Instability of the Cauchy horizon of Reissner-Nordström black holes

Richard A. Matzner and Nelson Zamorano*
Department of Physics, Center for Relativity, The University of Texas at Austin, Austin, Texas 78712

Vernon D. Sandberg
W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125
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The stability of the inner Reissner-Nordström geometry is studied with test massless integer-spin fields. In contrast to previous mathematical treatments we present physical arguments for the processes involved and show that ray tracing and simple first-order scattering suffice to elucidate most of the results. Monochromatic waves which are of small amplitude and ingoing near the outer horizon develop infinite energy densities near the inner Cauchy horizon (as measured by a freely falling observer). Previous work has shown that certain derivatives of the field in a general (nonmonochromatic) disturbance must fall off exponentially near the inner (Cauchy) horizon ($r = r_-$) if energy densities are to remain finite. Thus the solution is unstable to physically reasonable perturbations which arise outside the black hole because such perturbations, if localized near past null infinity ($\mathscr{I}^-$), cannot be localized near r_+, the outer horizon. The mass-energy of an infalling disturbance would generate multipole moments on the black hole. Price, Sibgatullin, and Alekseev have shown that such moments are radiated away as "tails" which travel outward and are rescattered inward yielding a wave field with a time dependence t^{-p}, $p > 0$. This decay in time is sufficiently slow that the tails yield infinite energy densities on the Cauchy horizon. (The amplification of the low-frequency tails upon interacting with the time-dependent potential between the horizons is an important feature guaranteeing the infinite energy density.) The interior structure of the analytically extended solution is thus disrupted by finite external disturbances. Gürsel *et al.* have further shown that even perturbations which are localized as they cross the outer horizon produce singularities at the inner horizon. By a ray-tracing scheme we are able to show that this singularity arises when the incoming radiation is first scattered for $r \lesssim r_+$ (i.e., just inside the outer horizon), whence the exponentially small scattered radiation is efficiently rescattered when the potential becomes strong. The exponentially small first scattering near the outer horizon is translated by the second scattering into exponentially decaying waves near the inner horizon. Their exponential decay is, however, so slow that the resultant energy density is singular on the horizon.

I. INTRODUCTION

The Reissner-Nordström (RN) geometry[1] represents the unique asymptotically flat spherically symmetric solution to the coupled Einstein-Maxwell equations describing the spacetime geometry outside of a spherical star with charge Q and mass M. The analytically extended electrovacuum solution (Fig. 1) possesses a topology which is not Euclidean (provided $M^2 > Q^2$, which we assume throughout this paper). The extended solution has at least two null hypersurfaces acting as horizons. The surface at $r = r_+$ is an event horizon causally separating the interior (region III) from the exterior (region I). The surface at r_- is a "Cauchy horizon" for initial data in region I (Ref. 2). In this paper we are concerned with the evaluation of test fields starting with initial data in region I and evolving through region III up to the Cauchy horizon.

In the uncharged Schwarzschild case the timelike (or null) histories of test objects falling inside the outer horizon reach a curvative singularity in a finite amount of time; the singularity lies to the future of any point inside the future Schwarzschild horizon. In the Reissner-Nordström solution with $Q^2 < M^2$ the radial geodesic timelike path of an uncharged black-hole penetrator in fact escapes the singularity, which it passes at a finite proper distance and which can causally influence the timelike line only for a finite proper time after it penetrates the inner horizon[3] at r_-. Even more surprisingly, the timelike path may eventually emerge into a distinct asymptotically flat cosmology. (See for instance curve A of Fig. 1.)

A freely falling observer in region III will see the entire history of region I as he crosses r_- in a finite lapse of proper time. This property of the r_- surface has led to the speculation that it is unstable, i.e., small perturbations in the initial-value data of the RN solution will destroy the Cauchy horizon. To verify this, Simpson and Penrose[4] numerically investigated the evolution of a spin-1 test field on the RN background. Their results suggest that, independent of the initial values, their test field and its associated energy density become singular on r_-. This they interpret as a generic property of region III and conclude that the interior RN solution is unstable. (Mon-

