Interior magnetohydrodynamic structure of a rotating relativistic star

Jacob D. Bekenstein and Eliezer Oron
Department of Physics, Ben Gurion University, Beer Sheva, 84120 Israel
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We study the interior magnetohydrodynamic structure of a rotating stationary axisymmetric neutron star. We assume the fluid is ideal, infinitely conducting, and flows only azimuthaly. We justify this assumption by considering in detail the superfluid physics in the interior. We obtain some of our results by taking a certain limit of previously discovered magnetohydrodynamic conservation laws. We show that the angular velocity, electric and magnetic potentials, and the red-shifted chemical potential are constant on magnetic surfaces. We demonstrate that the absence of meridional circulation implies the vanishing of the toriodal magnetic field. This clashes with previous arguments from the probable evolution of the magnetic field during the collapse to the neutron star. We solve completely Maxwell's equations for the distribution of magnetic field strength, and we show that the magnetic surfaces are the equipotentials of a simple geometrical invariant. With neglect of gravitational effects the magnetic field must be uniform in the interior in accordance with the Deutsch model, but at variance with numerous other models which have been proposed for ordinary stars. Gravitation causes the magnetic surfaces to flare out toward the polar regions and enhances the central field as compared to the polar field. The star must be charged; the charge distribution depends on the magnetic field strength and on the angular velocity relative to the local inertial frames.

I. INTRODUCTION

It is widely agreed that pulsars and pulsating x-ray sources are rotating magnetized neutron stars.[1] In a neutron star (NS) the gravitational binding energy per particle can be a tenth of the rest energy. Thus general-relativistic effects are sizable, and a definitive investigation of the magnetic structure in the interior and magnetosphere of a pulsar or pulsating x-ray source will have to be based on general-relativistic magnetohydrodynamics (GRM). Although the outlines of GRM have existed for years,[2,3] there have not been concrete methods for solving the complicated coupled equations of the theory for a particular model. This is perhaps the reason almost all investigations of magnetized NS's (for example, Refs. 4–6) have been based on special-relativistic or Newtonian magnetohydrodynamics (but see Ref. 7). The model usually adopted[4] is that of a rotating conductor endowed with a uniform interior magnetic field which may or may not be parallel to the rotation axis. Aside from not being general-relativistic, this model has two drawbacks: The shape of the magnetic lines is assumed, not calculated, and the magnetic structure is treated separately from the fluid structure. Clearly the situation needs improvement. A fully consistent GRM model for a magnetized rotating NS is desirable, not only in the interest of realism, but also to make a check of relativistic gravitational theory in the strong-field limit possible once other aspects of the problem are well understood.

Our purpose is to show that a GRM treatment of a magnetized rotating NS model is feasible, and to derive a number of exact results bearing on the interior magnetohydrodynamic structure of an NS having its rotation and magnetic axes aligned. This restriction is, of course, overly stringent; in pulsars and pulsating x-ray sources the axes are unaligned—otherwise there would be no pulses. Yet one need only recall the progress made in understanding pulsars as a result of the classic Goldreich and Julian aligned-axis model[4] to realize that one could profit greatly from even such an oversimplified model. None of our results depend on Einstein's equations; they are equally valid in any metric theory of gravitation. The gravitational theory comes in only in determining the metric. Hence observational testing of various gravitational theories by comparison of the model with observations should be comparatively straightforward.

The jumping-off point of our analysis is a collection of conservation laws for stationary axisymmetric GRM flow which we deduced earlier.[8] We review these laws in Sec. II. In Sec. III we justify the assumptions of our model: that the star's material can be regarded as a single fluid, that the fluid is ideal and infinitely conducting, that the flow is purely azimuthal, and that there is stationary and axial symmetry. In Sec. IV we demonstrate that the fluid's angular velocity and the electromagnetic potentials are constant on the surfaces containing equivalent magnetic lines—the magnetic surfaces (MS's). The first result is the relativistic generalization of Ferraro's classical theorem[9] which was earlier considered by Yodzis.[3] In Sec. V we show that the appropriately red-shifted chemical potential of the fluid is constant on each

MS. In Sec. VI we prove that there can be no toroidal magnetic field in our model, and that consequently the electric current is purely azimuthal and of a convective nature. The magnetic field strength is the subject of Sec. VII, where we explicitly integrate Maxwell's equations to determine its distribution throughout the star. The shape of the MS's is determined in Sec. VIII, where we show that the MS's are equipotentials of an invariant combination of the Killing vectors and the angular velocity field. In Sec. IX we examine in detail the magnetic structure of a Newtonian model star: We show that the field must be uniform, and we discuss departure from this behavior due to first-order relativistic effects. Finally, in Sec. X we find some relations between MS constants, and also discuss the electric charge distribution within the star.

A word about conventions. Our signature is $+2$. Greek indices run from 0 to 3 with $x^\alpha = (t, x^1, x^2, \phi)$; t denotes time, ϕ is the azimuthal angle; Latin indices run over the remaining coordinates. We take $c=1$.

II. CONSERVATION LAWS IN GRM

Assuming the permittivity and permeability of the fluid are unity, one can describe the electromagnetic field by a single antisymmetric tensor $F_{\alpha\beta}$ obeying Maxwell's equations

$$F_{[\alpha\beta,\gamma]} = 0, \tag{1}$$

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha . \tag{2}$$

One can choose a reference velocity u^α, which we shall identify with the baryon velocity, and define the electric and magnetic field vectors referred to u^α (see Ref. 3):

$$E_\alpha = F_{\alpha\beta} u^\beta , \tag{3}$$

$$B^\alpha = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta} u_\beta F_{\gamma\delta} , \tag{4}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor.

