SU(4) × U(1) gauge theory. III. New approach to Cabibbo mixing

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A new approach to Cabibbo mixing is proposed in the framework of an SU(4) × U(1) gauge theory. Instead of introducing Cabibbo mixing in the quark mass matrix as is usually done in SU(2) × U(1), we take advantage of the existence of the 12 additional gauge bosons and mix the gauge bosons in their mass matrix. It is found that in the most general case there are four mixing angles which are in principle observable. Among them one is the usual Cabibbo angle; another one is like it but is in a different sector. Two angles are associated with *CP* nonconservation. By demanding that the Higgs potential have the most general form, we show that in our theory Cabibbo mixing and *CP* and muon-number nonconservation are inevitable. An intimate connection between the Cabibbo angle and the muon-number-nonconserving $\mu \to e\gamma$ process is obtained. Possible experimental tests of our theory are suggested.

I. INTRODUCTION

This is the third in a series of papers[1] on the unified gauge theory based on a spontaneously broken SU(4)×U(1) symmetry. The subject concerns the development of a general framework in which the Cabibbo mixing in the conventional SU(2) ×U(1) sector can be understood as a result of a global mixing of the gauge bosons in the larger symmetry.

The origin of the Cabibbo angle is a mystery that has not been fully understood up to now. It is generally regarded as a measure of the mismatch between the classification of the quark states according to the strong and weak interactions. Taken as a phenomenological fact, this mismatch can conveniently be parametrized by a mixing angle between the d and s quark fields. Insofar as the weak-interaction processes in the usual SU(2) ×U(1) gauge theory are concerned, the mixing could equally well have been between the u and c quark fields, and there would be no difference. This is not true if there are other gauge bosons in a theory with higher symmetry, as we have shown in the case of SU(4) × U(1) in paper I, even though the difference is hard to detect experimentally. Thus there is a "freedom" in the description of the Cabibbo phenomenon that needs to be fully investigated.

In usual gauge theories the mixing of quark fields is achieved by an appropriate choice of the Higgs couplings which appear in the quark mass matrix. While the same procedure could also be used for SU(4) × U(1) and, in fact, was adopted in paper I, we note that the group is rich enough to permit the Cabibbo angle to be introduced by an alternative mechanism. Because a large gauge group has more gauge bosons, the procedure is to introduce effective Cabibbo mixing by mixing the gauge bosons. To illustrate this idea which is central to our present approach, we give here a simple example.

Consider the interactions of two of the SU(4) ×U(1) gauge fields W_+ and V_+ with the quarks; it is given in paper I, Eq. (2.4),

$$W_+(\bar{u}d+\bar{c}s)+V_+(\bar{u}s-\bar{c}d)\,, \tag{1.1}$$

where γ matrices and Lorentz indices have been suppressed. Now if the physical boson fields $\tilde{W}_+$ and $\tilde{V}_+$ are linear combinations of W_+ and V_+ according to

$$\begin{aligned}\tilde{W}_+ &= \cos\theta_C W_+ + \sin\theta_C V_+\,,\\ \tilde{V}_+ &= -\sin\theta_C W_+ + \cos\theta_C V_+\,,\end{aligned} \tag{1.2}$$

then (1.1) can be reexpressed in terms of $\tilde{W}_+$ and $\tilde{V}_+$ as

$$\begin{aligned}&\tilde{W}_+[\bar{u}(d\cos\theta_C+s\sin\theta_C)+\bar{c}(s\cos\theta_C-d\sin\theta_C)]\,,\\ &+\tilde{V}_+[\bar{u}(s\cos\theta_C-d\sin\theta_C)-\bar{c}(d\cos\theta_C+s\sin\theta_C)]\,.\end{aligned} \tag{1.3}$$

Evidently, a mixture of the gauge fields identified as the physical bosons leads to the usual Cabibbo mixing. Note that the quark fields are eigenstates of the quark mass matrix, and are "mixed" in (1.3) only as a result of diagonalizing the boson mass matrix. This example contains the essence of our approach. In this paper we consider the most general mixing of the bosons that is possible. In such a theory the "freedom" mentioned above, of course, disappears.

Our strategy is to break SU(4)×U(1) down to the SU(2)×U(1) of Weinberg and Salam.[2] In general there are many candidate subgroups. However, since we require the charge operator to be one of the generators of SU(2)×U(1), only a subset of them is acceptable. The various SU(2)×U(1) subgroups in this subset are related by transformations which leave the charge operator invariant.

It will not come as a surprise to find that these transformations form a subgroup themselves, comprising two disjoint SU(2) subgroups, one acting in the (d, s) subspace, the other in the (u, c) subspace. We shall make the most general rotations in this SU(2) ×SU(2) subspace, and examine the observable consequences of mixing the gauge bosons of SU(4). The usual Cabibbo angle becomes just one of four possible angles that are in principle measurable when viewed in this more general framework. Two of the others are associated with *CP* nonconservation. As in the conventional picture of the Cabibbo angle, these four angles give a measure of the mismatch between the diagonal bases of the mass matrices for the quarks and the gauge bosons.

We shall study the nature of the Higgs fields and determine how their expectation values give rise to the pattern of symmetry breaking that we want. Moreover, we shall examine the geometrical properties of the Higgs multiplets and find a connection between the mixing angles and the angles that specify the orientations of the Higgs multiplets, i.e., the orientation of the vacuum. We also study the most general SU(4) × U(1)-invariant Higgs potential, and it is of particular significance that the potential forces upon the theory the existence of Cabibbo mixing and *CP* nonconservation. Thus in our theory both of these physical phenomena have their origin in the spontaneous breakdown of a higher symmetry.

This paper is organized as follows. In Sec. II we discuss the algebra of SU(4) in a basis that is particularly suitable for our considerations in the paper. The general rotations in the subspace that leave the charge operator invariant are then discussed in Sec. III. The rotation angles are related to the observable angles such as the Cabibbo angle in Sec. IV, where it is found that there are at most four observable angles. In Sec. V the connection of the foregoing with the Higgs fields is established. We develop there the relationship between the Cabibbo angle and the orientations of the Higgs fields, and in Sec. VI the connection with *CP* nonconservation is investigated. In Sec. VII the Higgs potential is considered; we find there that in our theory Cabibbo mixing and *CP* nonconservation are inevitable. Some simple examples of the general mixings are considered in detail in Sec. VIII, where we also give the phenomenological consequences of our theory of the mixing of the gauge bosons. Conclusions are given in Sec. IX.

II. ALGEBRA

The algebra of SU(4) in the canonical basis is

$$[F_i, F_j] = if_{ijk}F_k\,, \quad i,j,k=1,\ldots,15 \tag{2.1}$$

where the structure constants f_{ijk} are given, for example, in Ref. 3. For our purposes in this paper it is more convenient to consider the algebra in a different basis, which for later reference we shall call the *standard* basis. In that basis the 15 elements of the algebra are denoted by the set

$$\{G\} \equiv \{G^{\pm}_{\alpha}, G^3, \mathcal{G}_i, \mathcal{G}'_i\}\,, \tag{2.2}$$

where $\alpha = 0,1,2,3$ and $i=1,2,3$. They are defined in terms of the F_i by

$$\begin{aligned} G_0^{\pm} &= (F_1+F_{13}) \pm i(F_2-F_{14})\,,\\ G_1^{\pm} &= \pm i[(F_4+F_{11}) \pm i(F_5-F_{12})]\,,\\ G_2^{\pm} &= (F_4-F_{11}) \pm i(F_5+F_{12})\,,\\ G_3^{\pm} &= \pm i[(F_1-F_{13}) \pm i(F_2+F_{14})]\,, \end{aligned} \tag{2.3}$$

$$G^3 = F_3 + \frac{1}{\sqrt{3}}F_8 - (\tfrac{2}{3})^{1/2}F_{15}\,, \tag{2.4}$$

$$\mathcal{G}_1 = F_6\,, \quad \mathcal{G}_2 = F_7\,, \quad \mathcal{G}_3 = \tfrac{1}{2}(-F_3+\sqrt{3}F_8)\,, \tag{2.5}$$

$$\mathcal{G}'_1 = F_9\,, \quad \mathcal{G}'_2 = F_{10}\,, \quad \mathcal{G}'_3 = \frac{1}{2}\left[F_3 + \frac{1}{\sqrt{3}}F_8 + 2(\tfrac{2}{3})^{1/2}F_{15}\right]. \tag{2.6}$$

This is the same set of generators introduced already in paper I, written here in more convenient notation.

We first consider the following set of commutation relations that can be established from (2.1):

$$\begin{aligned} &[G^+_{\alpha}, G^-_{\alpha}] = 2G^3\,, \quad \alpha \text{ not summed}\\ &[G^3, G^{\pm}_{\alpha}] = \pm G^{\pm}_{\alpha}\,, \end{aligned} \tag{2.7}$$

$$[G^3, \mathcal{G}_i] = [G^3, \mathcal{G}'_i] = 0\,, \tag{2.8}$$

$$\begin{aligned} &[\mathcal{G}_i, \mathcal{G}_j] = i\epsilon_{ijk}\mathcal{G}_k\,,\\ &[\mathcal{G}'_i, \mathcal{G}'_j] = i\epsilon_{ijk}\mathcal{G}'_k\,,\\ &[\mathcal{G}_i, \mathcal{G}'_j] = 0\,. \end{aligned} \tag{2.9}$$

We see from (2.7) that for every α the set $\{G^{\pm}_{\alpha}, G^3\}$ forms an SU(2) algebra. Similarly, on account of (2.9), $\{\mathcal{G}_i\}$ and $\{\mathcal{G}'_i\}$ are the elements of the algebra of SU(2) ×SU(2). The charge operator Q is[1]

$$Q = G^3 + F_0\,, \tag{2.10}$$

where F_0 is the U(1) generator of the SU(4) × U(1). We therefore see from (2.8) that transformations under the group SU(2) ×SU(2) mentioned above leave Q invariant. We shall find that this particular group is the one relevant for introducing Cabibbo mixing in the most general way.

Let us, for definiteness, use $\mathcal{G}$ to denote the group SU(2) ×SU(2) generated by $\{\mathcal{G}_i, \mathcal{G}'_j\}$. It is the covering group of SO(4), whose generators $L_{\alpha\beta}$ are

$$\begin{aligned} L_{0i} &= -\mathcal{G}_i + \mathcal{G}'_i\,,\\ L_{ij} &= -\epsilon_{ijk}(\mathcal{G}_k + \mathcal{G}'_k)\,, \end{aligned} \tag{2.11}$$

with $L_{\alpha\beta}=-L_{\beta\alpha}$, and $\alpha,\beta=0,\ldots,3$. They, of course, satisfy the algebraic relations

$$[L_{\alpha\beta},L_{\gamma\delta}]=i(\delta_{\beta\gamma}L_{\alpha\delta}+\delta_{\alpha\delta}L_{\beta\gamma}-\delta_{\alpha\gamma}L_{\beta\delta}-\delta_{\beta\delta}L_{\alpha\gamma}). \quad (2.12)$$

It is straightforward to establish that under the group $\mathcal{G}$ the generators $G^{\pm}_{\alpha}$ defined in (2.3) transform as four-dimensional vectors

$$[L_{\alpha\beta},G^{\pm}_{\gamma}]=i(\delta_{\beta\gamma}G^{\pm}_{\alpha}-\delta_{\alpha\gamma}G^{\pm}_{\beta}). \quad (2.13)$$

It is now clear that the advantage of the standard basis (2.2) is that the set $\{\mathcal{G}_i,\mathcal{G}'_j\}$ generates the subgroup $\mathcal{G}$ under which G^3 (or Q) is invariant, and under which $G^{\pm}_{\alpha}$ rotate as vectors.

The remaining commutation relations in the standard basis are

$$\begin{aligned}&[G^{\pm}_{\alpha},G^{\mp}_{\beta}]=\pm\delta_{\alpha\beta}2G^3-2iL_{\alpha\beta},\\&[G^{\pm}_{\alpha},G^{\pm}_{\beta}]=0.\end{aligned} \quad (2.14)$$

It now follows that the algebra of (2.7) is invariant under $\mathcal{G}$, for if under $\mathcal{G}$ we have

$$G^{\pm}_{\alpha}\to\tilde{G}^{\pm}_{\alpha}=\Lambda_{\alpha\beta}G^{\pm}_{\beta}, \quad (2.15)$$

where $\Lambda_{\alpha\beta}$ specifies a four-dimensional rotation, satisfying

$$\sum_{\alpha}\Lambda_{\alpha\beta}\Lambda_{\alpha\gamma}=\delta_{\beta\gamma}, \quad (2.16)$$

then the antisymmetry of $L_{\alpha\beta}$ in (2.14) guarantees that for any fixed α

$$\begin{aligned}&[\tilde{G}^{+}_{\alpha},\tilde{G}^{-}_{\alpha}]=2G^3,\quad \alpha \text{ not summed}\\&[G^3,\tilde{G}^{\pm}_{\alpha}]=\pm\tilde{G}^{\pm}_{\alpha}.\end{aligned} \quad (2.17)$$

The fact that this SU(2) structure is invariant under $\mathcal{G}$ is important for our theory in which we exploit the freedom in identifying the eight physical charged gauge bosons with the eight generators $\tilde{G}^{\pm}_{\alpha}$. The Cabibbo and other mixing angles result from this freedom. After the SU(4)×U(1) symmetry is broken down and the heavy gauge bosons acquire their masses, there remains an SU(2)×U(1) symmetry whose generators we shall take to be $\tilde{G}^{\pm}_0$, G^3, and F_0 without loss of generality.

