_0+is_1$ into the above formulas with s_0 and s_1 arbitrary real constants. Note that from (4.51) $\bar r\to\infty$ as $r\to 0$ and $P_c\to\infty$, $\rho_c\to\infty$, and $L_c\to 0$ at the center. In this respect this solution is not reasonable at the center but could satisfy our physical reasonableness conditions for finite $\bar r$.

Solution V

For $r_0^2<0$ we have found

$$z(\bar r)=\sqrt{\bar r}\,[C_1(\bar r)^{r_0}+C_2(\bar r)^{-r_0}],\quad r=(s/a_0)[c_1(\bar r)^{r_0+1/2}+c_2(\bar r)^{-r_0+1/2}]^{-1}, \tag{4.58}$$

$$h(\bar r)=-\left(\frac{a_0}{s}\right)\frac{c_1(\bar r)^{r_0}+c_2(\bar r)^{-r_0}}{(r_0+\frac12)c_1(\bar r)^{r_0}+(\frac12-r_0)c_2(\bar r)^{-r_0}}, \tag{4.59}$$

$$f(\bar r)=\frac{s}{\sqrt{\bar r}}[c_1(\bar r)^{r_0}+c_2(\bar r)^{-r_0}]^{-1}, \tag{4.60}$$

$$m(t)=c_0 e^{(s/2)t},$$

$$8\pi P(\bar{r},t)=\frac{e^{-st}}{c_0^2}\left\{\frac{[(c_1/2+c_1r_0)(\bar{r})^{r_0}+(c_2/2-c_2r_0)(\bar{r})^{-r_0}]^2}{(a_0/s)[c_1(\bar{r})^{r_0}+c_2(\bar{r})^{-r_0}]}\times\left[\frac{\bar{r}[c_1(\bar{r})^{r_0}+c_2(\bar{r})^{-r_0}]}{(s/a_0)}-\frac{h'(\bar{r})^{1/2}}{h}\right]-\frac{\bar{r}}{4}[c_1(\bar{r})^{r_0}+c_2(\bar{r})^{-r_0}]^2\right\}, \tag{4.61}$$

$$8\pi\rho(\bar{r},t)=\frac{e^{-st}}{c_0^2}\left\{\left[\left(\frac{c_1}{2}+c_1r_0\right)(\bar{r})^{r_0}+\left(\frac{c_2}{2}-c_2r_0\right)(\bar{r})^{-r_0}\right]^2(3\bar{r})+\left(2\frac{a_0^2}{s^2}-\frac{3}{4}\right)[c_1(\bar{r})^{r_0+1/2}+c_2(\bar{r})^{-r_0+1/2}]^2\right\}, \tag{4.62}$$

where

$$\frac{1}{h}\frac{dh}{dr}=\frac{1}{h}\frac{dh}{d\bar{r}}\frac{1}{(dr/d\bar{r})}$$
$$=-\frac{a_0}{s}\sqrt{\bar{r}}\,\frac{[r_0c_1(\bar{r})^{r_0}-c_2r_0(\bar{r})^{-r_0}][c_1(\bar{r})^{r_0}+c_2(\bar{r})^{-r_0}]}{(c_1/2+c_1r_0)(\bar{r})^{r_0}+(c_2/2-c_2r_0)(\bar{r})^{-r_0}}$$
$$+\frac{a_0}{s}\sqrt{\bar{r}}\,\frac{[(c_1/2+c_1r_0)(\bar{r})^{r_0}-(c_2/2-c_2r_0)(\bar{r})^{-r_0}][c_1(\bar{r})^{r_0}+c_2(\bar{r})^{-r_0}]^2}{[(c_1/2+c_1r_0)(\bar{r})^{r_0}+(c_2/2-c_2r_0)(\bar{r})^{-r_0}]^2}, \tag{4.63}$$

$$L(\bar{r},t)=+\frac{s^4}{2Ea_0^3}\frac{[(r_0+\frac{1}{2})c_1(\bar{r})^{r_0}+(\frac{1}{2}-r_0)c_2(\bar{r})^{-r_0}]^2}{\sqrt{\bar{r}}\,[c_1(\bar{r})^{r_0}+c_2(\bar{r})^{-r_0}]^3}. \tag{4.64}$$

We have the condition

$$-\frac{a_0^2}{s^2}-\frac{1}{4}<0$$

with the above equations. This solution is also unphysical at the origin but otherwise it is well behaved.