The electric current can be decomposed into conduction and convection parts,[2,3]

$$J^\alpha = \sigma E^\alpha + \epsilon u^\alpha \tag{5}$$

where σ is the conductivity and $\epsilon = - J^\alpha u_\alpha$ is the charge density measured by an observer with velocity u^α. In most astrophysical situations σ is very large and may be regarded as infinite. Thus J^α will be finite only if the frozen-in field condition

$$E_\alpha = F_{\alpha\beta} u^\beta = 0 \tag{6}$$

is satisfied.[2,3]

Assuming the plasma into which the field is frozen is ideal and nondissipative, one can write Euler's equations as[8]

$$(\rho + p + B^2/4\pi) a_\alpha = - h_\alpha{}^\beta [(p + B^2/8\pi)_{,\beta} - (B_\beta B^\gamma)_{;\gamma}/4\pi], \tag{7}$$

where ρ and p are the mass-energy density and pressure, respectively, $a_\alpha \equiv u_{\alpha;\beta} u^\beta$ is the plasma's acceleration, $B^2 \equiv B_\alpha B^\alpha$, and $h_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta$. By combining (7) with the homogeneous Maxwell equations, the local conservation of energy, and the conservation of baryon number, we derived the useful result[8]

$$(\mu B^\alpha)_{;\alpha} = 0, \tag{8}$$

where $\mu \equiv d\rho/dn = (\rho + p)/n$ is the plasma's chemical potential, and n is the scalar baryon density.

The results that we now summarize follow from the preceding theory for the case of stationary and axisymmetric flow.[8] Let us denote by t and ϕ the corresponding time and axial coordinates; all physical quantities are independent of t and ϕ. Then

$$F_{t\phi} = 0, \tag{9}$$

$$F_{at}/F_{a\phi} = A \quad (a = 1, 2;\ \text{no sum over } a), \tag{10}$$

$$B^\alpha = -Cn[(u_t - Au_\phi) u^\alpha + \delta_t^\alpha - A\delta_\phi^\alpha], \tag{11}$$

where the coordinates x^a are those independent of t and ϕ, and where A and C are conserved along each flowline (but may differ from flowline to flowline). In addition we showed that E, L, and D are also conserved along flowlines, where

$$D \equiv \mu (u_t - Au_\phi), \tag{12}$$

$$-E \equiv \chi u_t + C(u_t - Au_\phi) B_t/4\pi, \tag{13}$$

$$L \equiv \chi u_\phi + C(u_t - Au_\phi) B_\phi/4\pi, \tag{14}$$

and $\chi = \mu + B^2/4\pi n$. E and L are the enthalpy and enthalpic angular momentum per baryon of the plasma and magnetic field, as measured from infinity. We found the relation

$$D = -E - AL, \tag{15}$$

which together with (12)–(14) implies that

$$B^2/n + C(B_t - AB_\phi) = 0. \tag{16}$$

The conservation laws (10)–(14) for A, C, E, L, and D are the basis for our investigation of the magnetic structure of a rotating NS's interior. Before we get into that let us pose our model.

III. MODEL AND ASSUMPTIONS

It is widely accepted[5,10] that an NS's interior is divided into three distinct regimes: a solid crust, a thick neutron superfluid layer, and a hyperon core. It has been suggested that the core is solid,[11] but the concensus opinion is that there is no good theoretical support for solidification.[10] We shall thus assume that our star is fluid throughout the interior. We now wish to argue that under the conditions of interest, the flow inside the star may be regarded as that of a single, ideal, infinitely conducting fluid.

The material in the interior (core excepted) is mostly neutrons with a small admixture of protons and an equal number of electrons.[10] It is well established, both theoretically[10] and observationally,[5] that the neutrons form a superfluid. As is well known a superfluid flows inviscidly and is an excellent heat conductor.[12] Thus the bulk of the material is an ideal fluid. However, it does not follow that one can speak about a single baryon velocity under all circumstances. Because of the superfluidity, the neutron and proton fluids couple only very weakly; thus in general one would have to consider both a neutron and a proton velocity. For example, after a "glitch" (pulsar speed-up) the neutrons will flow with their preglitch velocity while the protons, being tied to the crust via the magnetic field threading the star, will flow faster.[5] Similarly, if the crust is subject to a sizable braking torque due to pulsar radiation, the proton fluid will flow slower than the neutron fluid.[13]

Our considerations will be confined to an NS whose crust is not subject to glitches, or appreciable radiation, or accretion torques. In this case the weak coupling of proton and neutron fluids due to the interaction of the protons with the normal component of the neutron fluid[5] (which must be present at finite temperature) will eventually ensure that the two fluids flow with one velocity. The same effect is brought about by a different factor—the existence of an array of vortex lines in the superfluid due to the rotation.[5,10,13] Although superfluid flow is irrotational, the effect of the vortex lines is to endow the superfluid with a "vorticity in the large" which mimics that of an ordinary rotating fluid. It is quite probable that the vortex lines are pinned at the crust.[13] Thus by virtue of their intrinsic tension they will drag the superfluid at the same angular velocity as the crust. (The assumption that the crust is unaccelerated ensures that the vortex lines do not get bent or tangled.) The proton fluid is dragged by the magnetic lines (which are anchored to the crust) at the same angular velocity. Thus macroscopically neutrons and protons flow with the same velocity. The second mechanism we discussed remains operative even if, as may well happen,[5,10] the protons form a superconductor, in which case the first mechanism is ineffective. We may conclude that macroscopically the neutron-proton fluid may be treated as an ideal fluid described by a single velocity field. Furthermore, the angular velocity will be uniform.

According to (5) the conduction current σE^{α} will be solely electronic; protons contribute only to ϵu^{α}. The electrons cannot collide with the neutrons which are superfluid (except for the rare normal component in the vortex cores). They may collide with the protons if these are normal. However, if the protons are superconducting, the only sources of electric resistivity are the interaction of electrons with the magnetic moments of normal neutrons, and with charge density fluctuations of other electrons. Clearly the electron conductivity will be high. Calculations have shown[14] that even if proton superconductivity does not set in, the conductivity is so high that the magnetic diffusion time can be much larger than 10^{10} yr. Over shorter times the field will be frozen into the fluid, and condition (6) will hold. If the protons are superconducting the diffusion time scale is lengthened. However, a superconductor strives to expel the magnetic field (Meissner-Ochsenfeld effect).[12] It appears that in neutron stars the expulsion time is so long that the field simply concentrates into a multitude of quantized vortices (type-II superconductor).[10,15] Macroscopically, the field will be effectively frozen in. We conclude that the superfluid behaves as an infinitely conducting fluid with a frozen-in magnetic field.