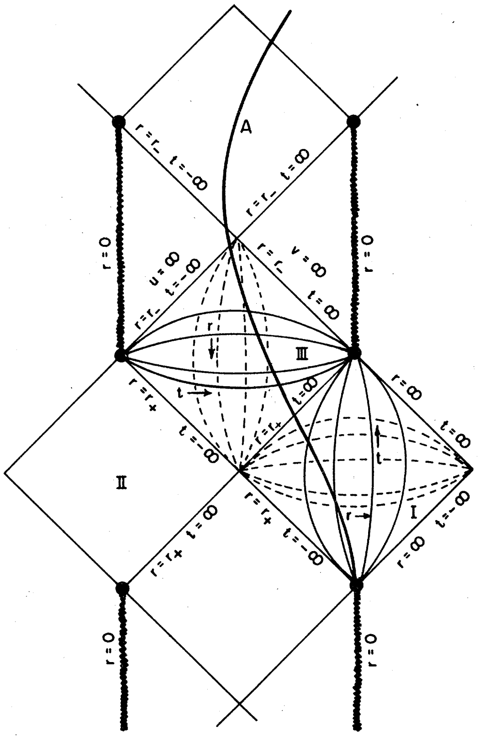

FIG. 1. Penrose-conformal diagram of an analytically extended Reissner-Nordström geometry. Region I ($r > r_+$) corresponds to an exterior static asymptotically flat spacetime, region III ($r_+ > r > r_-$) is spatially homogeneous geometry evolving with a temporal coordinate r. The curve A schematically gives the orbit of a geodesic observer who falls into the hole, passes a finite spacelike distance from the timelike singularity at $r = 0$, and emerges into another asymptotically flat region. Region III is a homogeneous cosmology of the Kantowski-Sachs type.

crief,[5] Zerilli,[6] and Chandrasekhar[7] have developed analytical formalisms for gravitational and electromagnetic perturbations on the RN background and have shown that the exterior RN solution is stable. The evolution of the fields and geometry outside of a star undergoing nearly spherical gravitational collapse has been given by Price[8] for the uncharged case and by Bičák[9] and Sibgatullin and Alekseev[10] for the charged case. They find that the fields damp out as power laws in time leaving a stable exterior geometry. Novikov and Doroshevich[11] have extended Price's analysis to the interior of a Schwarzschild black hole left in the wake of a collapse and they also find a stable interior geometry up to the curvature singularity.) In a situation where an arbitrarily small perturbation in the exterior region leads to a curvature singularity on r_-, a large change has occurred from the regular behavior of the Reissner-Nordström solution there, and in this sense *the solution is unstable* to such physically natural perturbations.

There have recently been several papers on this problem,[12-14] and these show that the Reissner-Nordström solution is not stable (in the sense just described) to external perturbations. In this paper we present a heuristic view of the processes that lead to this instability. We show that much of the behavior of perturbing waves in this geometry can be understood by simple ray tracing, and a first-order Green's function analysis gives details sufficient to see the fundamental instability.

II. MATHEMATICAL PRELIMINARIES

We consider perturbations of the Reissner-Nordström spacetime, which is described by the metric

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + dr^2\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} + r^2 d\Omega^2 . \tag{2.1}$$

The perturbations consist of integer-spin massless wave modes (i.e., massless scalar, or coupled electromagnetic and gravitational wave modes). Such perturbations may always be written as $\phi = (1/r)\,\phi_\omega(r)e^{-i\omega t}$ where any angular part has been factored out using the spherical symmetry and where ϕ_ω is a radial mode function for the perturbation in question.[5]

A general perturbation is a superposition of these ω modes. The equation solved by ϕ_ω is

$$\frac{d^2\phi_\omega}{dr^{*2}} + [\omega^2 - V(r)]\phi_\omega = 0 , \tag{2.2}$$

where[3]

$$r^* = r + (2\kappa_+)^{-1}\ln|r - r_+| - (2\kappa_-)^{-1}\ln|r - r_-| \tag{2.3}$$

and r and t are the usual Reissner-Nordström coordinates [extended across the horizons $r_\pm = M \pm (M^2 - Q^2)^{1/2}$] and $\kappa_\pm = (r_+ - r_-)/2r_\pm^{\,2}$ is the surface gravity at $r_\pm$. The important property of $V(r)$ is that it vanishes exponentially in r^* at the horizons.