III. GENERAL ROTATIONS

We now consider general rotations under $\mathcal{G}$ and examine the effects of the rotations on the quark fields and on the gauge-boson fields in our standard basis.

Let a transformation under $\mathcal{G}$ be specified by two two-dimensional matrices A and B, where $A\in$SU(2) generated by $\mathcal{G}_i$, and $B\in$SU(2) generated by $\mathcal{G}'_i$. For every set $\{A,B\}$ there corresponds a definite transformation $\Lambda\in$SO(4), satisfying

$$B^{\dagger}\tau_{\alpha}A=\Lambda_{\alpha\beta}\tau_{\beta}, \quad (3.1)$$

where

$$\tau_{\alpha}=(1,i\sigma_i),\quad \text{for } \alpha=(0,i). \quad (3.2)$$

Note that $\tau_{\alpha}\in$SU(2), but are not Hermitian.

We now apply this transformation to our interaction Lagrangian. In the canonical basis the coupling of the quarks to the SU(4) gauge bosons is

$$\mathcal{L}_{\text{int}}=\sum_{i=1}^{15}W_i\bar{q}\lambda_i q, \quad (3.3)$$

where

$$q=\begin{bmatrix}u\\d\\s\\c\end{bmatrix}. \quad (3.4)$$

The γ matrices, Lorentz indices, and coupling constant $g/\sqrt{2}$ have been suppressed for brevity. Throughout this paper the fields of (3.4) refer to the physical quarks. No mixing in the quark mass matrix will be introduced.

In analogy to (2.2) we express the gauge bosons in the standard basis, i.e.,

$$\{W\}=\{C^{\pm}_{\alpha},R,S_i,T_j\}, \quad (3.5)$$

where

$$C^{\pm}_{\alpha}=\begin{bmatrix}W_{\pm}\\\mp iU_{\pm}\\V_{\pm}\\\mp iX_{\pm}\end{bmatrix},\quad S_i=\begin{bmatrix}W_6\\W_7\\S\end{bmatrix},\quad T_j=\begin{bmatrix}W_9\\W_{10}\\T\end{bmatrix}. \quad (3.6)$$

Their relationships to the W_i have already been defined in paper I, Eq. (2.3), and apart from normalization are analogous to those given for the generators in (2.6). For the quark fields it is more convenient to use the two-dimensional representation

$$q_1=\begin{pmatrix}d\\s\end{pmatrix},\quad q_2=\begin{pmatrix}u\\c\end{pmatrix}. \quad (3.7)$$

The currents are then

$$\begin{aligned}&J^{+}_{\alpha}=\bar{q}_2\tau_{\alpha}q_1,\quad J^{-}_{\alpha}=(J^{+}_{\alpha})^{\dagger},\\&K=\frac{1}{\sqrt{2}}(\bar{q}_2q_2-\bar{q}_1q_1),\\&M_i=\bar{q}_1\sigma_iq_1,\quad N_i=\bar{q}_2\sigma_iq_2.\end{aligned} \quad (3.8)$$

In terms of these quantities, (3.3) now has the form

$$\mathcal{L}_{\text{int}}=C^{+}_{\alpha}J^{+}_{\alpha}+C^{-}_{\alpha}J^{-}_{\alpha}+RK+S_iM_i+T_iN_i. \quad (3.9)$$

In the exact symmetry limit $\mathcal{L}_{\text{int}}$ is invariant under the transformations $\Lambda(A,B)$ in $\mathcal{G}$. The charged bosons $C^{\pm}_{\alpha}$ and currents $J^{\pm}_{\alpha}$, being vectors in the four-dimensional space of SO(4), transform as follows:

$$U^{\dagger}(\Lambda)C^{\pm}_{\alpha}U(\Lambda)=\Lambda_{\alpha\beta}C^{\pm}_{\beta}=\tilde{C}^{\pm}_{\alpha}, \qquad (3.10)$$

$$U^{\dagger}(\Lambda)J^{\pm}_{\alpha}U(\Lambda)=\Lambda_{\alpha\beta}J^{\pm}_{\beta}=\tilde{J}^{\pm}_{\alpha}. \qquad (3.11)$$

The neutral bosons S_i and T_i form vectors in two separate three-dimensional spaces, the rotations in which are generated by $\mathcal{G}_i$ and $\mathcal{G}'_i$, respectively. Let those rotations be denoted by $R_{ij}(A)$ and $R_{ij}(B)$, respectively. To every matrix $M\in$SU(2) there corresponds a unique $R(M)\in$SO(3) given by

$$M^{\dagger}\sigma_i M=R_{ij}(M)\sigma_j, \qquad (3.12)$$

from which follows the explicit dependence

$$R_{ij}(M)=\tfrac{1}{2}\mathrm{tr}(M^{\dagger}\sigma_i M\sigma_j). \qquad (3.13)$$

In terms of these R_{ij} matrices we have

$$U^{\dagger}(\Lambda(A,B))S_iU(\Lambda(A,B))=U^{\dagger}(A)S_iU(A)=R_{ij}(A)S_j=\tilde{S}_i, \qquad (3.14)$$

$$U^{\dagger}(\Lambda(A,B))T_iU(\Lambda(A,B))=U^{\dagger}(B)T_iU(B)=R_{ij}(B)T_j=\tilde{T}_i. \qquad (3.15)$$

Since M_i and N_i transform similarly, it follows from (3.8) and (3.12) that

$$\tilde{M}_i=R_{ij}(A)M_j=\bar{q}_1A^{\dagger}\sigma_iAq_1, \qquad (3.16)$$

$$\tilde{N}_i=R_{ij}(B)N_j=\bar{q}_2B^{\dagger}\sigma_iBq_2. \qquad (3.17)$$

Thus we see that the quarks transform under $\mathcal{G}$ as

$$U^{\dagger}(A)q_1U(A)=Aq_1, \quad U^{\dagger}(B)q_2U(B)=Bq_2. \qquad (3.18)$$

Together with (3.8) and (3.11) they imply

$$\tilde{J}^{\pm}_{\alpha}=\bar{q}_2B^{\dagger}\tau_{\alpha}Aq_1 \qquad (3.19)$$

which reaffirms (3.1). Indeed, the gauge bosons and the two quark doublets q_1 and q_2 form a realization of the $\{A,B\}\to\Lambda(A,B)$ homomorphism.

The invariance of $\mathcal{L}_{\text{int}}$ under $\mathcal{G}$ therefore implies

$$\mathcal{L}_{\text{int}}=\tilde{C}^{+}_{\alpha}\tilde{J}^{+}_{\alpha}+\tilde{C}^{-}_{\alpha}\tilde{J}^{-}_{\alpha}+RK+\tilde{S}_i\tilde{M}_i+\tilde{T}_i\tilde{N}_i. \qquad (3.20)$$

The fact that R and K are invariant can be inferred from (2.8).

IV. OBSERVABLE ANGLES

We now make use of the formalism developed above to discuss the Cabibbo angle and its generalizations, and to determine the maximum number of observable angles in the SU(4)×U(1) gauge theory.

Let us begin by recalling the usual SU(2)×U(1) theory in which there is only one charged current. The Cabibbo coupling may be written in the form

$$W_+(\bar{u}\,\bar{c})\begin{bmatrix}\cos\theta_C & \sin\theta_C\\ -\sin\theta_C & \cos\theta_C\end{bmatrix}\begin{bmatrix}d\\ s\end{bmatrix}+\text{H.c.}, \qquad (4.1)$$

where θ_C is the Cabibbo angle and the γ matrices have been suppressed. It is clear that the mixing matrix can act on either q_1 on its right or $\bar{q}_2$ on its left, reaffirming the conventional wisdom that in the SU(2)×U(1) gauge theory (d,s) mixing is equivalent to (u,c) mixing. The matrix represents the mismatch between the quark eigenstates of strong interactions and the weak hadronic current that couples to W_+.

Our interest in this paper is to effect Cabibbo mixing not by mixing the quarks in their mass matrix but by mixing the gauge bosons. This possibility is open to us now in SU(4)×U(1) but was not available in a theory in which the only charged bosons are $W_\pm$. To see how it works consider the charged sector of (3.20),

$$C^{+}_{\alpha}J^{+}_{\alpha}=\tilde{C}^{+}_{\alpha}\tilde{J}^{+}_{\alpha}=\tilde{C}^{+}_{\alpha}\bar{q}_2B^{\dagger}\tau_{\alpha}Aq_1, \qquad (4.2)$$

where (3.19) has been substituted for $\tilde{J}^{+}_{\alpha}$. If the Higgs scalars are such that $\tilde{C}^{+}_{\alpha}$ are the eigenstates of the mass matrix of the gauge bosons, then the usual intermediate-vector-boson W coupling, which is now the $\alpha=0$ term of (4.2), becomes

$$\tilde{W}_+\bar{q}_2B^{\dagger}Aq_1, \qquad (4.3)$$

where the notation for bosons with a tilde is the same as (3.6). Since $B^{\dagger}A\in$SU(2), Euler decomposition implies the existence of three angles (ϕ,θ,ϕ') such that

$$B^{\dagger}A=Z(\phi')Y(\theta)Z^{\dagger}(\phi), \qquad (4.4)$$

where

$$Z(\phi)=e^{i\phi\sigma_3/2}=\begin{bmatrix}e^{i\phi/2} & 0\\ 0 & e^{-i\phi/2}\end{bmatrix}, \qquad (4.5)$$

$$Y(\theta)=e^{i\theta\sigma_2/2}=\begin{bmatrix}\cos\theta/2 & \sin\theta/2\\ -\sin\theta/2 & \cos\theta/2\end{bmatrix}. \qquad (4.6)$$

If we define two new quark doublets

$$q'_1\equiv Z^{\dagger}(\phi)q_1=\begin{bmatrix}e^{-i\phi/2}d\\ e^{i\phi/2}s\end{bmatrix}, \qquad (4.7)$$

$$\bar{q}'_2=\bar{q}_2Z(\phi'),\ \text{i.e.,}\ q'_2=\begin{bmatrix}e^{-i\phi'/2}u\\ e^{i\phi'/2}c\end{bmatrix}, \qquad (4.8)$$

we can write (4.3) in the form

$$\tilde{W}_+\bar{q}'_2Y(\theta)q'_1. \qquad (4.9)$$

This can be brought to the conventional form for Cabibbo mixing and is equivalent to (4.1) by a redefinition of the phases of the quark fields which we are free to make. This completes our proof of the possibility of Cabibbo mixing through the mixing of the gauge bosons, while establishing

$$\theta_C = \theta/2\,. \tag{4.10}$$

It should now be clear than in the context of SU(4)×U(1) symmetry the Cabibbo angle is a measure of the mismatch between the diagonal bases of the quark and gauge-boson mass matrices. This mismatch can be completely described by a rotation in $\mathcal{G}$.

Since we have many more bosons beside the conventional charged W bosons now identified as $\tilde{W}_\pm$, there can be other observable angles in addition to the Cabibbo angle θ_C. By focusing on the $\alpha = 0$ sector we obtained (4.10) with the phases of the quark fields specified by (4.7) and (4.8). Turning now to the $\alpha \neq 0$ sector but adhering to the same phases for the quark fields as in (4.7) and (4.8), we have from (4.2)

$$\tilde{C}_i^+ \tilde{J}_i^+ = \tilde{C}_i^+ \bar{q}_2 B^\dagger \tau_i A q_1 = \tilde{C}_i^+ \bar{q}_2' K_i q_1', \quad i = 1, 2, 3 \tag{4.11}$$

where

$$\begin{aligned} K_i &= Z^\dagger(\phi') B^\dagger \tau_i A Z(\phi) \\ &= Y(\theta)[AZ(\phi)]^\dagger \tau_i [AZ(\phi)]\,. \end{aligned} \tag{4.12}$$

Since $AZ(\phi)$ is an element of SU(2), it can be a function of at most three independent angles, which in general are independent of θ. Hence out of a total of six parameters specifying a general rotation in $\mathcal{G}$, there can at most be four (in principle) observable angles in the charged sector, of which the Cabibbo angle $\theta_C = \theta/2$ is one.

One way of understanding the three additional angles besides θ_C is to put (4.11) into a more recognizable form. In (4.11) it is assumed that $\tilde{C}_i^+$ are eigenvectors of the boson mass matrix, while q_1' and q_2' are eigenvectors of the quark mass matrix; the deviation of K_i from τ_i represents the mismatch. Now $AZ(\phi)$ induces a rotation of τ_i, i.e.,

$$[AZ(\phi)]^\dagger \tau_i [AZ(\phi)] = \Omega_{ij} \tau_j\,, \tag{4.13}$$

where Ω_{ij} is the $R_{ij}(A)$ in (3.12) preceeded by a rotation around the z axis through ϕ. Thus (4.11) and (4.12) can be rewritten as

$$\tilde{C}_i^+ \tilde{J}_i^+ = \tilde{C}_i^+ \Omega_{ij} [\bar{q}_2' Y(\theta) \tau_j q_1']\,. \tag{4.14}$$

The quantity inside the square brackets is a natural generalization of the current in (4.9). Note that because of the position of $Y(\theta)$ the effective Cabibbo mixing acts on the (u, c) quarks (again, in $\mathcal{L}_{\text{int}}$ not in the quark mass matrix). Evidently, the boson mass eigenstates $\tilde{C}_1^+$ couple to linear combinations of the currents in the square brackets which do not involve the Cabibbo current of (4.9). Because these bosons are heavy, the three mixing angles are not susceptible to easy detection.