The core baryons are most likely nonsuperfluid.[10] Their strong interaction guarantees that they may be described by a single baryon velocity. All fermions in the core should be highly degenerate. This will suppress dissipative effects such as viscosity or thermal resistivity. Thus it is a good approximation to regard the fluid as ideal. The free electrons and muons will flow with little impediment—the Pauli principle suppresses their scattering. Furthermore, a negative pion condensate may form in the core[16]; it will be superconducting.[10] There are thus two factors that make the core medium an excellent conductor. If pion condensation occurs, the magnetic field will probably not be expelled, but will nucleate into vortices, being effectively frozen in. It will be the agent which couples the core to the crust. We may conclude that the core fluid may also be regarded as an ideal, infinitely conducting fluid with a frozen-in magnetic field, and probably corotating with the crust.

We assume our model star to be stationary and axisymmetric. This immediately implies that the magnetic and rotation axes are aligned. We note that this is a necessary restriction if we are to avoid a radiation-induced braking torque on the crust as specified earlier. Although our assumption is unrealistic for pulsars and pulsating x-ray sources, it may hold for old NS's in which the magnetic symmetry axis has already been aligned with the rotation axis by a radiation torque.[17] We hope that our results will also be representative of the actual situation for pulsars and pulsating x-ray sources if we exclude questions about the pulsed emission. The assumed symmetry would still allow meridional circulation (nonazimuthal flow) such as tends to be induced in ordinary rotating stars by centrifugal forces.[18] The situation being considered here is, however, unique because of the presence of the superfluid with its array of vortex lines. It is hard to see how the vortex structure could coexist with meridional circulation. And if there is no vortex array, the superfluid does not rotate even on a macroscopic level. We thus assume the flow is strictly azimuthal.

This assumption has a strong bearing on the form of the metric. By the symmetries we must have $g_{\alpha\beta,t}=g_{\alpha\beta,\phi}=0$. If the effects of nonazimuthal flow in the star's magnetosphere are negligible, it follows from a theorem proved by Carter[19] within general relativity that

$$g_{ta}=g_{\phi a}=0, \quad a=1,2. \tag{17}$$

(Of course $g_{t\phi}\neq 0$ because of the rotation.) These results may well hold in other metric theories of gravity, and we shall assume them throughout. We note here that the influence of the magnetic field on the metric will be negligible for realistic situations. From estimates based on flux conservation, and from analysis of the observed rotational braking, a value of order 10^{12} G is inferred for the magnetic field of pulsars.[17] The same value may be typical of pulsating x-ray sources. Thus the magnetic energy density and pressure are both of order 4×10^{22} erg cm^{-3}. These are negligible compared to the typical fluid quantities in NS interiors, $\rho=10^{35}$ erg cm^{-3}, $p=10^{33}$ erg cm^{-3}.[20] Thus the magnetic field contributes negligibly to the stress-energy tensor and may be ignored in calculating the metric. Henceforth we assume the metric is known.

IV. ANGULAR VELOCITY AND ELECTROMAGNETIC POTENTIALS

From now on we shall confine our attention to the fluid interior of the NS. We made it plausible that it rotates uniformly. However, for the sake of discussion let us consider the more general case of differential rotation. We denote by Ω the angular velocity $d\phi/dt=u^\phi/u^t$; it is a function of the x^a only. Let us introduce the electromagnetic vector potential A_α defined by $F_{\alpha\beta}=A_{\beta,\alpha}-A_{\alpha,\beta}$ and let $\Phi\equiv A_t$ and $\Psi\equiv A_\phi$. By exploiting the gauge freedom one can make Φ and Ψ depend only on the x^a. It follows from (6) with $u^a=0$ (no meridional flow) that

$$\Phi_{,\alpha}+\Omega\Psi_{,\alpha}=0\ . \tag{18}$$

Taking the curl we get

$$\Omega_{,\beta}\Psi_{,\alpha}-\Omega_{,\alpha}\Psi_{,\beta}=0\,. \tag{19}$$

This shows that the normals to the equipotentials of Ω and Ψ are parallel. By (18) the same is true of the equipotentials of Φ and Ψ. Hence Ω, Φ, and Ψ have common equipotentials. This also means that these quantities are functions of one another; the functional relation follows from (18):

$$d\Phi/d\Psi=-\Omega. \tag{20}$$

The equipotentials mentioned are also the MS's—those axisymmetric surfaces tangent to B^α—as we show now. Let us use (4) with $u_a=0$ to calculate the components B^a. We get

$$B^a=\epsilon^{ab}(-g)^{-1/2}(u_\phi F_{bt}-u_tF_{b\phi}), \tag{21}$$

where $\epsilon^{12}=-\epsilon^{21}=1$, $\epsilon^{11}=\epsilon^{22}=0$. Now the frozen-in field condition (6) tells us that $F_{bt}/F_{b\phi}=-u^\phi/u^t$. Recalling the normalization condition $u_tu^t+u_\phi u^\phi=-1$ we can rewrite (21) as

$$B^a=\epsilon^{ab}(-g)^{-1/2}(u^t)^{-1}\Psi_{,b}. \tag{22}$$

It follows that $B^\alpha\Psi_{,\alpha}=B^a\Psi_{,a}=0$ by the antisymmetry of ϵ^{ab}. Thus Ψ is constant along the magnetic lines which means the lines lie in the equipotentials of Ψ. Therefore, the common equipotentials of Ω, Φ, and Ψ are also the MS's.

There is an alternative very consequential way of establishing this. Consider for the moment a general stationary axisymmetric flow with $u^a\neq0$. Let Q be some conserved quantity such as A, C, E, L, or D. Regarded as a field, it is a function only of the x^a. Because of this, we have by virtue of (11)

$$Q_{,\alpha}B^\alpha=-Cn(u_t-Au_\phi)Q_{,\alpha}u^\alpha. \tag{23}$$

By the conservation law $Q_{,\alpha}u^\alpha=0$, and therefore $Q_{,\alpha}B^\alpha=0$. Thus Q is constant on the MS's of the flow. Let us regard our model star with $u^a=0$ as the limit $u^a\to0$ of some general stationary axisymmetric flow. Then the equipotentials of Q must coincide with the MS's of the model.