In fact, the potential consists of a factor $(r - r_+)(r - r_-)$ times terms which are nonzero and finite at the horizons. For instance, the potential for scalar perturbations ϕ is

$$V(r) = \frac{(r_+ - r)(r_- - r)}{r^2}\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4}\right] . \tag{2.4}$$

(l is the angular momentum eigenvalue.)

We shall be concerned with propagation between the two horizons r_+ and r_-. An important point to notice is that the potential vanishes just outside the outer horizon $r = r_+$. Because r^* depends logarithmically on $|r-r_+|$ and $|r-r_-|$, $V(r(r^*))$ vanishes exponentially in r^* near the horizon:

$$V(r^*)\propto \exp(2\kappa_+ r^*), \quad \begin{matrix} r^*\to -\infty \\ (r\to r_+) \end{matrix} \tag{2.5}$$

$$V(r^*)\propto \exp(-2\kappa_- r^*), \quad \begin{matrix} r^*\to \infty \\ (r\to r_-) \end{matrix} \tag{2.6}$$

and peaks at an r finitely separated from the horizons. The results of our analysis depend on the rate of the exponential decrease of the potential. The technique used here is applicable for arbitrary exponential falloff. [$V(r)$ is also significant outside r_+, and falls off as $l(l+1)/r^2$ as $r\to\infty$.]

The linearly independent solutions near the horizons are $\phi \equiv e^{\pm i\omega r^*} e^{-i\omega t} f(r) \sim g(t\pm r^*)$ where $f(r)$ is a slow function of r and $g(t\pm r^*)$ shows the general (nondispersive) behavior for a general perturbation near the horizons r_+ and r_-. It will be useful to introduce null coordinates $v=r^*+t$, $u=r^*-t$. The inner horizon $r=r_-$ consists of two branches, the "right" branch which has coordinate $v=\infty$, and the "left" branch with $u=\infty$ (Fig. 1).

The energy density in the scalar field as measured by a freely falling observer near a horizon with four-velocity U^μ will be proportional to $\rho = (\phi_{,\alpha}U^\alpha)(\phi_{,\beta}U^\beta)+\frac{1}{2}\phi_{,\alpha}\phi^{*,\alpha}$. Because $t\pm r^* = \text{const}$ are null surfaces, the form $g(t\pm r^*)$ of the solutions near the horizons means that the energy density is dominated by the WKB form $|\phi_{,\alpha}U^\alpha|^2$. The same form holds for the electromagnetic stress tensor and the effective stress tensor of the gravitational perturbations.

We consider the energy density measured by freely falling observers when radiation falls into the hole from outside r_+. (We exclude radiation energy from region II into region III; region II is causally disconnected from region I.) Thus our "initial" conditions are

$$e^{-i\omega t}\phi_\omega\big|_{r_+} \sim T(\omega)e^{-i\omega(t+r^*)}, \tag{2.7}$$

corresponding to waves propagating "leftward" (Fig. 2) across r_+ from region I to region III. We solve the differential equation (2.2) for the amplitudes of right- and left-going waves on r_-:

$$e^{-i\omega t}\phi_\omega\big|_{r_-} \sim e^{-i\omega(t+r^*)} + R(\omega)e^{-i\omega(t-r^*)}. \tag{2.8}$$

Our boundary conditions require ingoing waves at the horizon r_+. Our modes are normalized as in Eq. (2.8) by demanding that the coefficient of the $(t+r^*)$ term near r_- be unity. The differential equation for a particular frequency then determines the coefficients $T(\omega)$ and $R(\omega)$.

Because of the structure of the r_- horizon, and because the scattering effectively occurs at finite $r>r_-$, the two terms in (2.8) refer to separated regions of the extended spacetime. The total wave may be written as $\phi = \int (1/r)\phi_\omega d\omega\, \mathfrak{A}(\omega)e^{-i\omega t}$ with $\mathfrak{A}(\omega)$ a mode constant. Via (2.8), the resultant wave near the inner horizon has the form

$$\phi\big|_{r_-} \sim [g^{(-)}(t-r^*)+g^{(+)}(t+r^*)]/r. \tag{2.9}$$

Via (2.8), the Fourier transforms of $g^{(-)}$ and $g^{(+)}$ are

$$\hat{g}^{(-)}_\omega = R(\omega)\hat{g}^{(+)}_\omega = \mathfrak{A}(\omega)R(\omega). \tag{2.10}$$

The remainder of this paper will concentrate on the evaluation of $g^{(-)}$ and $g^{(+)}$ near the inner horizon.