For completeness we give here also the description in which the effective Cabibbo mixing in $\mathcal{L}_{\text{int}}$ can be rewritten to appear in the (d, s) sector. It is straightforward to express (4.12) in the alternative form

$$K_i = [BZ(\phi')]^\dagger \tau_i [BZ(\phi')] Y(\theta)\,. \tag{4.15}$$

Thus proceeding as before with the recognition that

$$[BZ(\phi')]^\dagger \tau_i [BZ(\phi')] = \Omega_{ij}' \tau_j\,, \tag{4.16}$$

where Ω_{ij}' is the $R_{ij}(B)$ in (3.12) preceeded by a rotation around the z axis through ϕ', we obtain another expression for (4.14):

$$\tilde{C}_i^+ \tilde{J}_i^+ = \tilde{C}_i^+ \Omega_{ij}' [\bar{q}_2' \tau_j Y(\theta) q_1']\,. \tag{4.17}$$

This time the effective Cabibbo mixing is between the (d, s) quarks, and the currents are mixed by a different three-dimensional rotation.

In the neutral sector the situation is similar. Consider first the interactions of the neutral currents M_i:

$$\begin{aligned} S_i M_i = \tilde{S}_i \tilde{M}_i &= \tilde{S}_i \bar{q}_1 A^\dagger \sigma_i A q_1 \\ &= \tilde{S}_i \bar{q}_1' [AZ(\phi)]^\dagger \sigma_i [AZ(\phi)] q_1' \\ &= \tilde{S}_i \Omega_{ij} (\bar{q}_1' \sigma_j q_1')\,. \end{aligned} \tag{4.18}$$

Similarly, for N_i we have

$$\begin{aligned} T_i N_i = \tilde{T}_i \tilde{N}_i &= \tilde{T}_i \bar{q}_2 B^\dagger \sigma_i B q_2 \\ &= \tilde{T}_i \Omega_{ij} [\bar{q}_2' Y(\theta) \sigma_j Y(-\theta) q_2']\,. \end{aligned} \tag{4.19}$$

The above equations can be reexpressed in terms of Ω_{ij}' as follows:

$$S_i M_i = \tilde{S}_i \Omega_{ij}' [\bar{q}_1' Y(-\theta) \sigma_j Y(\theta) q_1']\,, \tag{4.20}$$

$$T_i N_i = \tilde{T}_i \Omega_{ij}' (\bar{q}_2' \sigma_j q_2')\,. \tag{4.21}$$

Clearly, the neutral sector does not involve any new angles that have not already been accounted for in the charged sector. Thus the total number of angles remains at four.

Having concluded that there are four angles that are in principle observable, we now need to identify them from among the six angles which specify a general four-dimensional rotation in $\mathcal{G}$. Let us parametrize A and B by

$$A(\alpha, \beta, \gamma) = Z(\alpha) Y(\beta) Z^\dagger(\gamma)\,, \tag{4.22}$$

$$B(\alpha', \beta', \gamma') = Z(\alpha') Y(\beta') Z^\dagger(\gamma')\,. \tag{4.23}$$

Moreover, in terms of the three angles $\alpha - \alpha'$, β, and β', we define three other angles α'', β'', and γ'' by setting

$$Y^\dagger(\beta') Z(\alpha - \alpha') Y(\beta) = Z(\alpha'') Y(\beta'') Z^\dagger(\gamma'')\,, \tag{4.24}$$

so that

$$\cos\beta\cos\beta' + \sin\beta\sin\beta' \cos(\alpha - \alpha') = \cos\beta'' ,$$
$$\cos(\alpha - \alpha') = \cos\alpha'' \cos\gamma'' + \cos\beta'' \sin\alpha'' \sin\gamma'' ,$$
$$\cos(\alpha - \alpha')\cos\beta\cos\beta' + \sin\beta\sin\beta' = \cos\beta'' \cos\alpha'' \cos\gamma'' + \sin\alpha'' \sin\gamma'' . \quad (4.25)$$

In terms of these new angles $B^\dagger A$ takes the form

$$B^\dagger A = Z(\gamma' + \alpha'')Y(\beta'')Z^\dagger(\gamma + \gamma'') . \quad (4.26)$$

Comparison of (4.26) with (4.4) yields the relations

$$\theta = \beta'' , \quad (4.27)$$

$$\phi = \gamma + \gamma'', \quad \phi' = \gamma' + \alpha'' . \quad (4.28)$$

It then follows that

$$AZ(\phi) = Z(\alpha)Y(\beta)Z(\gamma'') ,$$
$$BZ(\phi') = Z(\alpha')Y(\beta')Z(\alpha'') , \quad (4.29)$$

Through (4.13) and (4.16) they determine Ω_{ij} and Ω'_{ij}. Note that because α'', β'', and γ'' only depend on β, β', and $\alpha - \alpha'$, the rotations Ω_{ij} and Ω'_{ij} depend on α, β, α', and β' and not on γ or γ'; the same is true for θ on account of (4.27). Hence, the interactions of the charged bosons [(4.9), (4.14), and (4.17)] and of the neutral bosons [(4.18)–(4.21)] are completely specified by the four angles α, β, α', and β'. In every term γ and γ' appear only in the phases of q'_1 and q'_2, which are unobservable. We leave the relationship between the three obervables angles of Ω_{ij} and the angles α, β, α', and β' in the implicit form contained in (4.13), (4.16), (4.25), and (4.29), and note that the Cabibbo angle satisfies

$$\cos 2\theta_C = \cos\beta\cos\beta' + \sin\beta\sin\beta' \cos(\alpha - \alpha') . \quad (4.30)$$

In summarizing this section we remark that by a general transformation in $\mathcal{G}$ the gauge fields $\{W\}$ are transformed into the physical boson fields $\{\tilde{W}\}$. A Cabibbo angle appears in the usual $\tilde{W}_\pm$ coupling, (4.9), to the physical quarks. Three other angles describe the couplings of the other bosons to the same quark fields. Mixing of quarks in their mass matrix is completely circumvented. We have identified the four rotation angles of $\mathcal{G}$, which will be related to the Higgs fields in the next section. Of the four associated mixing angles, one is the Cabibbo angle θ_C, while the other three will be given phenomenological interpretation in Sec. VIII.

V. HIGGS FIELDS AND THE CABIBBO ANGLE

Thus far we have related a general transformation in $\mathcal{G}$ that diagonalizes the boson mass matrix to observable angles. Since the mass matrix depends entirely on the Higgs scalars, it is natural to investigate next how the scalars transform under $\mathcal{G}$ with the aim of obtaining a direct connection between the rotated Higgs fields and the Cabibbo angle.

As described in paper I the SU(4) × U(1) symmetry is spontaneously broken down to a residual SU(2) × U(1) symmetry by the introduction of two Higgs multiplets, ϕ_i and ψ_i, in adjoint representations. Since they transform in the same way as do the gauge bosons, they can be described in the standard basis analogously to (3.5),

$$\{\phi\} = \{\phi^\pm_\alpha, \phi_R, \Phi_i, \Phi'_j\} , \quad (5.1)$$

$$\{\psi\} = \{\psi^\pm_\alpha, \psi_R, \Psi_i, \Psi'_j\} , \quad (5.2)$$

where

$$\Phi_i = \begin{bmatrix} \phi_6 \\ \phi_7 \\ \phi_S \end{bmatrix} , \quad \Phi'_j = \begin{bmatrix} \phi_9 \\ \phi_{10} \\ \phi_T \end{bmatrix} , \quad (5.3)$$

and

$$\Psi_i = \begin{bmatrix} \psi_6 \\ \psi_7 \\ \psi_S \end{bmatrix} , \quad \Psi'_j = \begin{bmatrix} \psi_9 \\ \psi_{10} \\ \psi_T \end{bmatrix} . \quad (5.4)$$

All the Higgs fields above are normalized. The charged components cannot have nonzero vacuum expectation values, so

$$\langle\phi^\pm_\alpha\rangle = \langle\psi^\pm_\alpha\rangle = 0 . \quad (5.5)$$

Furthermore, because these Higgs scalars are in the adjoint representation, the way in which they transform can be obtained from (2.14),

$$[G^\pm_\alpha, \phi^\mp_\beta] = \pm\, \delta_{\alpha\beta}\sqrt{2}\,\phi_R - i\phi_{\alpha\beta} , \quad (5.6)$$

where $\phi_{\alpha\beta}$ is related to Φ_i and Φ'_i in the same way as $L_{\alpha\beta}$ is related to $\mathcal{G}_i$ and $\mathcal{G}'_i$, that is,

$$\phi_{0i} = -\Phi_i + \Phi'_i ,$$
$$\phi_{ij} = -\epsilon_{ijk}(\Phi_k + \Phi'_k) . \quad (5.7)$$

Equations similar to (5.6) and (5.7), of course, also hold for the other adjoint Higgs multiplet ψ. Now if ϕ and ψ are to cause breaking down to a residual SU(2) × U(1) symmetry which includes a charge changing generator, then (5.6) implies

$$\langle\phi_R\rangle = 0, \quad \langle\psi_R\rangle = 0 . \quad (5.8)$$

It follows from (5.5) and (5.8) that the nonvanishing elements of $\langle\phi\rangle$ and $\langle\psi\rangle$ span a subspace invariant under $\mathcal{G}$. Accordingly, we introduce

$$\eta_i = \langle\Phi_i\rangle, \quad \eta'_i = \langle\Phi'_i\rangle ,$$
$$\zeta_i = \langle\Psi_i\rangle, \quad \zeta'_i = \langle\Psi'_i\rangle . \quad (5.9)$$

On account of (3.14) and (3.15), they transform under $\mathcal{G}$ as

$$\eta \to R(A)\eta, \quad \eta' \to R(B)\eta' ,$$
$$\zeta \to R(A)\zeta, \quad \zeta' \to R(B)\zeta' . \tag{5.10}$$

Thus η and η' (and similarly ζ and ζ') are three-dimensional vectors which rotate separately under $\Lambda(A,B)\in\mathcal{G}$ by the actions of $R(A)$ and $R(B)$, respectively, specified in (3.12). It is this feature which allows us to describe a general rotation in $\mathcal{G}$ by specifying the rotated η, η', ζ, and ζ' relative to some reference vectors.

In the canonical basis the mass term of the gauge bosons has the structure (apart from a coupling constant)

$$W_i^\dagger M_{ij} W_j = (f_{kil} W_i \langle\phi_l\rangle)^\dagger (f_{kjm} W_j \langle\phi_m\rangle) + (\phi \to \psi) . \tag{5.11}$$

For our present purpose we restrict our attention to the charged bosons and to the Higgs in (5.9). Then in the standard basis the mass term is

$$C_\alpha^+ M_{\alpha\beta}(\eta,\eta',\zeta,\zeta') C_\beta^-, \quad \alpha = 0,\ldots,3 \tag{5.12}$$

where $M_{\alpha\beta}$ is the mass matrix.

In paper I we have chosen the expectation values of ϕ and ψ to be

$$\langle\phi_8\rangle = \sqrt{2}\,\langle\phi_{15}\rangle \neq 0, \quad \text{all other } \langle\phi_i\rangle = 0 ,$$
$$\langle\psi_6\rangle = \langle\psi_9\rangle \neq 0, \quad \text{all other } \langle\psi_i\rangle = 0 , \tag{5.13}$$

which in the standard basis corresponds to

$$\eta = \eta' = \bar\eta, \quad \text{where } \bar\eta \equiv \eta_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$
$$\zeta = \zeta' = \bar\zeta, \quad \text{where } \bar\zeta \equiv \zeta_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{5.14}$$

The notation $\bar\eta$ and $\bar\zeta$ is to refer to the particular set of expectation values indicated. With this choice $M_{\alpha\beta}$ is diagonal, and $W_\pm$ are the massless charged bosons. However, we can choose the expectation values differently. Under a general SU(2) $\times$SU(2) rotation (5.6) transforms into

$$[\tilde G_\alpha^\pm, \tilde\phi_\beta^\mp] = \pm\delta_{\alpha\beta}\sqrt{2}\,\phi_R - i\tilde\phi_{\alpha\beta} \tag{5.15}$$

in an obvious notation. In order that $\tilde C_0^\pm$ be identified with the massless $\tilde W_\pm$ of the new basis we require $\langle\tilde\phi_{0i}\rangle = 0$, so we identify the rotated vectors with $\bar\eta$ and $\bar\zeta$ this time, i.e.,

$$R(A)\eta = R(B)\eta' = \bar\eta ,$$
$$R(A)\zeta = R(B)\zeta' = \bar\zeta . \tag{5.16}$$

Thus we assign to $\eta, \eta', \zeta, \zeta'$ new specific values $\hat\eta, \hat\eta', \hat\zeta, \hat\zeta'$, i.e.,

$$\eta = \hat\eta = R(A^\dagger)\bar\eta, \quad \eta' = \hat\eta' = R(B^\dagger)\bar\eta ,$$
$$\zeta = \hat\zeta = R(A^\dagger)\bar\zeta, \quad \zeta' = \hat\zeta' = R(B^\dagger)\bar\zeta \tag{5.17}$$

to replace (5.14). It follows from the invariance of $\mathcal{L}_{\text{int}}$ under $\mathcal{G}$ that

$$C_\alpha^+ M_{\alpha\beta}(\hat\eta,\hat\eta',\hat\zeta,\hat\zeta') C_\beta^- = \tilde C_\alpha^+ M_{\alpha\beta}(\bar\eta,\bar\zeta)\tilde C_\beta^- , \tag{5.18}$$

where $\tilde C_\alpha^\pm$ are exactly as defined in (3.10). Since $\bar\eta$ and $\bar\zeta$ have the form given in (5.14), we recognize $\tilde C_\alpha^\pm$ to be the new eigenstates of the mass matrix, with $\tilde C_0^\pm$ ($=\tilde W_\pm$) being massless.