Solution VI

For $r_0=0$ we have found

$$z(\bar{r})=\sqrt{\bar{r}}\,(c_1+c_2\ln\bar{r}),\quad r=\frac{s}{a_0}(\bar{r})^{-1/2}(c_1+c_2\ln\bar{r})^{-1}. \tag{4.65}$$

The physical variables become

$$h(\bar{r})=-\frac{2a_0}{s}\frac{c_1+c_2\ln\bar{r}}{c_1+2c_2+c_2\ln\bar{r}}, \tag{4.66}$$

$$f(\bar{r})=s(\bar{r})^{-1/2}(c_1+c_2\ln\bar{r})^{-1}, \tag{4.67}$$

$$8\pi P(\bar{r},t)=\frac{e^{-st}}{c_0^2}\left\{\frac{\bar{r}(c_1+2c_2+c_2\ln\bar{r})^2}{4}\times\left[1+4\left(\frac{c_1+c_2\ln\bar{r}}{c_1+2c_2+c_2\ln\bar{r}}\right)\left(\frac{1}{c_1+c_2\ln\bar{r}}-\frac{1}{c_1+2c_2+c_2\ln\bar{r}}\right)\right]-\frac{\bar{r}}{4}(c_1+c_2\ln\bar{r})^2\right\}, \tag{4.68}$$

$$8\pi\rho(\bar{r},t)=\frac{e^{-st}}{c_0^2}\left[-\frac{3\bar{r}}{4}(c_1+2c_2+c_2\ln\bar{r})^2+\left(\frac{2a_0^2}{s^2}-\frac{3}{4}\right)\bar{r}(c_1+c_2\ln\bar{r})^2\right], \tag{4.69}$$

$$L=\frac{s^4}{8Ea_0^3}(\bar{r})^{-1/2}\frac{(c_1+2c_2+c_2\ln\bar{r})^2}{(c_1+c_2\ln\bar{r})^3}. \tag{4.70}$$

The condition $r_0=0$ gives

$$\frac{a_0^2}{s^2}=-\frac{1}{4}.$$

Again we have to consider s in the complex plane and at the end take the real part of the physical variables which are unphysical at the origin, otherwise well behaved.

Note that Eq. (4.23) is still separable even if we take l as a function of r, as long as $k(t)=m(t)$. The differential equations to be solved in this case become

$$h'\left(-\frac{f'}{f}-\frac{l'}{l}-\frac{1}{r}\right)=-\left(\frac{l''}{l}-\frac{l'^2}{l^2}+\frac{f''}{f}-\frac{f'l'}{fl}-\frac{f'}{fr}-\frac{1}{r^2}\right)h+\left(s\frac{f'}{f^2}\right)h^2-\left(\frac{1}{l^2r^2}\right)h^3 \tag{4.71}$$

and

$$2\dot{m}/g=s\,. \tag{4.72}$$

Equation (4.71) is again an Abel equation for $h(r)$ and we need two more relations between h, f, and l to solve it. As our first relation we take

$$\frac{f'}{f}+\frac{l'}{l}+\frac{1}{r}=0\,, \tag{4.73}$$

which immediately makes it possible to write h and f in terms of l as

$$h=\left(\frac{l'}{l}+\frac{1}{r}\right)\left(-\frac{l^2r^2}{2}\right)\times\left[\frac{slr}{c_0}\mp\left(\frac{s^2l^2r^2}{c_0^{\,2}}-\frac{8}{l^2r^2}\right)^{1/2}\right] \tag{4.74}$$

and

$$f=\frac{c_0}{lr}\,, \tag{4.75}$$

where c_0 is an arbitrary constant. Hence, if we choose $l(r)$ as our second relation, (4.74) and (4.75) will give us the other metric coefficients and the problem will be solved. Of course whether these solutions we generated by arbitrarily choosing $l(r)$ are physically reasonable or not has to be settled later.

V. SUMMARY AND CONCLUSIONS

We have discussed radiating fluid spheres in general relativity and used the same technique discussed by Bayin[11] to solve them. The differential equation to be solved for isotropic coordinates is immediately separable in space and time coordinates; see Eqs. (4.3) and (4.5). In these coordinates we have found one new solution (solution I) which is physically reasonable throughout the entire star. Solution II is already known in cosmology and leads us to the Robertson-Walker metric. This solution is not good for representing radiating fluid spheres.