Consider a radially freely falling observer. His four-velocity is given by

$$U^t = \frac{dt}{d\tau} = \frac{Er^2}{(r-r_+)(r-r_-)} \tag{2.11}$$

and

$$U^r = \frac{dr}{d\tau} = p\,|E^2 - (r-r_+)(r-r_-)/r^2|^{1/2}, \tag{2.12}$$

where $p=\pm 1$ gives the sign of the square root. The value of p inverts at the turning points in r. However, because of the timelike character of r in region III, $p=-1$ in all of region III. From (2.11), (2.12), and Fig. 1 we see that if $E>0$ the world line enters region III from region I and exits region III through the left-hand ($t=-\infty$, i.e., $u=\infty$) branch of the inner horizon r_-. If $E<0$ the world line enters region III from region II and exits via the right-hand ($t=+\infty$, i.e., $v=\infty$) branch r_-. The case $E=0$ gives a world line which moves (at constant t) through region III (and its copies) passing through the bifurcation points of the horizon.

The energy measured near the horizon by one of these observers in region II is proportional to the square of

$$U^\alpha g^{(\pm)}_{,\alpha} = \frac{rg^{(\pm)\prime}}{(r-r_+)(r-r_-)}\left[E \mp \left|E^2 - \frac{(r-r_+)(r-r_-)}{r^2}\right|^{1/2}\right], \tag{2.13}$$

where the prime denotes a differentiation with respect to the argument of $g^{(\pm)}$ and the upper (lower) signs are to be taken together. For $E>0$ (an infalling observer) (2.13) calculated for $g^{(+)}$ is finite at the horizons. $g^{(+)}$ corresponds to a wave packet infalling with the observer. On the other hand, the calculation for $g^{(-)}$, with $E>0$, diverges at $r=r_-$ if the radiation is monochromatic, or in fact if the radiation defined by $g^{(-)}$ falls off slowly enough. As the observer crosses the left branch of the r_- horizon, his coordinates obey $dr^*/dt \simeq -1$ [cf. Eqs. (2.3), (2.11) and (2.12)]. Hence

$$\begin{aligned} -u &\equiv t - r^* \approx -2r^* + \text{const} \\ &\approx \kappa_-^{-1}\ln|r-r_-| + \text{const}. \end{aligned}$$

Reverting to null coordinates:

$$U^\alpha g^{(-)}_{,\alpha} \simeq \text{const} \times g^{(-)\prime}(u) \exp(+\kappa_- u) \qquad (2.14)$$

as the observer crosses the horizon $u=\infty$ (v remains finite). Hence $g^{(-)\prime}$ must decay at least as fast as $\exp(-\kappa_- u)$ as $u=\infty$ is approached; otherwise unbounded energy densities arise.

Similar analysis for an observer with $E<0$ who crosses the right ($v=\infty$) horizon at constant u shows that $g^{(-)\prime}$ gives finite-energy densities there but $g^{(+)\prime}$ must fall as fast as $\exp(-\kappa_- v)$ if unbounded energy densities are to be avoided.

Returning to Eqs. (2.11), (2.12), and (2.13), we note that an observer of the $E=0$ type sees

$$u^\alpha \phi_{,\alpha}\big|_{r_-} \propto (r-r_-)^{-1/2} g^{(\pm)\prime}(t \pm r^*).$$

Since such an observer has $t=$ const, we require that g' fall off faster than $(r-r_-)^{1/2} \propto \exp[\kappa_-(-r^*)]$.

III. STABILITY OF THE INNER HORIZON

Gürsel *et al.*[14] and McNamara[12,13] have shown that initial data consisting of an inverse-power-law decay in time after an initial onset reaches the inner horizon with a decaying power-law component in v which guarantees that the $v=\infty$ horizon is singular. Since power-law tails are a ubiquitous feature of propagation *outside* the black hole,[8-11] this result guarantees the instability of the inner structure of real holes which are in contact with the radiation content of the universe.