The kernel of our approach in this paper is to be seen in the mass and interaction terms of the gauge bosons

$$\mathcal{L} = C_\alpha^+ M_{\alpha\beta}(\hat\eta,\hat\eta',\hat\zeta,\hat\zeta') C_\beta^- + C_\alpha^+ J_\alpha^+ + \cdots$$
$$= \tilde C_\alpha^+ M_{\alpha\beta}(\bar\eta,\bar\zeta)\tilde C_\beta^- + \tilde C_\alpha^+ \tilde J_\alpha^+ + \cdots . \tag{5.19}$$

The mismatch mentioned earlier is now clearly exhibited in the fact that the diagonal gauge bosons couple to the rotated currents $\tilde J_\alpha^+$ of (4.2). If we substitute (5.17) into the first form of (5.19), the Lagrangian becomes one explicitly specified by the angles of A and B instead of the Cabibbo and other angles.

We now show how the Cabibbo angle can be extracted directly from $\hat\eta$ and $\hat\eta'$. To that end it is convenient to consider a three-dimensional space in which $\hat\eta, \hat\eta', \hat\zeta, \hat\zeta'$ are all vectors; that is, let the (6, 9), (7, 10), and (S, T) components of the Higgs fields be projections along the x, y, and z axes of this space, respectively. Then from (5.17) we have

$$\hat\eta' = R(B^\dagger A)\hat\eta ,$$
$$\hat\zeta' = R(B^\dagger A)\hat\zeta . \tag{5.20}$$

We recall our conclusion at the end of the preceding section that γ and γ' appear only in the unobservable phases of the quark fields and do not enter into the relationship between θ_C and the other rotation angles. They are therefore freely adjustable. For our present purpose of finding a geometrically transparent way of relating the vectors in (5.20), it is advantageous to set

$$\gamma = -\gamma'' \quad \text{and} \quad \gamma' = -\alpha'' \tag{5.21}$$

so that one gets, according to (4.28),

$$\phi = \phi' = 0 \tag{5.22}$$

together with the implication that the quark fields no longer undergo any phase transformation in (4.7) and (4.8). Then (4.26) attains the simple form

$$B^\dagger A = Y(\beta'') \tag{5.23}$$

Since (4.10) and (4.27) give, in general,

$$\theta_C = \beta''/2 , \tag{5.24}$$

we obtain from (5.20) the result

$$\hat\eta' = R_y(-2\theta_C)\hat\eta, \quad \hat\zeta' = R_y(-2\theta_C)\hat\zeta \tag{5.25}$$

with the obvious notation that R_y is the three-dimensional rotation matrix around the y axis.[4] This result indicates that with an appropriate choice of γ and γ' the vectors $\hat{\eta}$, $\hat{\eta}'$, $\hat{\xi}$, and $\hat{\xi}'$ can collectively be oriented in such a way that a pure rotation around the y axis by $-2\theta_C$ takes $\hat{\eta}$ to $\hat{\eta}'$ and $\hat{\xi}$ to $\hat{\xi}'$.

Particular simplicity occurs in the case $\alpha=\alpha'$, for in that case (4.25) implies

$$2\theta_C=\beta''=\beta-\beta',\quad \alpha''=\gamma''=0\,, \tag{5.26}$$

which in turn implies through (5.21) that $\gamma=\gamma'=0$. It then follows from (5.17) that

$$\begin{aligned}\hat{\eta}&=\eta_0R_y(\beta)\hat{u}_z,\quad \hat{\eta}'=\eta_0R_y(\beta')\hat{u}_z,\\ \hat{\xi}&=\zeta_0R_y(\beta)R_z(\alpha)\hat{u}_x,\quad \hat{\xi}'=\zeta_0R_y(\beta')R_z(\alpha)\hat{u}_x.\end{aligned} \tag{5.27}$$

Since $\hat{\eta}$ and $\hat{\eta}'$ are now in the x-z plane, $2\theta_C$ is therefore the angle between them. This is not true for $\hat{\xi}$ and $\hat{\xi}'$ if $\alpha=\alpha'\neq 0$. Their relationship is shown in Fig. 1.

In the special case where $\alpha=\alpha'=0$, then all four vectors are in the x-z plane, and the angle between $\hat{\xi}$ and $\hat{\xi}'$ is also $2\theta_C$ (see Fig. 2). Since now $A=Y(\beta)$ and $B=Y(\beta')$, we have $\Omega=R_y(-\beta)$ and $\Omega'=R_y(-\beta')$. With $\theta_C=(\beta-\beta')/2$, either β or β' is the only other observable angle. In Sec. VIII we shall examine the phenomenology of this special case for the purpose of illustrating the physical implications of our present approach.

In paper I, we parametrized the Higgs fields in a different fashion from the one we have just used here, and so for completeness we shall now derive the relationship between the formalisms. Specifically, in paper I we set

$$\langle\phi_{ab}\rangle=\sum_{i=1}^{15}\lambda^i_{ab}\langle\phi_i\rangle = U_\phi\begin{bmatrix}\frac{1}{\sqrt2}\langle\phi_R\rangle+\eta_0' & 0 & 0 & 0\\ 0 & -\frac{1}{\sqrt2}\langle\phi_R\rangle+\eta_0 & 0 & 0\\ 0 & 0 & -\frac{1}{\sqrt2}\langle\phi_R\rangle-\eta_0 & 0\\ 0 & 0 & 0 & \frac{1}{\sqrt2}\langle\phi_R\rangle-\eta_0'\end{bmatrix}U_\phi\,, \tag{5.28}$$

$$\langle\psi_{ab}\rangle=U_\psi\begin{bmatrix}\frac{1}{\sqrt2}\langle\psi_R\rangle+\zeta_0' & 0 & 0 & 0\\ 0 & -\frac{1}{\sqrt2}\langle\psi_R\rangle+\zeta_0 & 0 & 0\\ 0 & 0 & -\frac{1}{\sqrt2}\langle\psi_R\rangle-\zeta_0 & 0\\ 0 & 0 & 0 & \frac{1}{\sqrt2}\langle\psi_R\rangle-\zeta_0'\end{bmatrix}U_\psi\,, \tag{5.29}$$

where

$$U_\phi=\begin{bmatrix}\cos\frac12\beta_1' & 0 & 0 & \sin\frac12\beta_1'e^{-i\alpha_1'}\\ 0 & \cos\frac12\beta_1 & \sin\frac12\beta_1e^{-i\alpha_1} & 0\\ 0 & \sin\frac12\beta_1e^{i\alpha_1} & -\cos\frac12\beta_1 & 0\\ \sin\frac12\beta_1'e^{i\alpha_1'} & 0 & 0 & -\cos\frac12\beta_1'\end{bmatrix},$$
$$U_\psi=\begin{bmatrix}\cos\frac12\beta_2' & 0 & 0 & \sin\frac12\beta_2'e^{-i\alpha_2'}\\ 0 & \cos\frac12\beta_2 & \sin\frac12\beta_2e^{-i\alpha_2} & 0\\ 0 & \sin\frac12\beta_2e^{i\alpha_2} & -\cos\frac12\beta_2 & 0\\ \sin\frac12\beta_2'e^{i\alpha_2'} & 0 & 0 & -\cos\frac12\beta_2'\end{bmatrix}. \tag{5.30}$$

In general, the above structures for $\langle\phi_{ab}\rangle$ and $\langle\psi_{ab}\rangle$ break SU(4)×U(1) down beyond SU(2)×U(1), and so we shall discuss this more general case first. We replace $\vec{\eta}$ and $\vec{\xi}$ by a nonorthogonal set of vectors of arbitrary lengths so that Eqs. (5.17) are replaced by

$$\eta=\eta_0 R(A^\dagger)\begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \eta'=\eta_0' R(B^\dagger)\begin{pmatrix}0\\0\\1\end{pmatrix},$$
$$\zeta=\zeta_0 R(A^\dagger)\begin{bmatrix}\sin\lambda\cos\mu\\ \sin\lambda\sin\mu\\ \cos\lambda\end{bmatrix}, \quad \zeta'=\zeta_0' R(B^\dagger)\begin{bmatrix}\sin\lambda'\cos\mu'\\ \sin\lambda'\sin\mu'\\ \cos\lambda'\end{bmatrix}. \tag{5.31}$$

For a given $A(\alpha,\beta,\gamma)$ we note that

$$R_{ij}(A)=\begin{bmatrix}\cos\alpha\cos\beta\cos\gamma+\sin\alpha\sin\gamma, & -\cos\alpha\cos\beta\sin\gamma+\sin\alpha\cos\gamma, & -\cos\alpha\sin\beta\\ -\sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\gamma, & \sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma, & \sin\alpha\sin\beta\\ \sin\beta\cos\gamma, & -\sin\beta\sin\gamma, & \cos\beta\end{bmatrix}. \tag{5.32}$$

Consequently by direct comparison we find that

$$\beta_1=\beta, \quad \alpha_1=-\gamma$$
$$\cos\beta_1\cos\beta_2+\sin\beta_1\sin\beta_2\cos(\alpha_1-\alpha_2)=\cos\lambda,$$
$$\cos\beta_1\sin\beta_2\cos(\alpha_1-\alpha_2)-\sin\beta_1\cos\beta_2=\sin\lambda\cos(\mu+\alpha),$$
$$\sin\beta_2\sin(\alpha_1-\alpha_2)=-\sin\lambda\sin(\mu+\alpha), \tag{5.33}$$

with analogous relations for the primed quantities. Thus we can replace the eight degrees of freedom $\beta_1,\beta_2,\alpha_1,\alpha_2,\beta_1',\beta_2',\alpha_1',\alpha_2'$ by $\beta,\gamma,\lambda,\mu+\alpha,\beta',\gamma',\lambda',\mu'+\alpha'$. (The parameters μ and α, and also μ' and α', are not separately observable since there are two consecutive rotations around the z axis in the definitions of ζ and ζ'. We shall therefore absorb μ,μ' in the definitions of α,α'.) Now we have previously noted that the phases of the quarks are unobservable. Thus we can eliminate α_1 and α_1' and we are left with six degrees of freedom, β_1, β_2, $\alpha_1-\alpha_2$, β_1', β_2', and $\alpha_1'-\alpha_2'$ in general.

If we now return to the specific situation in which the structure of the adjoint potential breaks SU(4)×U(1) down only as far as SU(2)×U(1), we then recover (5.17) because of the constraints $\lambda=\lambda'=\pi/2$. So finally only four of the angles of $\langle\phi_{ab}\rangle$ and $\langle\psi_{ab}\rangle$ are observable, in agreement with the earlier analysis of this section. Using Eqs. (4.25), we can now extract out the physically relevant angles, for instance

$$\cos 2\theta_C=\cos\beta_1\cos\beta_1'+\cos\beta_2\cos\beta_2' +\sin\beta_1\sin\beta_1'\sin\beta_2\sin\beta_2' \times\sin(\alpha_1-\alpha_2)\sin(\alpha_1'-\alpha_2') \tag{5.34}$$

(with the angles satisfying $\cos\lambda=\cos\lambda'=0$).

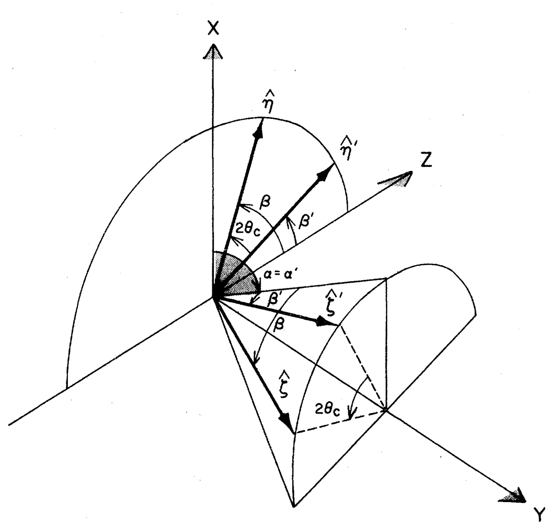

FIG. 1. Orientations of the vectors $\hat\eta$, $\hat\eta'$, $\hat\xi$, and $\hat\xi'$ for the case $\alpha=\alpha'\neq 0$, $\beta\neq 0$, $\beta'\neq 0$.