As an example let us take A. By comparing (10) with (6) in the limit $u^a\to0$ we see that $\Omega=-A$. Thus the equipotentials of Ω must coincide with the MS's of the star. We now turn our attention to Φ and Ψ. For general stationary axisymmetric flow the condition (6) with $\alpha=t,\phi$ gives $\Phi_{,\beta}u^\beta=\Psi_{,\beta}u^\beta=0$. These conservation laws imply that for our model star the equipotentials of Φ and Ψ are also MS's. We thus recover our previous results with little effort. Clearly our procedure applies only if the model with $u^a=0$ is the limit of a sequence of models with $u^a \neq 0$. There may exist "isolated" solutions of the equations which cannot be obtained by such a limit; for them the method would be misleading. However, we would not expect isolated solutions to be of physical interest since small perturbations of their velocity fields would change them drastically, and in the real world u^a is never exactly zero. Thus our procedure, which we dub the "method of limiting magnetic surfaces," should be widely applicable.

The result that Ω is constant on MS's is the relativistic verision of Ferraro's theorem.[9] It was given earlier by Söderholm.[21] Related theorems have been proved by Yodzis,[3] Banerji,[2] and Ciubotariu.[22] Physically, Ferraro's theorem holds because the only way for the fluid to rotate differentially without continually winding up the magnetic lines frozen into it is for each MS which contains the lines to rotate rigidly. We note that each such MS is an electric equipotential.

V. CHEMICAL POTENTIAL

The distribution of μ—the chemical potential—inside the star can easily be deduced from our previous results. Substituting (22) into (8) and recalling the symmetries we get

$$\Psi_{,\alpha}(\mu/u^t)_{,\beta} - \Psi_{,\beta}(\mu/u^t)_{,\alpha} = 0\,. \tag{24}$$

This shows that the equipotentials of μ/u^t coincide with those of Ψ, i.e., with the MS's. The same result follows from the method of limiting MS's. According to it the quantity $\mu(u_t - Au_\phi)$ appearing in (12) must be constant on MS's. We already saw that for $u^a \to 0$, $A = -\Omega$. Thus in the limit

$$\begin{aligned} u_t - Au_\phi &= (u^t)^{-1}(u^t u_t + u^\phi u_\phi) \\ &= -(u^t)^{-1}. \end{aligned} \tag{25}$$

Hence μ/u^t is constant on MS's as we concluded previously. The method of limiting MS's has the advantage of determining the *value* of μ/u^t on each MS. By (12) and (25)

$$\begin{aligned} \mu_* &\equiv \mu(-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2)^{1/2} \\ &= -D = E - \Omega L, \end{aligned} \tag{26}$$

where we have evaluated u^t explicitly by means of the normalization condition $u^\alpha u_\alpha = -1$. Thus, if we know the parameters E, L, and Ω for an MS, we know μ everywhere on it. From (26) it is clear that $D < 0$.

The square root on the left-hand side of (26) is the red-shift factor with respect to infinity. Thus (26) states that the red-shifted chemical potential μ_* is constant on MS's. This is physically reasonable. The magnetic field cannot impede motion of the highly conducting fluid along its lines. Thus equilibrium can be established only when the chemical potential including gravitational and rotation effects, μ_*, becomes constant along the magnetic lines. We note that the condition (26) does not contain B^α explicitly. It thus applies as well to an unmagnetized star; in that case the surfaces of constant μ_* are those of constant Ω. We shall show in Sec. VIII that for a uniformly rotating magnetized star D is independent of MS. Thus μ_* is constant in the interior. This result is identical to that for a uniformly rotating unmagnetized star.[23]

VI. VANISHING OF THE TOROIDAL FIELD AND OF THE CONDUCTION CURRENT

The B_a represent the poloidal field while B_t and B_ϕ represent the toroidal field. It is easily shown that due to the symmetry of our model the toroidal field must be absent. From (4) it follows that $B_\alpha u^\alpha = 0$ in general. For a stationary axisymmetric flow with $u^a = 0$ this gives

$$B_t - AB_\phi = 0 \tag{27}$$

since $u^t \neq 0$ and $A = -\Omega$. We now regard the star as the limit $u^a \to 0$ of a sequence of general stationary axisymmetric flows. Then (16) will be valid for each flow in the sequence. In the limit B^2 should be nonvanishing throughout the star (except possibly on isolated surfaces). In light of (27) it follows from (16) that $C \to \infty$ throughout the stellar interior. However, because of their physical interpretation E and L will always be bounded. It then follows from (13), (14), and (17) that $B_t = B_\phi = B^t = B^\phi = 0$ inside our model star (by continuity this will be true even on the isolated surfaces). Thus there cannot be a toroidal field. It is clear that this conclusion depends critically on the postulated absence of meridional circulation.

The exclusion of the toroidal field is surprising. Models of ordinary stars with toroidal fields can be constructed,[24] and it has been argued that a purely poloidal configuration is magnetohydrodynamically unstable.[25] Furthermore, it is known that the sun, and probably other main sequence stars, have strong toroidal fields. Because of these points, it has usually been assumed that a toroidal field is a must for a neutron star.[5,10] This conclusion has been buttressed by the argument that differential rotation in the progenitor stellar core as it collapses to the neutron star should wind some of its poloidal field into a toroidal component that would augment the primordial one.[5] Our conclusion indicates that some of these expectations are unwarranted. For a neutron star in a stationary state the purely poloidal configuration must be stable as it is the only one possible. Perhaps the disagreement arises from the widespread assumption of earlier works[25,26] that in the star's interior there is no convection current. We shall now see that such an assumption would be inconsistent with our model.

Since $u^a = 0$ the result $B_t = B_\phi = 0$ implies by (4) that $F^{12} = 0$. Substituting this into Maxwell's equations (2) we get, in view of the symmetries, $J^a = 0$: the current is purely azimuthal just as the velocity. We now show that $J^\alpha \propto u^\alpha$.