There is a straightforward physical explanation of the fact that the tails can disrupt the horizon by propagating past the spacetime region where the inner potential is strong. From the viewpoint of the radial equation, each frequency mode scatters off a simple, bounded, localized (in r^*) potential, which falls off exponentially on both sides of its region of significance. The coordinate r is a timelike coordinate in the region between the horizons, and the potential (which displays the nature of propagation in this spacetime) is thus actually spatially homogeneous and time dependent. This in no way changes the mathematical analysis of wave propagation in this region, but the physical requirements dictate a change in boundary condition from the simple one-dimensional scattering problem suggested by Eq. (2.2). For waves falling inward from outside the black hole, the natural behavior near the r_+ horizon is $e^{-i\omega r^*}e^{-i\omega t}$, since the potential near $r \sim r_+$ (i.e., $r^* \sim -\infty$) is proportional to $\exp(+2\kappa_+ r^*)$. Near r_- we expect waves (after traveling through the interaction region) to be traveling across both branches of the r_- horizon. It is seen that the equivalent one-dimensional scattering problem has the incident wave from the $r \gtrsim r_-$ side of the potential. The conservation of the Wronskian in Eq. (2.2) implies $|T|^2+|R|^2=1$. [See Eqs. (2.7) and (2.8).] Hence the magnitude of the wave entering across r_+ (i.e. $|T|$) is always less than that of the wave which propagates leftward after the encounter with the effective region of the time-dependent potential.

The time dependence of the potential gives an *amplification* of the incident wave. This amplification is greatest for those frequencies most strongly affected by the potential, i.e., for low frequencies (roughly, $|\omega|<M$). These are just the frequencies which are important in the tails, which are low-frequency phenomena. Hence we have a physical explanation of the disruptiveness of the t^{-q} tails. The geometry amplifies just those waves which lead to its destruction. [Ref. 14 shows that $R(\omega)$ and $T(\omega)$ are finite at $\omega \to 0$.]

The results just quoted show that the inner regions of the extended Reissner-Nordström solution are disrupted by infinite local energy densities when physically reasonable radiation—with power-law tails in time—falls across the outer horizon. Gürsel *et al.*[14] have shown further that even the data which are bounded in time as they cross the outer horizon r_+ (e.g., a sharply peaked Gaussian in time near the outer horizon) yield divergent energy densities at the inner horizon. The Gaussian gives a finite energy density on the $u=\infty$ horizon (the left branch of $r=r_-$). But a divergent flux is obtained near $v=\infty$, the right branch of the horizon.

Our qualitative investigation of this behavior must thus consider the radiation following rays which are parallel to and near the horizons. Such rays correspond, in the scattering picture, to the energy scattered in the regions where the potential $V(r(r^*))$ is exponentially small. It turns out that a first-order calculation which treats the exponentially small potential as a perturbation—together with some intuition—leads to the elegant results of Refs. 12–14.

For this purpose, we notice that the solution to

$$\left(\frac{d^2}{dr^{*2}}+\omega^2\right) g_\omega(r^*,y^*) = \delta(r^*-y^*) \qquad (3.1)$$

with "outgoing" boundary conditions (assuming an $e^{-i\omega t}$ time dependence) is

$$g_\omega(r^*,y^*) = \begin{cases} \dfrac{1}{2i\omega} e^{i\omega(r^*-y^*)}, & r^*>y^* \\[2ex] \dfrac{1}{2i\omega} e^{-i\omega(r^*-y^*)}, & r^*<y^*. \end{cases} \qquad (3.2)$$

For a first-order process involving the potential V, g_ω acts as a Green's function. For instance, consider the deviation from a wave of the form $\phi_\omega = e^{-i\omega r^*}$ treating V as an infinitesimal perturbation. If the total wave is written as $\Phi_\omega = \phi_\omega + \psi_\omega$ we have to first order

$$\psi_\omega(r^*) = \int_{-\infty}^{\infty} dy^* g_\omega(r^*, y^*) V(y^*) \phi_\omega(y^*)\,. \tag{3.3}$$