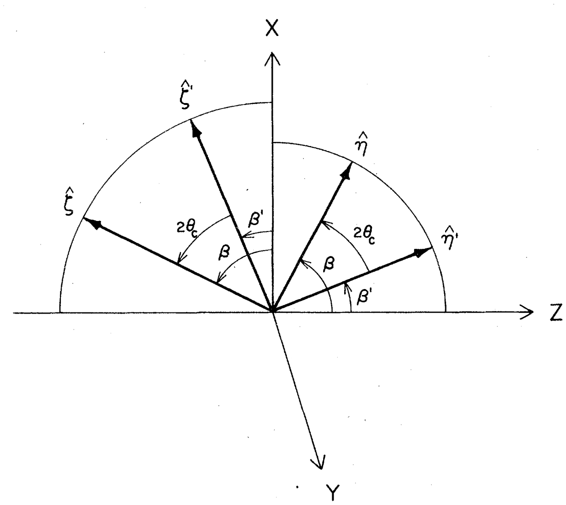

FIG. 2. Orientations of the vectors $\hat\eta$, $\hat\eta'$, $\hat\xi$, and $\hat\xi'$ for the case $\alpha=\alpha'=0$, $\beta\neq 0$, $\beta'\neq 0$.

We conclude this section by remarking again that we have presented a way to relate the observable angles such as the Cabibbo angle to the orientations of the Higgs multiplets. In Sec. VII we will show how these orientations are themselves determined by the full Higgs potential.

VI. *CP* NONINVARIANCE

In paper II we considered CP nonconservation as a consequence of W_6 and W_7 mixing and gave phenomenological bounds on the parameters of the theory. It was found that there would be no CP-violating effects if there were no Cabibbo angle. The discussion was given in the framework where the Cabibbo angle is introduced through mixing in the quark mass matrix, as considered in paper I. We now give a unified approach to Cabibbo mixing and CP noninvariance through mixing of the gauge bosons. As we shall presently see, the substance of this unification has already been developed in the preceding section. We need only identify here the CP-nonconserving part of the transformations in $\mathcal{G}$.

We have seen in Sec. III that under a general rotation $\Lambda(A,B)$ in $\mathcal{G}$ the interaction terms transform as

$$C_0^+J_0^+=\tilde{W}_+\bar{q}_2B^\dagger Aq_1\,, \tag{6.1}$$

$$C_i^+J_i^+=\tilde{C}_i^+\bar{q}_2B^\dagger\tau_iAq_1\,, \tag{6.2}$$

$$S_iM_i=\tilde{S}_i\bar{q}_1A^\dagger\sigma_iAq_1\,, \tag{6.3}$$

$$T_iN_i=\tilde{T}_i\bar{q}_2B^\dagger\sigma_iBq_2\,. \tag{6.4}$$

As explained in paper II, CP violation only occurs if it is impossible to remove a complex phase from all the terms (6.1) to (6.4) simultaneously. Thus if both A and B are diagonal the theory is CP conserving, as can be seen by defining

$$\begin{aligned} q_1' &= Aq_1 = Z(\alpha)q_1\,,\\ q_2' &= Bq_2 = Z(\alpha')q_2\,. \end{aligned} \tag{6.5}$$

It is also evident from (6.1) that in this case there is no Cabibbo angle either.

Consider next the case where A is not diagonal and B still diagonal, e.g.,

$$A=Z(\alpha)Y(\beta),\quad B=Z(\alpha')\,. \tag{6.6}$$

Let us now redefine the phases of the quark fields as

$$\begin{aligned} q_1'' &= q_1\,,\\ q_2'' &= Z(\alpha'-\alpha)q_2\,. \end{aligned} \tag{6.7}$$

We then obtain

$$C_0^+J_0^+=\tilde{W}_+[\bar{q}_2''Y(\beta)q_1'']\,, \tag{6.8}$$

$$C_i^+J_i^+=\tilde{C}_i^+R_z(\alpha)_{ij}[\bar{q}_2''\tau_iY(\beta)q_1'']\,, \tag{6.9}$$

$$S_iM_i=\tilde{S}_iR_z(\alpha)_{ij}[\bar{q}_1''Y^\dagger(\beta)\sigma_jY(\beta)q_1'']\,, \tag{6.10}$$

$$T_iN_i=\tilde{T}_iR_z(\alpha)_{ij}(\bar{q}_2''\sigma_jq_2'')\,. \tag{6.11}$$

Clearly, (6.8) involves the Cabibbo current with $\theta_C=\beta/2$. Defining the phases of q_1'' and q_2'' to be such that $\tilde{W}_\pm$ do not lead to CP-nonconserving transitions, our theory will only possess CP violation if there is a rotation around the z axis in the couplings to the other bosons, since this rotation leads to transitions between currents of opposite CP properties. Thus the interaction in (6.10) leads, for $\alpha\neq 0$, to CP nonconservation in the K-meson system, while (6.11) leads to CP nonconservation in the D-meson system.

In the general case

$$\begin{aligned} A&=Z(\alpha)Y(\beta)Z(\gamma'')\,,\\ B&=Z(\alpha')Y(\beta')Z(\alpha'')\,, \end{aligned} \tag{6.12}$$

where (5.21) has been used, there is no redefinition of the quark phases. It follows from (4.9), (4.14), (4.18), and (4.19) that the interactions are

$$C_0^+J_0^+=\tilde{W}_+\bar{q}_2Y(2\theta_C)q_1\,, \tag{6.13}$$

$$C_i^+J_i^+=\tilde{C}_i^+R_{ij}(A)[\bar{q}_2Y(2\theta_C)\tau_jq_1]\,, \tag{6.14}$$

$$S_iM_i=\tilde{S}_iR_{ij}(A)(\bar{q}_1\sigma_jq_1)\,, \tag{6.15}$$

$$T_iN_i=\tilde{T}_iR_{ij}(A)[\bar{q}_2Y(2\theta_C)\sigma_jY(-2\theta_C)q_2]\,. \tag{6.16}$$

If $\alpha=\alpha'=0$, A would become a pure rotation around the y axis, and there would be no CP violation. Hence, we conclude that though the nonvanishing of α or α' does not in itself guarantee CP violation, their vanishing guarantees CP invariance.

One final case to note is that if $A=B=Z(\alpha)Y(\beta)$ so that $\beta''=2\theta_C=0$, there are still nontrivial rotations on the currents which couple to $\tilde{S}_i$ and $\tilde{T}_i$, causing CP violation. Thus a necessary condition for CP nonconservation is not $\theta_C\neq 0$ but rather the nonvanishing of either β or β'.

From the above discussion it is clear that the sufficient condition for CP noninvariance is the simultaneous nonvanishing of both α and β or alternatively of both α' and β'.

VII. HIGGS POTENTIAL AND MIXING ANGLES

In the preceding sections we have described how the nonconservation of the strangeness and of CP can be obtained in weak interactions by mixing the gauge bosons and can be specified by the orientations of the adjoint Higgs bosons. We explain in this section how the orientations of the Higgs bosons are determined by the potential.

In paper I we introduced the potential $V_1(\phi,\psi)$ of the adjoints ϕ and ψ that breaks $SU(4)\times U(1)$ down to $SU(2)\times U(1)$, viz. (in the tensor notation $\phi=\phi_i\lambda_i$),

$$V_1(\phi,\psi) = -\mu_1^2\mathrm{Tr}(\phi^2) + \lambda_1[\mathrm{Tr}(\phi^2)]^2 + \lambda_2\mathrm{Tr}(\phi^4) - \mu_2^2\mathrm{Tr}(\psi^2) + \lambda_3[\mathrm{Tr}(\psi^2)]^2 + \lambda_4\mathrm{Tr}(\psi^4) + g_1\mathrm{Tr}(\phi^2\psi^2) + g_2\mathrm{Tr}(\phi^2)\mathrm{Tr}(\psi^2) + g_3[\mathrm{Tr}(\phi\psi)]^2 + g_4\mathrm{Tr}(\phi\psi\phi\psi), \quad (7.1)$$

and noted that the parameters in V_1 can be chosen so that it is minimized by $\vec{\eta}\cdot\vec{\zeta} = 0$ and $\vec{\eta}'\cdot\vec{\zeta}' = 0$. The Cabibbo and other angles, being related to the relative orientations of $\vec{\eta}$ and $\vec{\eta}'$ (and $\vec{\zeta}$ and $\vec{\zeta}'$), are allowed but not specified by V_1. In particular, if $\vec{\eta}$, etc. are given as in (5.14), i.e.,

$$\vec{\eta} = \vec{\eta}' = \bar{\eta}, \quad \vec{\zeta} = \vec{\zeta}' = \bar{\zeta}, \quad (7.2)$$

then there is no Cabibbo angle and no CP violation.

In paper I we also introduced the potential $V_2(\chi)$ of the four fundamentals χ^s $(s = a, b, c, d)$ which breaks SU(2)×U(1), viz.,

$$V_2(\chi) = \sum_s [-\mu_s^2\chi^{s\dagger}\chi^s + \lambda_{ss}^{ss}(\chi^{s\dagger}\chi^s)^2] + \sum_{s,t}[\lambda_{tt}^{ss}(\chi^{s\dagger}\chi^s)(\chi^{t\dagger}\chi^t) + \lambda_{ts}^{st}(\chi^{s\dagger}\chi^t)(\chi^{t\dagger}\chi^s)] + \kappa\,\mathrm{Re}[\det(\chi^a\chi^b\chi^c\chi^d)], \quad (7.3)$$

and noted that the parameters in V_2 can be chosen so that it is minimized by

$$\sum_i \langle\chi_i^s\rangle^\dagger\langle\chi_i^{s'}\rangle = \delta_{ss'}\chi_0^{s2}. \quad (7.4)$$

Here χ_0^s is the normalization of the expectation value of each χ^s. To limit the couplings of the Higgs fields to the fermions, we introduce two separate sets of discrete symmetries, viz.,

$$\{\chi^b \to e^{i(\pi/4)}\chi^b, \chi^c \to e^{-i(\pi/4)}\chi^c, d_R \to e^{-i(\pi/4)}d_R, s_R \to e^{i(\pi/4)}s_R, e_R \to e^{-i(\pi/4)}e_R, \mu_R \to e^{i(\pi/4)}\mu_R\} \quad (7.5)$$

and

$$\{\chi^a \to e^{i\pi/4}\chi^a, \chi^d \to e^{-i\pi/4}\chi^d, u_R \to e^{-i\pi/4}u_R, c_R \to e^{i\pi/4}c_R\}. \quad (7.6)$$

These symmetries have in fact already been imposed on $V_2(\chi)$ to give (7.3). In the presence of these symmetries the interaction of the fundamentals with the fermions is given by

$$\mathcal{L}_{f\chi} = g_{u,a}\bar{u}_R\chi^{a\dagger}q_L + g_{d,b}\bar{d}_R\chi^{b\dagger}q_L + g_{s,c}\bar{s}_R\chi^{c\dagger}q_L + g_{c,d}\bar{c}_R\chi^{d\dagger}q_L + g_{e,b}\bar{e}_R\chi^{b\dagger}l_L + g_{\mu,c}\bar{\mu}_R\chi^{c\dagger}l_L + \mathrm{H.c.}, \quad (7.7)$$

where

$$q_L = \begin{bmatrix} u \\ d \\ s \\ c \end{bmatrix}_L, \quad l_L = \begin{bmatrix} \nu_e \\ e \\ \mu \\ \nu_\mu \end{bmatrix}_L. \quad (7.8)$$

Our motivation for introducing the discrete symmetries is to produce a completely flavor-conserving theory. While we have built our weak-interaction theory by expressly identifying the weak- and strong-interaction flavor groups, we have only made the identification for the left-handed fermions. Our discrete symmetries prevent SU(4) ×U(1)-conserving terms such as $\bar{s}_R\chi^{b\dagger}q_L$ to appear in $\mathcal{L}_{f\chi}$ while allowing terms such as $\bar{d}_R\chi^{b\dagger}q_L$. Thus we can now distinguish between different right-handed quarks just as we could between the left-handed ones, and the Lagrangian is completely flavor-conserving. Since the elimination of these terms also eliminates the conventional method for introducing the Cabibbo angle, we are now in a position to take advantage of the approach developed in this paper, namely mixing of the gauge bosons.

It can be shown that in the presence of (7.4) the interaction $\mathcal{L}_{f\chi}$ does not lead to any flavor changing processes mediated by the Higgs scalars, i.e., processes in which a Higgs scalar is emitted and readsorbed. (The discussion of Appendix C in paper I explains as an example why the Higgs scalars do not contribute to the K_L-K_S mass difference.) The Higgs scalars are flavor-conserving because no matter what basis we use to satisfy (7.4), we have to diagonalize the mass eigenstates of $V_2(\chi)$ accordingly, a diagonalization which is exactly duplicated in $\mathcal{L}_{f\chi}$. The most convenient basis to use for (7.4) is

$$\langle\chi^a\rangle = \chi_0^a\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \langle\chi^b\rangle = \chi_0^b\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \langle\chi^c\rangle = \chi_0^c\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \langle\chi^d\rangle = \chi_0^d\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \quad (7.9)$$

In this basis the quark and lepton mass matrices are now diagonal in the (u, d, s, c) and (ν_e, e, μ, ν_μ) bases, respectively. (In other orthogonal bases there would be fermion mass mixing though no flavor-changing processes, a point we will discuss further below.)