Let us consider a general stationary axisymmetric flow for which not both u^a vanish. We use the freedom inherent in choosing the coordinates x^a to make one, r, constant on the axisymmetric surfaces tangent to the velocity, and the other, z, constant along the normals to the surfaces. Thus by definition $u^r = 0$. For future convenience we identify $x^1 = z$ and $x^2 = r$. For the present we regard the conductivity σ as finite. The result (9) $F_{t\phi} = 0$ is actually valid in this case also as may be verified from (1) together with suitable boundary conditions. Thus we have

$$\sigma E_t = \sigma F_{tz} u^z. \tag{28}$$

We shall recover our star by passing to the limit $\sigma \to \infty$ and $u^z \to 0$. However, this limit must be taken while respecting the condition that the diffusion time for the magnetic field T be large compared to the time scales of interest. Now

$$T = 4\pi\sigma l^2, \tag{29}$$

where l is the typical size of the system. We clearly want $\Omega T \gg 2\pi$ so that the field will not diffuse out in a few revolutions. Likewise, we want $T u^z \gg l$ so that the fluid traverses the region of interest before sizable diffusion takes place. Thus $\sigma\Omega \gg l^{-2}$ and $\sigma u^z \gg l^{-1}$. Thus in taking our limit we must first let $\sigma \to \infty$ and then $u^z \to 0$. In order that J_t remain bounded during the first limit we must have $F_{tz} = O(\sigma^{-1})$. Thus upon taking the second limit $\sigma E_t \to 0$. In exact analogy we can show that $\sigma E_\phi \to 0$. Thus for our model $J_t = \epsilon u_t$, $J_\phi = \epsilon u_\phi$, and since $J_a = 0$ [see (17)],

$$J^\alpha = \epsilon u^\alpha. \tag{30}$$

We conclude that the current is purely a convection current. We must clearly have $\epsilon \neq 0$, for if ϵ were to vanish everywhere in the interior, $J^\alpha = 0$, and by Maxwell's equations there would be no magnetic field. Thus a rotating magnetized NS must be electrically charged just as predicted by the Newtonian models.[4] Comparing (30) with (6) we see that $F_{\alpha\beta} J^\beta = 0$: The Lorentz force vanishes. Thus due to the high symmetry of the problem, the magnetic field obeys the force-free condition. A different argument showing that this condition should hold in an NS interior has been given earlier by Easson[27] for a Newtonian nonrotating star.

VII. DISTRIBUTION OF THE MAGNETIC FIELD

In the limit $\sigma \to \infty$ of Sec. VI the constant r surfaces coincide with the MS's since by (11) the B^a are parallel to the u^a; the constant z surfaces are orthogonal to the MS's. This will remain true as $u^z \to 0$ and we recover our model star. Since r and z are orthogonal coordinates, the line element for the star can be written as

$$ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{zz} dz^2 + g_{rr} dr^2, \tag{31}$$

where account has been taken of (17). Now Φ, Ψ, Ω, D, E, and L can depend only on r since they are constant on MS's. For any such quantity Q, $B^\alpha Q_{,\alpha} = 0$, and since $Q_{,r} \neq 0$ in general, it follows that $B^r = 0$.

In view of this (8) takes the form

$$[(-g)^{1/2} \mu B^z]_{,z} = 0, \tag{32}$$

where $-g = g_{rr} g_{zz} \Delta$ and $\Delta = g_{t\phi}{}^2 - g_{tt} g_{\phi\phi} > 0$. We also note that $g_{zz}{}^{1/2} B^z = (B^2)^{1/2} \equiv B$. Recalling (26) and the fact that $D_{,z} = 0$ we can integrate (32) to get

$$B = B_0(r)(g_{rr}\Delta)^{-1/2}(u^t)^{-1}, \tag{33}$$

where $B_0(r)$ is an undetermined function, and

$$u^t = (-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2)^{-1/2}. \tag{34}$$

Equation (33) determines B everywhere within each MS in terms of its value at one point which fixes B_0 for that MS. To connect the different MS's we need to determine $B_0(r)$.

We shall write the nontrivial Maxwell equations (2) in terms of B. Since $\Phi_{,z} = \Psi_{,z} = 0$ we have $F^{tz} = F^{\phi z} = F_{tz} = F_{\phi z} = 0$ while $F_{\phi r} = -\Psi_{,r}$ and $F_{tr} = -\Phi_{,r} = \Omega\Psi_{,r}$, the last equality following from (20). A simple calculation gives

$$F^{tr} = -(g_{rr}\Delta)^{-1}(g_{t\phi} + g_{\phi\phi}\Omega)\Psi_{,r} = -(g_{rr}\Delta u^t)^{-1} u_\phi \Psi_{,r}, \tag{35}$$

$$F^{\phi r} = (g_{rr}\Delta)^{-1}(g_{tt} + g_{t\phi}\Omega)\Psi_{,r} = (g_{rr}\Delta u^t)^{-1} u_t \Psi_{,r}, \tag{36}$$

where we have used the relations

$$g^{tt} g^{rr} = -(\Delta g_{rr})^{-1} g_{\phi\phi}, \text{ etc.}, \tag{37}$$

$$u_t = (g_{tt} + g_{t\phi}\Omega) u^t, \tag{38}$$

$$u_\phi = (g_{t\phi} + g_{\phi\phi}\Omega) u^t. \tag{39}$$

But from (22) and our definition of B we have

$$\Psi_{,r} = u^t(-g)^{1/2} B^z = u^t (\Delta g_{rr})^{1/2} B. \tag{40}$$

Then Maxwell's equations take the form

$$(g_{zz}{}^{1/2}u_\phi B)_{,r} = -4\pi J^t(-g)^{1/2} = -4\pi\epsilon u^t(-g)^{1/2}, \tag{41}$$

$$(g_{zz}{}^{1/2}u_t B)_{,r} = 4\pi J^\phi(-g)^{1/2} = 4\pi\epsilon u^\phi(-g)^{1/2}. \tag{42}$$

By combining these we get our basic result

$$\Omega(g_{zz}{}^{1/2}u_\phi B)_{,r} + (g_{zz}{}^{1/2}u_t B)_{,r} = 0. \tag{43}$$

Let us now carry out the differentiation in (43) explicitly. After rearrangement we get

$$(\ln B)_{,r} = -\tfrac{1}{2}(\ln g_{zz})_{,r} - (u_{t,r} + \Omega u_{\phi,r})(u_t + \Omega u_\phi)^{-1}. \tag{44}$$

We now make use of $u^\alpha u_\alpha = -1$ in the form

$$u_t + \Omega u_\phi = -(u^t)^{-1} \tag{45}$$

to write

$$(u_{t,r} + \Omega u_{\phi,r})(u + \Omega u_\phi)^{-1} = u^t[u_\phi \Omega_{,r} + (1/u^t)_{,r}]. \tag{46}$$