An important calculation is the one in which there is an exponentially decaying potential $V_0 e^{\kappa y^*}$ for $y^* \to -\infty$. For simplicity of calculation we suppose the potential vanishes for $y^* \geq 0$. Then, if $r^* > 0$, a trivial integration yields

$$\psi_\omega(r^*) = \frac{V_0 e^{i\omega r^*}}{2i\omega\,(\kappa - 2i\omega)}\,. \tag{3.4}$$

A similar calculation for the same incident ϕ may be carried out for $r^* > 0$, but assuming a potential $V = V_0 e^{-\kappa y^*}$, with $V = 0$ for $y^* < 0$. Then

$$\psi_\omega(r^*) = \frac{V_0}{2i\omega} e^{-i\omega r^*} e^{-\kappa r^*} \left(\frac{-1}{\kappa + 2i\omega} + \frac{1}{\kappa}\right) + \frac{V_0 e^{i\omega r^*}}{2i\omega(\kappa + 2i\omega)}\,. \tag{3.5}$$

A partial diagram of the ray traces that are important at the left branch of r_- is shown in Fig. 2. These are the waves which Gürsel *et al.*[13] found are not divergent at this horizon. The rays which are parallel to the horizon are associated with waves produced by interaction with the potential near there. Because the potential is exponentially decreasing $[\propto \exp(-2\kappa_- r^*)]$ near the horizon, the situation is like that described by Eq. (3.5). The zero-order wave ϕ_ω is multiplied by $e^{-i\omega t}$ (to give a wave moving across r_+ to the left), and the resultant scattering toward the right branch of the r_- horizon is the $e^{i\omega(r^*-t)}$ term appearing in $\psi_\omega e^{-i\omega t}$ with the ψ_ω of Eq. (3.5).

For purposes of evaluating the scattered waves we may take a pulse in ϕ which is simply $\phi = \delta(r^* + t)$. The correction to the wave will be $\psi = (1/2\pi)\int \psi_\omega d\omega$. To calculate the energy density associated with the rightward-moving wave, we differentiate the Fourier transform of the last term in Eq. (3.5) with respect to u:

$$g^{(-)\prime}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{V_0}{4i}\,\frac{e^{i\omega u}}{(\kappa_- + i\omega)}\,d\omega\,. \tag{3.6}$$

This $g^{(-)\prime}$ enters Eq. (2.13) to yield the observed energy density. We shall evaluate this integral by complex contour integration. An observer measuring ψ can always be located near the $u = \infty$ horizon so that $u = r^* - t > 0$. Then the integral may be closed in the upper half plane. Closing the contour in the upper half plane picks up the residue at the pole $\omega = i\kappa_-/2$, and for large u

$$g^{(-)\prime} \propto e^{-\kappa_- u}\,, \tag{3.7}$$

which is just rapid enough falloff that this wave

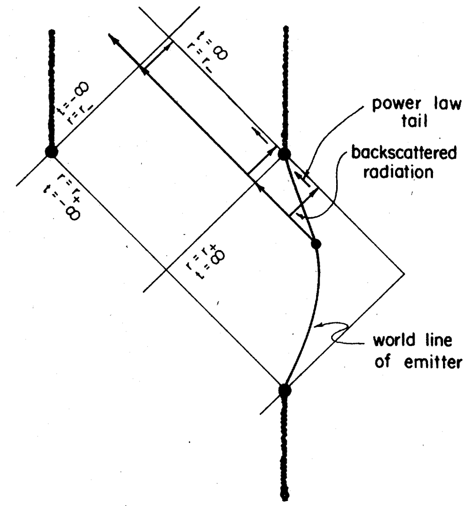

FIG. 2. A wave packet is scattered from the potential in the exterior region I, then proceeds into region III where further scattering occurs. The scattered radiation in region I will be rescattered into the hole, to give a "tail" which has a power-law (in time) decay. The energy density near the right branch of the r_- horizon (i.e., near $v = \infty$ where $v = r^* + t$) due to these tails decays sufficiently slowly so that infinite energy densities are developed near the $v = \infty$ horizon. Even if such tails are somehow suppressed so that the pulse is manufactured to have finite duration as it crosses r_+, further scattering near r_+ gives waves which are then rescattered at $r^* \approx 0$ to give waves traveling near the $v = \infty$ horizon. Such waves have infinite energy density there. (The waves scattered just before the initial pulse crosses r_- travel rightward parallel to the $u = \infty$ horizon, i.e., parallel to the left branch of the r_- horizon, yielding a *finite* energy density as $u \to \infty$.)

does give finite energy at the horizon.