Next we studied the field equations for the metric (4.28) which can be considered as the time-dependent generalization of the Schwarzschild metric. Eq. (2.26) for this metric turned out to be separable and led to (4.25) and (4.26) to be solved for r and t, respectively. Equation (4.25) is an Abel equation and is quite difficult to solve. We reduced it to one of the second order (4.40). This led to three new solutions (solutions IV, V, VI). They are not reasonable at the origin but otherwise satisfy our physical reasonableness conditions. Also, for the metric

$$ds^2=-h(r)^2m(t)^2dr^2-l^2(r)m(t)^2(r^2d\theta^2+r^2\sin^2\theta d\phi^2)+f(r)^2g(t)^2dt^2\,,$$

we managed to write $h(r)$ and $f(r)$ in terms of $l(r)$. This reduced the problem to choosing a functional form for $l(r)$ which will lead to physically reasonable pressure, density, and luminosity distributions; see Eqs. (4.74) and (4.75). The solutions we have considered correspond to fluids at rest and have radius independent of time. Hence they represent fluid spheres in equilibrium.

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APPENDIX A

Since photons have to follow null geodesics, we obtain the following two equations to be satisfied by w^μ:

$$w^\mu w_\mu=0\,,$$

which gives

$$w^4=e^{(\alpha-\gamma)/2}w^1\,, \tag{A1}$$

and

$$w^\mu{}_{;\nu}w^\nu=0\,. \tag{A2}$$

Eliminating w^4 among (A1) and (A2) we obtain

$$\frac{\partial w^1}{\partial r}+e^{(\alpha-\gamma)/2}\frac{\partial w^1}{\partial t}+w^1\left(\frac{\alpha'+\gamma'}{2}+e^{(\alpha-\gamma)/2}\dot{\alpha}\right)=0\,, \tag{A3}$$

to be solved for w^1. A solution of this differential equation with the particular time dependence of the metric we are using is

$$w^1 = Ee^{(-\alpha-\gamma)/2} \text{ and } w^4 = Ee^{-\gamma}, \tag{A4}$$

where E is an arbitrary constant.

Using (A4) and (2.23) with the line element, Eq. (4.28), we obtain the following expression for luminosity:

$$L(r,t) = \frac{s}{2E}\left(\frac{f'r^2}{h^2 f}\right). \tag{A5}$$

Similarly, in isotropic coordinates the luminosity is given by

$$L(r,t) = \frac{s}{E}\left(\frac{f'r^2}{f}\right). \tag{A6}$$

APPENDIX B

The differential equation to be solved for the time-dependent part of the metric from (4.4) is

$$\dot{k} = sg. \tag{B1}$$

We are looking for a solution where the radius of the star is independent of time.

From (2.22) and (2.14) with $\alpha=\beta$ for isotropic coordinates the pressure takes the following form:

$$P(r,t) = e^{-\beta}\left(\frac{\beta''}{2} + \frac{\gamma''}{2} + \frac{\gamma'^2}{2} + \frac{\beta'+\gamma'}{2r}\right) - e^{-\gamma}\left(\ddot{\beta} + \frac{3}{4}\dot{\beta}^2 - \frac{\dot{\beta}\dot{\gamma}}{2}\right). \tag{B2}$$

With the substitution (4.2) we obtain

$$P(r,t) = \frac{1}{h^2k^2}\left(\frac{h''}{h} - \frac{h'^2}{h^2} + \frac{f''}{f} + \frac{h'f+f'h}{hfr}\right) - \frac{1}{f^2g^2}\left(\frac{2\ddot{k}}{k} + \frac{\dot{k}^2}{k^2} - \frac{2\dot{k}\dot{g}}{kg}\right). \tag{B3}$$

Since the radius is defined by $P(R,t)=0$, R could be independent of time either by having

$$k = g,$$

which leads to

$$k = c_0 e^{st}, \tag{B4}$$

or by having

$$\frac{2\ddot{k}}{k} + \frac{\dot{k}^2}{k^2} - \frac{2\dot{k}\dot{g}}{kg} = 0. \tag{B5}$$

But simultaneous solution of (B5) and (B1) leads to a trivial answer:

$$g=0, \quad k = \text{const}. \tag{B6}$$

Hence (B4) is the only solution with desired property. A similar argument leads to (4.29).

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