Substituting (46) and (33) into (44) and rearranging terms we get the following equation for B_0:

$$(\ln B_0)_{,r} = \tfrac{1}{2}\{\ln[g_{rr}\Delta(u^t)^4/g_{zz}]\}_{,r} - u^t u_\phi \Omega_{,r}. \tag{47}$$

For a uniformly rotating star $\Omega_{,r} = 0$ and the equation can be integrated easily. Actually we can do better; according to Eq. (61) to be derived in Sec. VIII, the last term in (47) equals $[\ln(-D)]_{,r}$. Thus we can integrate (47) in general to get

$$B_0(r) = -\overline{B}(z)D(u^t)^2(g_{rr}\Delta/g_{zz})^{1/2}, \tag{48}$$

where $\overline{B}$ is some function of z. It cannot be taken as constant for the following reason. Since both the left-hand side and the last term of (47) depend only on r, the expression in curly brackets must be a sum of two functions, $g(r)$ and $h(z)$. Thus

$$g_{rr}\Delta(u^t)^4/g_{zz} = g(r)h(z). \tag{49}$$

Therefore, in order that B_0 in (48) depend only on r, we must take $\overline{B} = B{*}h^{-1/2}$ where $B{*}$ is a constant.

Substituting (48) into (33) we get

$$B = -B{*}Du^t(g_{zz}h)^{-1/2}. \tag{50}$$

This formula determines B at all points in the star's interior in terms of $B{*}$, an arbitrary parameter which fixes the overall strength of the field. We note that under the coordinate transformation $r \to F(r)$ and $z \to G(z)$, $g_{zz} \to g_{zz}(G')^{-2}$. But clearly D and Δ do not change. Thus by (49) $h \to h(G')^2$. Therefore, B is unaffected as we would expect. For numerical models (50) may not be a convenient formula because the determination of h by separation of variables in (49) may be unfeasible. In that case (48) may be used directly; for fixed z it determines B_0 for the various MS's up to a constant overall factor. (This factor is actually the arbitrary parameter of the model.) Then the distribution of B may be determined by using (33). Implicit in all we have said is the assumption that the shape of the MS's is known so that the coordinate system (r,z) can be constructed, and g_{rr} and g_{zz} determined. Thus we turn our attention to deducing an equation for the MS's.

VIII. THE EQUATION FOR THE MAGNETIC SURFACES

As mentioned in Sec. VII we can always replace the coordinate z by some function of itself. It will be advantageous here to use this freedom to choose a z coordinate which molds itself to the magnetic field strength. To this end consider the expression $\epsilon^{\alpha\beta\gamma\delta}B_{\gamma,\delta}B_\beta$. Because $B_t = B_\phi = 0$ and t and ϕ derivatives vanish, it vanishes identically: B_α is vorticity-free. By Frobenius's theorem it follows that B_α must be orthogonal to a family of surfaces:

$$B_\alpha = fz_{,\alpha}, \tag{51}$$

where z labels the surfaces and f is some scalar. According to (51) z is constant along the normals to MS's (vectors orthogonal to B_α). Thus we may identify z as defined here with the coordinate z of Sec. VII. We see that $B_z = f$; also, $B_z B^z = B^2$ so $B^z = B^2/f$. It then follows that $g_{zz} = f^2/B^2$.

To determine f let us substitute this form of g_{zz} into (44) and simplify with the help of (46), (34), and (39). We get after some cancellations

$$f_{,r}/f = \tfrac{1}{2}(u^t)^2(g_{tt,r} + 2g_{t\phi,r}\Omega + g_{\phi\phi,r}\Omega^2). \tag{52}$$

We compare this with the acceleration vector. Because of the symmetries and because $u^z = 0$,

$$a_\alpha = -\Gamma_{\delta\alpha\beta}u^\delta u^\beta = -\tfrac{1}{2}(u^t)^2(g_{tt,\alpha} + 2g_{t\phi,\alpha}\Omega + g_{\phi\phi,\alpha}\Omega^2). \tag{53}$$

Thus

$$a_r = -f_{,r}/f. \tag{54}$$

To evaluate a_r we bring in Euler's equations (7); because $B_r = B^r = 0$ the rth component takes the form

$$(\rho + p + B^2/4\pi)a_r = -(p + B^2/8\pi)_{,r} - \Gamma^z_{rz}B^2/4\pi. \tag{55}$$

Since $g_{zz} = f^2/B^2$ we have

$$\Gamma^z_{rz} = f_{,r}/f - \tfrac{1}{2}B^2{}_{,r}/B^2. \tag{56}$$

In view of (54) and (56), (55) simplifies to

$$(\rho + p)f_{,r}/f = p_{,r}. \tag{57}$$

We now recall the definition of μ (Sec. II), $d\rho = \mu dn$, and the equality $\rho + p = n\mu$. Differentiating the latter and substituting the former we get $nd\mu = dp$. Thus (57) is equivalent to

$$[\ln(f/\mu)]_{,r} = 0, \quad (58)$$

which states that f/μ depends on z only. We lose no generality in taking $f = \mu$, for if f equals μ multiplied by some function of z, we can absorb this function in the defining relation (51) by replacing z with an appropriate function of itself. Thus in arriving at

$$B_\alpha = \mu z_{,\alpha} \quad (59)$$

we have used up the freedom of coordinate transformation of z. That our z is well behaved can be seen directly from (59): $z_{,\alpha} z^{,\alpha} \neq 0$ provided $\mu \neq \infty$ and $B \neq 0$ which we always assume. Thus z grows monotonically along the magnetic lines as a good coordinate should.

Let us substitute $g_{zz} = \mu^2/B^2$ and (46) into (44) and rearrange terms to get

$$[\ln(\mu/u^t)]_{,r} = -u^t u_\phi \Omega_{,r}. \quad (60)$$

But according to (26) this is the same as

$$[\ln(-D)]_{,r} = -u^t u_\phi \Omega_{,r}. \quad (61)$$

This is the equation we wanted. Since D and Ω depend only on r, it states that $u^t u_\phi$ must be constant on MS's.[28] Defining the time and axial Killing vectors $\xi^\alpha = \delta^\alpha_t$ and $\eta^\alpha = \delta^\alpha_\phi$, respectively, and the auxiliary vector field $\zeta^\alpha = \xi^\alpha + \Omega\eta^\alpha$, we find that

$$u^t u_\phi = U \equiv (-\zeta_\alpha \zeta^\alpha)^{-1} \zeta_\beta \eta^\beta. \quad (62)$$

Thus the MS's of the star are the equipotentials of the coordinate invariant U. If the geometry is known, these can be calculated and the coordinates r and z defined. Then one can calculate B by means of (50). For this last purpose the coordinate z defined by (59) is ill suited, for it $g_{zz} \propto B^{-2}$ and so B cancels out of (50). We must thus use a "less perfect" z.