In order to give a meaning to the relative orientations of the adjoint and fundamental Higgs multiplets (so that the orientations of the gauge-boson eigenstates relative to the fermion eigenstates are determined), we introduce extra ϕ, ψ, χ cross terms into the potential. The most general renormalizable SU(4)× U(1)-invariant term which possesses the discrete symmetries introduced above is

$$V_3(\phi,\psi,\chi)=\sum_{s=a,b,c,d}\{p_s\chi^{s\dagger}\phi^2\chi^s+q_s\chi^{s\dagger}\psi^2\chi^s+r_s\chi^{s\dagger}\chi^s\mathrm{Tr}(\phi^2)+t_s\chi^{s\dagger}\chi^s\mathrm{Tr}(\psi^2)+f_s\chi^{s\dagger}\phi\chi^s+g_s\chi^{s\dagger}\psi\chi^s$$
$$+k_s\chi^{s\dagger}\tfrac{1}{2}i[\phi,\psi]\chi^s+l_s\chi^{s\dagger}\tfrac{1}{2}\{\phi,\psi\}\chi^s+m_s\chi^{s\dagger}\chi^s\mathrm{Tr}(\phi\psi)\}, \tag{7.10}$$

so that the full potential is

$$V_{\rm tot}=V_1(\phi,\psi)+V_2(\chi)+V_3(\phi,\psi,\chi). \tag{7.11}$$

Note that V_3 is not reflection invariant under separate $\phi\to-\phi$, $\psi\to-\psi$. While V_3 is the most general potential that we could write, it is not yet CP conserving. Under CP

$$\phi_{ab}\to\eta(\phi)\phi^*_{ab},\quad \psi_{ab}\to\eta(\psi)\psi^*_{ab}. \tag{7.12}$$

We choose the CP phases of ϕ_{ab} and ψ_{ab} so that $\eta(\phi)=-\eta(\psi)=1$. With this choice we set $g_s=l_s=m_s=0$, so that the resulting V_3 is CP conserving. Consequently at this stage the complete theory (i.e., gauge-boson, fermion, and Higgs-boson sectors) is both flavor-conserving and CP conserving. Thus all violations of these symmetries can only be introduced through spontaneous breakdown of the potential $V_{\rm tot}$. We shall show below the in the presence of the V_3 term it is impossible to maintain both Eqs. (7.2) and (7.9) and that there is both Cabibbo mixing and CP violation in the S matrix.

In order to proceed and minimize $V_{\rm tot}$ we must introduce a basis which is not a minimum of V_1+V_2. For χ^s we set

$$\langle\chi^a\rangle=\chi_0^a\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\quad \langle\chi^b\rangle=\chi_0^b\begin{bmatrix}0\\1\\0\\0\end{bmatrix},$$
$$\langle\chi^c\rangle=\chi_0^c e^{i\psi_1}\begin{bmatrix}0\\ \sin\theta_1 e^{i\phi_1}\\ \cos\theta_1\\0\end{bmatrix},\quad \langle\chi^d\rangle=\chi_0^d e^{i\psi_2}\begin{bmatrix}\sin\theta_2 e^{i\phi_2}\\0\\0\\ \cos\theta_2\end{bmatrix}, \tag{7.13}$$

while for ϕ and ψ we use the bases given in (5.28) and (5.29), respectively. Any other basis can be reached by applying the rotations $A(\alpha,\beta,\gamma)$ and $B(\alpha',\beta',\gamma')$ to ϕ, ψ, and χ. In this basis V_1 is a function of η_0, η_0', $\langle\phi_R\rangle$, $\cos\lambda$, ζ_0, ζ_0', $\langle\psi_R\rangle$, and $\cos\lambda'$, while V_2 is a function of χ_0^a, χ_0^b, χ_0^c, χ_0^d, θ_1, θ_2, and $\psi_1+\psi_2$. Both of these functions were given explicitly in paper I. For V_3 we find

$$\begin{aligned}
V_3(\langle\phi\rangle,\langle\psi\rangle,\langle\chi\rangle)={}&p_a\chi_0^{a2}(\tfrac{1}{2}\langle\phi_R\rangle^2+\eta_0'^2+\sqrt2\langle\phi_R\rangle\eta_0'\cos\beta_1')+p_b\chi_0^{b2}(\tfrac{1}{2}\langle\phi_R\rangle^2+\eta_0^2-\sqrt2\langle\phi_R\rangle\eta_0\cos\beta_1)\\
&+p_c\chi_0^{c2}[\tfrac{1}{2}\langle\phi_R\rangle^2+\eta_0^2+\sqrt2\langle\phi_R\rangle\eta_0\cos\beta_1\cos2\theta_1-\sqrt2\langle\phi_R\rangle\eta_0\sin\beta_1\sin2\theta_1\cos(\alpha_1+\phi_1)]\\
&+p_d\chi_0^{d2}[\tfrac{1}{2}\langle\phi_R\rangle^2+\eta_0'^2-\sqrt2\langle\phi_R\rangle\eta_0'\cos\beta_1'\cos2\theta_2+\sqrt2\langle\phi_R\rangle\eta_0'\sin\beta_1'\sin2\theta_2\cos(\alpha_1'+\phi_2)]\\
&+q_a\chi_0^{a2}(\tfrac{1}{2}\langle\psi_R\rangle^2+\zeta_0'^2+\sqrt2\langle\psi_R\rangle\zeta_0'\cos\beta_2')+q_b\chi_0^{b2}(\tfrac{1}{2}\langle\psi_R\rangle^2+\zeta_0^2-\sqrt2\langle\psi_R\rangle\zeta_0\cos\beta_2)\\
&+q_c\chi_0^{c2}[\tfrac{1}{2}\langle\psi_R\rangle^2+\zeta_0^2+\sqrt2\langle\psi_R\rangle\zeta_0\cos\beta_2\cos2\theta_1-\sqrt2\langle\psi_R\rangle\zeta_0\sin\beta_2\sin2\theta_1\cos(\alpha_2+\phi_1)]\\
&+q_d\chi_0^{d2}[\tfrac{1}{2}\langle\psi_R\rangle^2+\zeta_0'^2-\sqrt2\langle\psi_R\rangle\zeta_0'\cos\beta_2'\cos2\theta_2+\sqrt2\langle\psi_R\rangle\zeta_0'\sin\beta_2'\sin2\theta_2\cos(\alpha_2'+\phi_2)]\\
&+2(\eta_0^2+\eta_0'^2+\langle\phi_R\rangle^2)(r_a\chi_0^{a2}+r_b\chi_0^{b2}+r_c\chi_0^{c2}+r_d\chi_0^{d2})+2(\zeta_0^2+\zeta_0'^2+\langle\psi_R\rangle^2)(t_a\chi_0^{a2}+t_b\chi_0^{b2}+t_c\chi_0^{c2}+t_d\chi_0^{d2})\\
&+f_a\chi_0^{a2}\left(\frac{1}{\sqrt2}\langle\phi_R\rangle+\eta_0'\cos\beta_1'\right)+f_b\chi_0^{b2}\left(-\frac{1}{\sqrt2}\langle\phi_R\rangle+\eta_0\cos\beta_1\right)\\
&+f_c\chi_0^{c2}\left[-\frac{1}{\sqrt2}\langle\phi_R\rangle-\eta_0\cos\beta_1\cos2\theta_1+\eta_0\sin\beta_1\sin2\theta_1\cos(\alpha_1+\phi_1)\right]\\
&+f_d\chi_0^{d2}\left[\frac{1}{\sqrt2}\langle\phi_R\rangle-\eta_0'\cos\beta_1'\cos2\theta_2+\eta_0'\sin\beta_1'\sin2\theta_2\cos(\alpha_1'+\phi_2)\right]\\
&+k_a\chi_0^{a2}\eta_0'\zeta_0'\sin\beta_1'\sin\beta_2'\sin(\alpha_1'-\alpha_2')+k_b\chi_0^{b2}\eta_0\zeta_0\sin\beta_1\sin\beta_2\sin(\alpha_1-\alpha_2)\\
&-k_c\chi_0^{c2}\eta_0\zeta_0[\sin\beta_1\sin\beta_2\sin(\alpha_1-\alpha_2)\cos2\theta_1+\sin\beta_1\cos\beta_2\sin(\alpha_1+\phi_1)\sin2\theta_1\\
&\qquad-\sin\beta_2\cos\beta_1\sin(\alpha_2+\phi_1)\sin2\theta_1]\\
&-k_d\chi_0^{d2}\eta_0'\zeta_0'[\sin\beta_1'\sin\beta_2'\sin(\alpha_1'-\alpha_2')\cos2\theta_2+\sin\beta_1'\cos\beta_2'\sin(\alpha_1'+\phi_2)\sin2\theta_2\\
&\qquad-\sin\beta_2'\cos\beta_1'\sin(\alpha_2'+\phi_2)\sin2\theta_2].
\end{aligned} \tag{7.14}$$

Before minimizing $V_{\rm tot}$ we recall that our whole program is that SU(2)×U(1) should be a much better symmetry than SU(4)×U(1). Consequently we have chosen the mass scales of V_1 to be much greater than those of V_2 so that

$$\eta_0 \sim \eta_0' \sim \zeta_0 \sim \zeta_0' \gg \chi_0^a \sim \chi_0^b \sim \chi_0^c \sim \chi_0^d \ . \tag{7.15}$$

In order that V_3 will not spoil this mass pattern, we must choose its parameters so that

$$\text{all } p_s, q_s, r_s, t_s \leq O\left(\frac{\chi_0^{a^2}}{\eta_0\zeta_0}\right) \tag{7.16}$$

and

$$\text{all } k_s, f_s/\zeta_0 \leq O\left(\frac{\chi_0^{a^6}}{\eta_0{}^3\zeta_0{}^3}\right) \tag{7.17}$$

with the stronger condition of (7.17) being chosen in order to ultimately produce observable angles of $O(1)$. Thus η_0, η_0', ζ_0, ζ_0', χ_0^a, χ_0^b, χ_0^c, and χ_0^d retain their previous values to this level of approximation. If we now vary $V_{\rm tot}$ we find a stationary point in which

$$\begin{aligned}
&\theta_1 = \theta_2 = 0, \quad \psi_1 + \psi_2 = 0 \ ,\\
&\cos\lambda, \cos\lambda' \sim O\left(\frac{\chi_0^{a^8}}{\eta_0{}^4\zeta_0{}^4}\right) ,\\
&\langle\phi_R\rangle \sim O\left(\frac{\chi_0^{a^4}}{\eta_0{}^2\zeta_0}\right) , \quad \langle\psi_R\rangle \sim O\left(\frac{\chi_0^{a^4}}{\eta_0\zeta_0{}^2}\right)
\end{aligned} \tag{7.18}$$

(thus making ϕ_1 and ϕ_2 unobservable) while the remaining angles are $O(1)$. The relationships for them are somewhat untractable, but can be solved in the simplifying case where $f_s = 0$ (a case which corresponds to imposing invariance under the joint reflection $\phi \to -\phi$, $\psi \to -\psi$). Then we obtain

$$\begin{aligned}
&(P_1{}^2Q_2{}^2 - Q_1{}^2P_2{}^2)\cos^2\beta_1/P_1{}^2 = Q_2{}^2 - Q_1{}^2\frac{(\lambda_2Q_2{}^2 - \lambda_4P_2{}^2)^2}{(\lambda_2Q_1{}^2 - \lambda_4P_1{}^2)^2} + \frac{(\lambda_2Q_2{}^2 - \lambda_4P_2{}^2)^2}{(Q_1{}^2P_2{}^2 - Q_2{}^2P_1{}^2)^2}(4\eta_0\zeta_0)^2(Q_1{}^2K_2{}^2 - Q_2{}^2K_1{}^2) ,\\
&P_1(\lambda_4P_2{}^2 - \lambda_2Q_2{}^2)\cos\beta_1' = P_2(\lambda_2Q_1{}^2 - \lambda_4P_1{}^2)\cos\beta_1 ,\\
&\cos^2\beta_2 = \sin^2\beta_1 - \left[\frac{4\lambda_2\eta_0\zeta_0K_1\cos\beta_1}{P_1(\cos\beta_1P_1 + \cos\beta_1'P_2)}\right]^2 ,\\
&\cos\beta_2' = \frac{P_1Q_2\cos\beta_1'\cos\beta_2}{P_2Q_1\cos\beta_1} ,\\
&\cos(\alpha_1 - \alpha_2) = -\cot\beta_1\cot\beta_2 ,\\
&\cos(\alpha_1' - \alpha_2') = -\cot\beta_1'\cot\beta_2' ,
\end{aligned} \tag{7.19}$$

where

$$\begin{aligned}
&P_1 = \sqrt{2}\,(p_b\chi_0^{b^2} - p_c\chi_0^{c^2}) ,\\
&P_2 = \sqrt{2}(p_d\chi_0^{d^2} - p_a\chi_0^{a^2}) ,\\
&Q_1 = \sqrt{2}(q_b\chi_0^{b^2} - q_c\chi_0^{c^2}) ,\\
&Q_2 = \sqrt{2}(q_d\chi_0^{d^2} - q_a\chi_0^{a^2}) ,\\
&K_1 = k_b\chi_0^{b^2} - k_c\chi_0^{c^2} ,\\
&K_2 = k_a\chi_0^{a^2} - k_d\chi_0^{d^2} .
\end{aligned} \tag{7.20}$$

Because of the complexity of these relations, we have not investigated the stability conditions on p_s, q_s, and k_s which would ensure that the stationary point is a minimum. Nonetheless our calculation does illustrate how an angle such as the Cabibbo angle could emerge as a stability condition on a potential.

Equations (7.19) constitute the result that we have sought, relating the angles to the parameters of the potential. The Cabibbo and other observable angles are thereby also determined in turn, as discussed already in previous sections. We do not exhibit the relations here specifically, leaving them in an implicit form for the reader.