The calculation of Ref. 14 shows that for a δ-function wave across the outer (r_+) horizon, there is near the *right* branch of the inner horizon a leftward-traveling wave falling off as $\exp[-\kappa_+ \times(r^* + t)]$ as the horizon is approached $[(r^* + t) \to \infty]$. Figure 2 shows how this wave arises. The inward-falling wave (leftward traveling in the diagram) at r_+ is scattered near r_+ by the exponentially small potential there. This produces radiation which travels rightward parallel to and just inside the outer horizon $r_+(r^* = -\infty)$. These waves travel toward larger values of r^*, and at $r^* \sim 0$, when the potential is strong, the waves are scattered again so that they are leftward traveling. The origin of time can be adjusted so that the inital scattering at $r^* \approx r_0^* \approx -\infty$ occurs for $r^* + t = 0$. After the first scattering the radiation travels along the ray $r^* - t \approx \text{const} \approx 2r_0^*$. The second scattering takes place in the strong potential near $r^* = 0$. After this second scattering the ray traveling leftward near the right branch of the $r = r_-$ horizon has $r^* + t = \text{const} \approx -2r_0^*$. Hence the second scattering translates a function of the form $r^* - t \approx 2r_0^*$ to one of the form $r^* + t \approx -2r_0^*$ (i.e., u is replaced by $-v$). This second scattering occurs in a *strong*-field region of the potential and may be assumed to be essentially 100% effective as indicated by the physics of the process. We make this assumption, which is justified by calculations[14,15] and now concentrate on the first scattering. We idealize the potential to a decaying exponential of the form $V_0 e^{2\kappa_+ r^*}$ ($r^* \to -\infty$) and assume that it vanishes for $r^* > 0$. The waves are scattered rightward can be investigated by evaluating the field (corresponding to $e^{i\omega r^*}$) in Eq. (3.4), at the point $r^* = 0 + \epsilon$. Since $\kappa = 2\kappa_+$ here, we find a Fourier component of the form

$$\frac{V_0 e^{i\omega(r^*-t)}}{4i\omega(\kappa_+ - i\omega)}. \quad (3.8)$$

A $\delta(t + r^*)$ input then yields a rightward-traveling wave

$$\psi^{(-)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{V_0 e^{i\omega(r^*-t)}}{4i\omega(\kappa_+ - i\omega)} d\omega. \quad (3.9)$$

The second scattering translates this into a leftward-traveling wave, which near the horizon (large v) has the behavior

$$g^{(+)}(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{V_0}{4i\omega} \frac{e^{-i\omega v}}{(\kappa_+ - i\omega)}.$$

Again we calculate the derivative

$$g^{(+)\prime}(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{V_0}{4i} \frac{e^{-i\omega v}}{(\kappa_+ - i\omega)} d\omega. \quad (3.10)$$

For large v the exponential decreases if ω has a negative imaginary part. Hence we evaluate this integral by contour integration in the lower half plane. This procedure encloses the pole at $\omega = -i\kappa_+$. The residue from this pole yields

$$g^{(+)\prime}(v) \propto e^{-v\kappa_+}. \quad (3.11)$$

These are the waves which Gürsel *et al.*[14] found to have a divergent energy density on the right horizon. Our result of Eqs. (3.7) and (3.11) demonstrates analytically—but with physically reasonable approximation—what was shown by computer analysis in Ref. 14.

IV. CONCLUSIONS

The lack of stability within perturbation theory of course does not logically imply that a perturbed Reissner-Nordström solution cannot have a causal structure "similar" to that in Fig. 1. Using the word instability, we meant that large changes from the background situation were implied by the infinite energy densities arising from finite perturbations. By now it is amply clear that the inner structure is unstable in that sense.

In this paper we have attempted to show that physical intuition and simple calculation can explain most of the results that have been found concerning the inner stability of the Reissner-Nordström solution. We have had to make some simplifying but physically reasonable assumptions. Nonetheless, we are excited and encouraged by the end simplicity of the results and by their direct physical explanation and interpretation.

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