We notice that for a uniformly rotating star ($\Omega_{,r} = 0$) Eq. (61) implies D is independent of MS (since ζ^α is parallel to u^α, it cannot be null so U cannot blow up). We made use of this fact in Sec. V. The fact that both sides of (61) vanish means that the equation by itself does not establish the constancy of U on MS's. To handle this case we consider a star which rotates nearly uniformly so that $\Omega_{,r}$ is small. Then $[\ln(-D)]_{,r}(\Omega_{,r})^{-1}$ will be constant on MS's. In the limit $\Omega_{,r} \to 0$ we recover our uniformly rotating star with U bounded and constant on MS's.

IX. MAGNETIC STRUCTURE OF A MODEL NEUTRON STAR

We mentioned in Sec. III that the magnetic field affects negligibly the metric of a realistic model NS. And we showed in Sec. VI that the Lorentz force on the fluid vanishes. It follows that the geometry and fluid structure of a magnetic NS model coincide with those of some nonmagnetic model star. The converse is not generally true. Suppose we are given a nonmagnetic model; if we determine the equipotentials of U and define r and z, there is no guarantee that the separability condition (49) will hold. If it does not, then the model cannot be "dressed" with a magnetic field. The strategy for constructing a magnetic NS model then is to select from the arsenal of nonmagnetic models[20] one which does satisfy (49), and to determine the magnetic field distribution by means of (50).

Assuming one has done this, what is the structure of the magnetic field? In order to make the essential points clear, we confine our treatment to post-Newtonian order, thus assuming that the star's gravitational field is weak. Then the metric can be written in cylindrical coordinates as[29,20]

$$ds^2 = -(1+2V)dt^2 - 2\omega R^2 dt d\phi + (1-2V)(R^2 d\phi^2 + dR^2 + dZ^2), \quad (63)$$

where V is the Newtonian potential ($V \ll 1$ by assumption), and ω is the angular frequency measuring the dragging of inertial frames. We shall assume the star rotates uniformly and slowly in the sense that $(\Omega R)^2 \ll |V|$ inside the star (an excellent assumption even for the fastest pulsars). As a consequence V and ω will both be spherically symmetric,[20] i.e., functions of $R^2 + Z^2$ only. We note that ω/Ω is $O(V)$.[29] To $O(V)$ and neglecting $(\Omega R)^2$ compared to V we get

$$U = \Omega R^2 (1 - 4V - \omega/\Omega). \quad (64)$$

Were we to neglect gravitation, we would get $U(r) = \Omega R^2$ showing that the MS's are right cylinders ($R = \text{const}$) in the Minkowski metric. In this limit the magnetic lines are straight and parallel to the rotation axis. Gravitation distorts the cylinders. In view of (64) we can take the coordinate r to be

$$r = R(1 - 2V - \omega/2\Omega) \quad (65)$$

[that is, $r = (U/\Omega)^{1/2}$]. To find out how the metric radius of an MS changes as we go from the center of the polar region, let us calculate g_{rr} to $O(V)$ according to $g_{RR} = g_{rr}(\partial r/\partial R)^2$:

$$g_{rr} = 1 + 2(V + 4V'R^2) + (\omega + 2\omega' R^2)/\Omega, \quad (66)$$

where a prime denotes the derivative with respect to the argument $R^2 + Z^2$. We know that V is negative inside the star and increases outward ($V' > 0$). For the nearly homogeneous interior of an NS,[20] V' should be nearly constant (Poisson equation). Thus $V + 4V'R^2$ will undoubtedly increase as we move from the central to the polar region. Numerical calculations of NS models have shown that $\omega/\Omega \leq |V|$.[20] Thus the V dependent term in (66) dominates the ω dependent one. Since $g_{rr}{}^{1/2}dr$ gives the metric distance between MS's separated by dr we conclude that the effect of gravitation is to cause the magnetic lines to flare out toward the polar regions.

What about the field strength? We shall employ (33) to see how B varies along an MS. Simple calculations to $O(V)$ with neglect of $(\Omega R)^2$ give

$$\Delta^{1/2} = R = r(1 + 2V + \omega/2\Omega), \tag{67}$$

$$u^t = 1 - V. \tag{68}$$

Were we to neglect V we would find upon substituting (66)–(68) into (33) that B is constant on each MS, which is consistent with the fact that the lines are straight in this approximation. If we now retain V we find

$$B = B_0(r)[1 - 2(V + 2V'R^2) - (\omega + \omega' R^2)/\Omega] r^{-1}. \tag{69}$$

Exactly the same argument used in the preceding paragraph shows that B decreases as we move from the central to the polar regions—just what would be expected since the lines flare out. Thus the effect of gravitation is to enhance the field strength in the central region as compared to the poles, and to cause a divergence of the lines toward the poles. What about the distribution of B normal to magnetic surfaces? This can be inferred from (50). We know that $-Du^t = \mu$. Since $h = h(z)$, B changes in the r direction as $\mu(g_{zz})^{-1/2}$. Neglecting gravitational effects we expect μ to be nearly constant since $p \ll \rho$ and thus μ approximates the baryon mass. Also, g_{zz} can be identified with $g_{ZZ} = 1$. Thus B is nearly constant in the r direction. Retaining V to first order should be reflected in a modest variation of B in the r direction.