Of particular interest is the relation $\theta_1 = \theta_2 = 0$ given in (7.18). This condition restores the orthogonality of $\langle\chi^s\rangle$ and recovers (7.9). It is of interest, now that we have the solution, to make what is now an unobservable rotation of our basis of fundamentals and adjoints through arbitrary A and B. Since we never referred the $\langle\chi^s\rangle$ to any particular quark basis, we shall not rotate the quarks. In terms of a fixed set of quark states we will find that there is now Cabibbo mixing in the quark mass matrix, though, as we have already noted, no flavor-changing processes mediated by the Higgs scalars.[5] However, there will now be an attendant change in the amount of gauge-boson mixing to the same set of quark states so that the S matrix is left invariant. Thus there is no physical effect which is not already contained in the relations of (7.9) and a completely diagonal quark mass matrix.

We note that (7.19) fixes all of the angles of $\langle\phi\rangle$ and $\langle\psi\rangle$ that we have shown in Sec. V to be independently observable. Had we chosen the CP phases of ϕ and ψ to be equal (so that a CP-conserving theory would have $f_s, g_s, l_s, m_s \neq 0$, $k_s=0$ in V_3), we would not have been able to obtain nontrivial values for $\alpha_1-\alpha_2$ and $\alpha_1'-\alpha_2'$. Thus the structure of the potential is such that our theory only possesses an observable CP violation if the intrinsic CP phases of the two adjoints are opposite. With ϕ and ψ having opposite CP phases, we should replace (5.14) by

$$\langle\phi\rangle=\eta_0\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},\qquad \langle\psi\rangle=\zeta_0\begin{bmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{bmatrix}. \tag{7.21}$$

This new basis leaves the vacuum CP invariant. If we now couple the adjoints to the fundamentals, both $\langle\phi\rangle$ and $\langle\psi\rangle$ have to rotate. Thus $\langle\psi\rangle$ has to rotate into a configuration where CP is violated in the vacuum, while $\langle\phi\rangle$ has to rotate into a configuration where there is a Cabibbo angle. Thus once the parameters p_s, q_s, and k_s are nonzero (which must be the case since in a renormalizable theory any term which is not forbidden by the symmetry eventually appears as a counterterm), the presence of $V_3(\phi,\psi,\chi)$ in the Higgs potential requires the existence of a Cabibbo angle and of CP nonconservation if the SU(4)×U(1) symmetry is to be completely broken down to only a residual U(1) symmetry associated with the photon. Hence, in our theory the nonconservations of strangeness, CP, and muon number are inevitable.

VIII. PHENOMENOLOGICAL APPLICATIONS

In this section we present a few simple examples of the general theory that we have developed to illustrate our approach, and to study some phenomenological implications. Though we have the freedom to choose four independent mixing angles, we restrict ourselves at first to one mixing angle,[6] which we will take to be θ_C. The Higgs adjoints that minimize the potential are chosen to be

$$\hat\eta=R(A^\dagger)\bar\eta,\quad \hat\eta'=R(B^\dagger)\bar\eta,\qquad \hat\xi=R(A^\dagger)\bar\xi,\quad \hat\xi'=R(B^\dagger)\bar\xi, \tag{8.1}$$

where

$$R(A^\dagger)=R(B)=\begin{bmatrix}\cos\theta_C&0&\sin\theta_C\\0&1&0\\-\sin\theta_C&0&\cos\theta_C\end{bmatrix}. \tag{8.2}$$

From the definition of R_{ij} in (3.13) we find that

$$A^\dagger=B=e^{-i\theta_C\sigma_2/2}=\begin{bmatrix}\cos\theta_C/2&-\sin\theta_C/2\\ \sin\theta_C/2&\cos\theta_C/2\end{bmatrix}. \tag{8.3}$$

In terms of the tensor notation (8.1) gives

$$\langle\phi\rangle=\eta_0\begin{bmatrix}\cos\theta_C&0&0&-\sin\theta_C\\0&\cos\theta_C&\sin\theta_C&0\\0&\sin\theta_C&-\cos\theta_C&0\\-\sin\theta_C&0&0&-\cos\theta_C\end{bmatrix}, \tag{8.4}$$

$$\langle\psi\rangle=\zeta_0\begin{bmatrix}\sin\theta_C&0&0&\cos\theta_C\\0&-\sin\theta_C&\cos\theta_C&0\\0&\cos\theta_C&\sin\theta_C&0\\ \cos\theta_C&0&0&-\sin\theta_C\end{bmatrix}. \tag{8.5}$$

To discover which gauge bosons diagonalize the mass matrix we recall (3.10), (3.14), and (3.15),

$$\tilde C_\alpha^\pm=\Lambda_{\alpha\beta}(A,B)C_\beta^\pm, \tag{8.6}$$

$$\tilde S_i=R_{ij}(A)S_j, \tag{8.7}$$

$$\tilde T_i=R_{ij}(B)T_j. \tag{8.8}$$

From (8.3) we find that

$$\Lambda_{\alpha\beta}=\tfrac{1}{2}\mathrm{Tr}(B^\dagger\tau_\alpha A\tau_\beta^\dagger)=\begin{bmatrix}\cos\theta_C&0&\sin\theta_C&0\\0&1&0&0\\-\sin\theta_C&0&\cos\theta_C&0\\0&0&0&1\end{bmatrix}. \tag{8.9}$$

Thus the eigenstates of the gauge-boson mass matrix are

$$\begin{aligned}\tilde W_\pm&=W_\pm\cos\theta_C+V_\pm\sin\theta_C,\\ \tilde V_\pm&=-W_\pm\sin\theta_C+V_\pm\cos\theta_C,\\ \tilde W_6&=W_6\cos\theta_C-S\sin\theta_C,\\ \tilde S&=W_6\sin\theta_C+S\cos\theta_C,\\ \tilde W_9&=W_9\cos\theta_C+T\sin\theta_C,\\ \tilde T&=-W_9\sin\theta_C+T\cos\theta_C,\end{aligned} \tag{8.10}$$

together with $X_\pm$, $U_\pm$, W_7, W_{10}, R, and W_0. In particular, $\tilde{W}_\pm$, R, and W_0 are the massless gauge bosons of the so far unbroken SU(2)×U(1) subgroup, so $\tilde{W}_\pm$ are to be identified with the usual intermediate vector bosons. Using the relations given in Appendix B of paper I we find from (8.4) and (8.5) that the gauge-boson mass term is given as

$$\mathfrak{L}_{\text{mass}} = \tfrac{1}{2}g^2\eta_0^2(2U_+U_- + \tilde{W}_6^2 + \tilde{W}_9^2 + 2\tilde{V}_+\tilde{V}_- + W_7^2 + W_{10}^2) + \tfrac{1}{2}g^2\zeta_0^2(2X_+X_- + \tilde{S}^2 + \tilde{T}^2 + 2\tilde{V}_+\tilde{V}_- + W_7^2 + W_{10}^2), \tag{8.11}$$

which reconfirms that the fields of (8.10) are the eigenstates.

Having found the gauge-boson eigenstates we can readily write the interaction Lagrangian in terms of these states using (3.20)

$$\mathfrak{L}_{\text{int}} \propto \tilde{C}^+_\alpha\Lambda_{\alpha\beta}J^+_\beta + \tilde{C}^-_\alpha\Lambda_{\alpha\beta}J^-_\beta + RK + \tilde{S}_iR_{ij}(A)M_j + \tilde{T}_iR_{ij}(B)N_j\,. \tag{8.12}$$

Here the currents are the ones previously given and are expressed in terms of the quartet $(u, d, s, c)_L$ of fields that are eigenstates of the quark mass matrix. The couplings of the gauge bosons to the leptons are given analogously. We take as the fundamental lepton quartet

$$(\cos\theta_C\nu_e + \sin\theta_C\nu_\mu, e, \mu, -\sin\theta_C\nu_e + \cos\theta_C\nu_\mu)_L \tag{8.13}$$

and have introduced the Cabibbo angle in the neutrino sector (which we are free to do since the neutrinos are massless) only to maintain the conventional definitions of ν_e and ν_μ in the couplings to $\tilde{W}_\pm$. There is no mixing of e and μ in the lepton mass matrix. After the Weinberg mixing of R and W_0 into Z and A which occurs once the SU(2)×U(1) symmetry is broken, we obtain for the interaction Lagrangian

$$\begin{aligned}\frac{\sqrt{2}}{g}\mathfrak{L}_{\text{int}} = &-\sqrt{2}\sin\theta_W(A + \tan\theta_W Z)J^{\text{em}} + (\sqrt{2}\cos\theta_W)^{-1}Z(\bar{\nu}_e\nu_e + \bar{\nu}_\mu\nu_\mu - \bar{e}e - \bar{\mu}\mu + \bar{u}u + \bar{c}c - \bar{d}d - \bar{s}s)\\
&+\tilde{W}_+[\bar{\nu}_e e + \bar{\nu}_\mu\mu + \bar{u}(d\cos\theta_C + s\sin\theta_C) + \bar{c}(-d\sin\theta_C + s\cos\theta_C)]\\
&+\tilde{V}_+[\bar{\nu}_e\mu - \bar{\nu}_\mu e + \bar{u}(-d\sin\theta_C + s\cos\theta_C) - \bar{c}(d\cos\theta_C + s\sin\theta_C)]\\
&+X_+[\cos\theta_C(\bar{\nu}_e e - \bar{\nu}_\mu\mu) + \sin\theta_C(\bar{\nu}_\mu e + \bar{\nu}_e\mu) + \bar{u}d - \bar{c}s]\\
&+U_+[\cos\theta_C(\bar{\nu}_e\mu + \bar{\nu}_\mu e) - \sin\theta_C(\bar{\nu}_e e - \bar{\nu}_\mu\mu) + \bar{u}s + \bar{c}d]\\
&+\tilde{W}_6[\cos\theta_C(\bar{e}\mu + \bar{\mu}e + \bar{d}s + \bar{s}d) - \sin\theta_C(\bar{e}e - \bar{\mu}\mu + \bar{d}d - \bar{s}s)]\\
&+\tilde{S}[\cos\theta_C(\bar{e}e - \bar{\mu}\mu + \bar{d}d - \bar{s}s) + \sin\theta_C(\bar{e}\mu + \bar{\mu}e + \bar{d}s + \bar{s}d)] + iW_7(\bar{\mu}e - \bar{e}\mu + \bar{s}d - \bar{d}s)\\
&+\tilde{W}_9\{\cos\theta_C[\cos2\theta_C(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e) + \sin2\theta_C(\bar{\nu}_\mu\nu_\mu - \bar{\nu}_e\nu_e) + \bar{u}c + \bar{c}u]\\
&\quad+\sin\theta_C[\cos2\theta_C(\bar{\nu}_e\nu_e - \bar{\nu}_\mu\nu_\mu) + \sin2\theta_C(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e) + \bar{u}u - \bar{c}c]\}\\
&+\tilde{T}\{\cos\theta_C[\cos2\theta_C(\bar{\nu}_e\nu_e - \bar{\nu}_\mu\nu_\mu) + \sin2\theta_C(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e) + \bar{u}u - \bar{c}c]\\
&\quad-\sin\theta_C[\cos2\theta_C(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e) + \sin2\theta_C(\bar{\nu}_\mu\nu_\mu - \bar{\nu}_e\nu_e) + \bar{u}c + \bar{c}u]\}\\
&+iW_{10}(\bar{\nu}_\mu\nu_e - \bar{\nu}_e\nu_\mu + \bar{c}u - \bar{u}c) + \text{H.c. of the charged sector}.\end{aligned} \tag{8.14}$$

In our notation $\bar{a}b$ denotes $\bar{a}\gamma_\lambda\frac{1}{2}(1-\gamma_5)b$ in the above.

In (8.14) we have yet to include the small mixings between the light and heavy gauge bosons which ensue once the SU(2)×U(1) symmetry is broken. However, before doing this we discuss the phenomenology which follows from (8.14). First of all, of course, it contains Cabibbo mixing in the couplings to $\tilde{W}_\pm$. Second, it is similar in structure to the interaction Lagrangian given in paper I, Eq. (3.7), except that now there is one common mixing angle in both the lepton and quark sectors. Thus we have related θ_C to the lepton angle θ_L introduced in paper I and hence reduced the number of free parameters compared to paper I. The couplings of $\tilde{W}_\pm$, Z, A, W_7, and W_{10} in $\mathfrak{L}_{\text{int}}$ have a form equivalent to that in paper I, Eq. (3.7), whereas there are modifications in the couplings of the remaining gauge bosons.