Our results provide a justification of the widely employed Deutsch model[30] of a magnetic star which assumes the field inside is uniform and parallel to the rotation axis. What we have shown is that this is indeed the only possible configuration if the flow is azimuthal and gravitational effects are neglected. That such a conclusion is not trivial can be seen from the numerous papers proposing more complicated interior configurations.[31] The relevance of such configurations for neutron stars is doubtful. Our results also demonstrate that calculations employing the Deutsch model for a gravitating NS[4,5] are off by only mild factors.[32] Recalling that for a typical NS V is of order 0.1, we see that the error incurred in assuming a uniform field is some 20–30% [see (69)]. This is not too bad given the present state of the art. The claim often made that fields inside pulsars may be considerably stronger than the polar values $\sim 10^{12}$ G[33] is seen to be without grounds. There is no toroidal field and gravitational effects cause only a modest central enhancement of the poloidal field. Finally, our results support the use of a dipole exterior field for an NS.[4-6] Such a field is the appropriate one for a uniform interior field in flat spacetime.[30] If gravitational effects were included, we would expect the exterior field to be distorted slightly, but to preserve its topology. There is no indication that a quadrupole field is to be expected.

X. RELATIONS BETWEEN MAGNETIC SURFACE CONSTANTS

Quantities such as E, L, Ω, Φ and Ψ that are constant on MS's can be regarded as functions of one another. To know the forms of these functions is to know a great deal about the structure of the star. One step in the direction of finding these functions is Eq. (20) relating Φ, Ψ, and Ω. It shows that for uniform rotation Φ and Ψ are linearly related. Let us seek some other such relations.

To this effect let us write E and L as given by (13) and (14) in the form

$$E = -\mu u_t + E_M = E_F + E_M, \tag{70}$$

$$L = \mu u_\phi + L_M = L_F + L_M, \tag{71}$$

where we have lumped into E_M and L_M all the magnetic contributions; thus E_F and L_F are the pure fluid contributions. By (26) $L_F = -Du^t u_\phi$ which is seen by (62) to be constant on MS's. Thus L_M is also constant on MS's. Let us now substitute (70) and (71) into (15) and compare with (12) to get (recall that $A = -\Omega$)

$$E_M = \Omega L_M, \tag{72}$$

$$E_F - \Omega L_F = -D. \tag{73}$$

These show that E_F and E_M are both constants on MS's. Thus the fluid and magnetic parts of E and L are separately MS constants subject to the relations (72) and (73). A further relation is obtained by taking the differential of (73),

$$dE_F - \Omega dL_F = L_F d\Omega - dD, \tag{74}$$

and noticing that the right-hand side vanishes by (61). Thus

$$dE_F/dL_F = \Omega. \tag{75}$$

This sort of relation is well known from the mechanics of rotating bodies where it applies to changes of the total energy and angular momentum. Here it refers to changes of E_F and L_F in going from MS to MS. For constant Ω we get from (72) and (75) a relation just like (75) for E and L themselves.

Although we know that E_M and L_M are MS constants, we have been unable to compute them explicitly because they involve CB_t and CB_ϕ which clearly are defined only by a suitable limiting procedure. Explicit calculation of E_M or L_M might yield some constant combination of B and metric factors which might be more transparent than (50) in describing the variation of the magnetic field. However, from the point of view of finding E and L we do not need E_M and L_M, for we expect $E_M \ll E_F$ and $L_M \ll L_F$ (magnetic energy density negligible compared to fluid energy density; see Sec. III).

A rather mysterious MS constant is B_0 of (33). Let us find out how it relates to the others. Let us calculate B^2 by means of (22); we get

$$B^2 = (g_{rr}\Delta)^{-1}(u^t)^{-2}(\Psi_{,r})^2. \tag{76}$$

Comparing with (33) we see that

$$B_0 = \Psi_{,r}, \tag{77}$$

where the sign has been chosen to be negative because $B_0 > 0$, and by (40) $\Psi_{,r} > 0$ granted the *convention* $B^z > 0$. Thus naturally enough the "scale of the magnetic field on a MS", B_0, is the derivative of the magnetic potential Ψ. Of interest in the theory of pulsars is the electrostatic potential difference between two points on the star's surface. This is just the difference of Φ at the two appropriate values of r, r_1, and r_2. Taking (20) and (76) into account we find

$$\Phi(r_2) - \Phi(r_1) = -\int_{r_1}^{r_2} \Omega B u^t (g_{rr}\Delta)^{1/2} dr, \tag{78}$$

which is the relativistic version of the famous Goldreich-Julian formula [Eq. (11) in Ref. 4].

Let us now consider an interesting relation between the charge density ϵ and L_M. Working with the special coordinate z of Sec. VIII we substitute $g_{zz} = \mu^2/B^2$ into (41) and rearrange the expression to get

$$\epsilon = -\frac{B L_{M,r}}{4\pi\mu u^t (\Delta g_{rr})^{1/2}}\,. \tag{79}$$

The overall sign of this expression is ultimately fixed by the convention implicit in (40) that the magnetic field points in the positive z direction. When the rotation is counterclockwise (clockwise) as seen from the north magnetic pole, L_M will be positive (negative). We also expect that if $|\Omega|$ does not decrease outward too fast, $|L_M|$ will grow outward. In this case ϵ is negative (positive). If we evaluate (79) in the Newtonian limit using (63) with $r = R$ and assuming $\Omega_{,R} = 0$ we get

$$\epsilon = -\Omega B/2\pi = \text{const}, \tag{80}$$

which is the classical result.[4,5]

To visualize the relativistic effects on the charge distribution we integrate (41) to find the total charge dq enclosed inside an MS labeled by r between z and $z+dz$. In view of (39) we get

$$\begin{aligned} dq &= -(4\pi)^{-1}\int_0^r \int_0^{2\pi} J^t(-g)^{1/2} d\phi\, dr\, dz \\ &= -\tfrac{1}{2} B u^t g_{\phi\phi}(\Omega-\omega)\big|_r dl, \end{aligned} \tag{81}$$

where we have used the fact that $g_{\phi\phi} \to 0$ at the axis of symmetry $r = 0$. In (81) $dl = g_{zz}{}^{1/2}|_r dz$ is the element of proper length along the MS, and $\omega = -g_{t\phi}/g_{\phi\phi}$ is again the frequency of dragging of inertial frames. Now $g_{\phi\phi}$ gives the squared circumferential radius of the MS. Thus the relativistic formula (81) differs from the integral of (80) only in the appearance of the red-shift factor u^t and in that $\Omega - \omega$, the angular velocity with respect to the local inertial frames, appears instead of Ω.

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