Since W_7 alone is responsible for the $K_L \to \mu e$ decay in lowest order we obtain the same constraint previously found in paper I:

$$M_7 > 450M_W\,. \tag{8.15}$$

The analysis of the $K_L - K_S$ mass difference involves more bosons than in paper I but is otherwise analogous and leads to

$$\left|\frac{\cos^2\theta_C}{M^2(\tilde{W}_6)} + \frac{\sin^2\theta_C}{M^2(\tilde{S})} - \frac{1}{M_7^2}\right| \leqslant \frac{3\times10^{-8}}{M_W^2}\,. \tag{8.16}$$

The bosons $\tilde{W}_6$ and $\tilde{S}$ mediate $\mu \to 3e$ and $\mu + N \to e + N$ in the tree approximation and give

$$R_{3e} = \frac{\Gamma(\mu \to 3e)}{\Gamma(\mu \to \text{all})} = \tfrac{1}{8}\sin^2 2\theta_C \left[\frac{M_W^2}{M^2(\tilde{S})} - \frac{M_W^2}{M^2(\tilde{W}_6)}\right]^2 \qquad (8.17)$$

and

$$R_{eN} = \frac{\sigma(\mu + N \to e + N)}{\sigma(\mu + N \to \nu + N')} = \tfrac{1}{4}C \sin^2 2\theta_C \left[\frac{M_W^2}{M^2(\tilde{S})} - \frac{M_W^2}{M^2(\tilde{W}_6)}\right]^2 , \qquad (8.18)$$

where C is the enhancement factor introduced in paper I. From these two relations we obtain the completely parameter-independent ratio

$$\rho'' = \frac{R_{eN}}{R_{3e}} = 2C \qquad (8.19)$$

so that $\rho'' = 10^3$ for N = copper. The current experimental upper limit on R_{eN} is 1.6×10^{-8} in copper.[7] From this limit, from the bounds of (8.15) and (8.16), and from the theoretical relation $M_7^2 = M^2(\tilde{W}_6) + M^2(\tilde{S})$ which follows from (8.11) we find that

$$M(\tilde{S}) > 225 M_W . \qquad (8.20)$$

Though it is not forced, the structure of (8.16) permits $M(\tilde{S})$ to be somewhat less than $M(\tilde{W}_6)$. For convenience in the following we shall make the very mild assumption that

$$\frac{M^2(\tilde{S})}{M^2(\tilde{W}_6)} < \tfrac{1}{2} . \qquad (8.21)$$

All the above discussed processes are permitted before the SU(2)×U(1) symmetry is broken. In order to break SU(2)×U(1) we introduce the four fundamentals of Higgs bosons, χ^s. Following Appendix B of paper I we find the mass term

$$\begin{aligned}
\frac{2}{g^2}\mathcal{L}_{\text{mass}} = {}& \tfrac{1}{2}[\langle\chi_1^a\rangle^2 + \langle\chi_2^b\rangle^2 + \langle\chi_3^c\rangle^2 + \langle\chi_4^d\rangle^2](2\tilde{W}_+\tilde{W}_- + 2\tilde{V}_+\tilde{V}_- + 2X_+X_- + 2U_+U_- + Z^2\sec^2\theta_W) \\
& + (\langle\chi_1^a\rangle^2 + \langle\chi_4^d\rangle^2)(\tilde{W}_9^2 + \tilde{T}^2 + W_{10}^2) + (\langle\chi_2^b\rangle^2 + \langle\chi_3^c\rangle^2)(\tilde{W}_6^2 + \tilde{S}^2 + W_7^2) \\
& + (\langle\chi_1^a\rangle^2 + \langle\chi_2^b\rangle^2 - \langle\chi_3^c\rangle^2 - \langle\chi_4^d\rangle^2)\Big\{(\cos\theta_C\tilde{W}_+ - \sin\theta_C\tilde{V}_+)X_- + \text{H.c.} \\
& \qquad + \frac{1}{\sqrt{2}} Z\sec\theta_W[(\tilde{T} - \tilde{S})\cos\theta_C + (\tilde{W}_9 + \tilde{W}_6)\sin\theta_C]\Big\} \\
& + (\langle\chi_1^a\rangle^2 - \langle\chi_2^b\rangle^2 + \langle\chi_3^c\rangle^2 - \langle\chi_4^d\rangle^2)\Big\{(\sin\theta_C\tilde{W}_+ + \cos\theta_C\tilde{V}_+)U_- + \text{H.c.} \\
& \qquad + \frac{1}{\sqrt{2}} Z\sec\theta_W[(\tilde{T} + \tilde{S})\cos\theta_C + (\tilde{W}_9 - \tilde{W}_6)\sin\theta_C]\Big\} .
\end{aligned} \qquad (8.22)$$

In order to extract some information from (8.22) we make a simplifying but inessential assumption,

$$\langle\chi_1^a\rangle^2 - \langle\chi_2^b\rangle^2 + \langle\chi_3^c\rangle^2 - \langle\chi_4^d\rangle^2 = 0 , \qquad (8.23)$$

which will enable us to study the $\mu \to e\gamma$ phenomenon directly. We define

$$\begin{aligned} p &= \tfrac{1}{4}g^2(\langle\chi_1^a\rangle^2 + \langle\chi_2^b\rangle^2 + \langle\chi_3^c\rangle^2 + \langle\chi_4^d\rangle^2) , \\ q &= \tfrac{1}{4}g^2(\langle\chi_2^b\rangle^2 - \langle\chi_3^c\rangle^2) . \end{aligned} \qquad (8.24)$$

As in paper I we then find that $\tilde{W}_\pm$ and $X_\pm$ mix through an angle ξ determined by the mass matrix

$$M_{\tilde{W}X} = \begin{bmatrix} p & 2q\cos\theta_C \\ 2q\cos\theta_C & M_X^2 \end{bmatrix} . \qquad (8.25)$$

The mixing angle is given by

$$\sin\xi = \frac{2q}{M_X^2}\cos\theta_C = 2x\cos\theta_C \frac{M_W^2}{M_X^2} , \qquad (8.26)$$

where $x = q/p$. The mixing in the neutral sector is rather complicated, but because of (8.21) we find that Z mixes predominantly with the field $B = (\tilde{T} - \tilde{S})/\sqrt{2}$ to give a mass matrix

$$M_{ZB} = \begin{bmatrix} p\sec^2\theta_W & 2q\cos\theta_C\sec\theta_W \\ 2q\cos\theta_C\sec\theta_W & M_X^2 \end{bmatrix} \qquad (8.27)$$

using the degeneracy of $X_\pm$ and $\tilde{S}$ given in (8.11). The Z, B mixing angle ξ' satisfies

$$\sin\xi' = \frac{\sin\xi}{\cos\theta_W} . \qquad (8.28)$$

In terms of the new eigenstates $\overline{W}_\pm$ and $\overline{Z}$ the piece of the interaction Lagrangian which interests us here is given by

$$\frac{\sqrt{2}}{g}\mathcal{L}_{\rm int} = -\sqrt{2}\sin\theta_W(A+\tan\theta_W\cos\xi'\,\overline{Z})J^{\rm em}$$
$$+\overline{Z}\{\cos\xi'(\sqrt{2}\cos\theta_W)^{-1}(\overline{\nu}_e\nu_e+\overline{\nu}_\mu\nu_\mu-\overline{e}e-\overline{\mu}\mu+\overline{u}u+\overline{c}c-\overline{d}d-\overline{s}s)$$
$$-\sin\xi'(\sqrt{2})^{-1}[\cos\theta_C(\overline{e}e-\overline{\mu}\mu+\overline{d}d-\overline{s}s)+\sin\theta_C(\overline{e}\mu+\overline{\mu}e+\overline{d}s+\overline{s}d)]\}$$
$$+\overline{W}_\pm\{\cos\xi[\overline{\nu}_e e+\overline{\nu}_\mu\mu+\overline{u}(d\cos\theta_C+s\sin\theta_C)+\overline{c}(-d\sin\theta_C+s\cos\theta_C)]$$
$$+\sin\xi[\cos\theta_C(\overline{\nu}_e e-\overline{\nu}_\mu\mu)+\sin\theta_C(\overline{\nu}_\mu e+\overline{\nu}_e\mu)+\overline{u}d-\overline{c}s]\}+\cdots, \qquad (8.29)$$

where

$$\overline{W}_\pm=\tilde{W}_\pm\cos\xi+X_\pm\sin\xi,$$
$$\overline{Z}=Z\cos\xi'+\frac{1}{\sqrt{2}}(\tilde{T}-\tilde{S})\sin\xi'. \qquad (8.30)$$

The phenomenology that follows from (8.29) is similar to the analysis of paper I, so we only give the results here. For the $\mu\to e\gamma$ decay one-loop $\overline{W}_\pm$ exchange and $\overline{Z}$ exchange give

$$R_{e\gamma}=\frac{\Gamma(\mu\to e\gamma)}{\Gamma(\mu\to {\rm all})}$$
$$=\frac{\alpha}{96\pi}\sin^2\theta_C[5\sin2\xi-\cos\theta_W(5C_V-1)\sin2\xi']^2$$
$$=\frac{\alpha}{24\pi}\sin^2 2\theta_C x^2\left(\frac{M_W}{M_X}\right)^4(6-5C_V)^2, \qquad (8.31)$$

where $C_V=1-4\sin^2\theta_W$. The tree approximation exchange of $\overline{Z}$ gives additional contributions to (8.17) and (8.18) so that finally we obtain

$$R_{3e}=\tfrac{1}{2}(|A|^2+|B|^2), \qquad (8.32)$$

where

$$A=\tfrac{1}{2}\sin2\theta_C\left[\frac{M_W^{\,2}}{M^2(\tilde{S})}-\frac{M_W^{\,2}}{M^2(\tilde{W}_6)}\right]$$
$$+\tfrac{1}{4}(1-2\sin^2\theta_W)\cos\theta_W\sin2\xi'\sin\theta_C, \qquad (8.33)$$

$$B=-\tfrac{1}{2}\sin\theta_C\sin2\xi'\sin^2\theta_W\cos\theta_W, \qquad (8.34)$$

and

$$R_{eN}=\tfrac{1}{4}C\sin^2 2\theta_C\left(\frac{M_W}{M(\tilde{S})}\right)^4$$
$$\times\left[1-\frac{M^2(\tilde{S})}{M^2(\tilde{W}_6)}-\frac{(C_VZ-N)}{4(Z+2N)}\frac{M^2(\tilde{S})}{M_W^{\,2}}\frac{\cos\theta_W}{\cos\theta_C}\sin2\xi'\right]^2. \qquad (8.35)$$

Since $|x|<\frac{1}{2}$ we find that the $\overline{Z}$ contributions to (8.32) and (8.35) are not as large as those due to $\tilde{S}$ and $\tilde{W}_6$ so that we recover (8.17) and (8.18) with the constraints on ρ'' and $M(\tilde{S})$ thus being unchanged. Consequently, we conclude that even though R_{eN} is constrained in our model by the strangeness-changing processes, it could still be close to its current upper limit. Thus some improvement in the experimental limit on R_{eN} might yield interesting results. Should R_{eN} actually be seen in the near future we would then also anticipate the possible observation of $K_L\to\mu e$.

Making futher use of the condition $|x|<\frac{1}{2}$ and of the approximate (8.21) we obtain the following interesting relations (N = copper):

$$\rho=\frac{R_{eN}}{R_{e\gamma}}>10^5, \qquad (8.36)$$

$$\rho'=\frac{R_{3e}}{R_{e\gamma}}>10^2. \qquad (8.37)$$

These relations make observation of $\mu\to e\gamma$ impossible at present though ultimately these relations would be very strong signatures of our model.

While it is characteristic of the approach both here and in paper I that $R_{e\gamma}<10^{-13}$, making the process somewhat academic, the relation in (8.31) is particularly interesting since it provides an intimate connection between muon-number nonconservation in the lepton sector and strangeness nonconservation in the quark sector [as do (8.17) and (8.18) of course]. It is of interest to impose some extra constraints on the couplings of the quarks and leptons to the fundamentals χ^s which give them their masses. If we require all the four quarks to couple with a universal strength to the fundamentals and all leptons to couple with a universal strength to the same fundamentals, we obtain the following mass formula for the contribution of the weak interaction to the fermion masses

$$\frac{m_d}{m_s}=\frac{m_e}{m_\mu}. \qquad (8.38)$$

Also from (8.31) we obtain

$$R_{e\gamma}=\frac{\alpha}{96\pi}\sin^2 2\theta_C(6-5C_V)^2\left(\frac{M_W}{M_X}\right)^4\frac{(1-m_e^{\,2}/m_\mu^{\,2})^2}{(1+m_u^{\,2}/m_s^{\,2})^2}. \qquad (8.39)$$

In (8.39) the parameter M_X is essentially the only unknown quantity. Moreover, we see that in our approach the $\mu\to e\gamma$ decay amplitude is related to the μ-e mass difference and the Cabibbo angle, an intriguing situation.

We turn our attention now to illustrate very briefly the phenomenological significance of the other three observable angles that we introduced in previous sections. In the absence of *CP* violation it is possible to introduce one further mixing angle. This would mix $X_\pm, \tilde{S}, \tilde{T}$ with $U_\pm, \tilde{W}_6, \tilde{W}_9$ so that

$$\tilde{X}_\pm = \cos\theta_X X_\pm + \sin\theta_X U_\pm \,, \quad \tilde{U}_\pm = -\sin\theta_X X_\pm + \cos\theta_X U_\pm \tag{8.40}$$

with analogous relations for the neutrals. These mixings are achieved if we replace (8.2) by

$$R(A^\dagger) = \begin{bmatrix} \cos(\theta_C - \theta_X) & 0 & \sin(\theta_C - \theta_X) \\ 0 & 1 & 0 \\ -\sin(\theta_C - \theta_X) & 0 & \cos(\theta_C - \theta_X) \end{bmatrix}, \quad R(B^\dagger) = \begin{bmatrix} \cos(\theta_C + \theta_X) & 0 & -\sin(\theta_C + \theta_X) \\ 0 & 1 & 0 \\ \sin(\theta_C + \theta_X) & 0 & \cos(\theta_C + \theta_X) \end{bmatrix}. \tag{8.41}$$

Further, (8.41) leaves untouched the fields $\tilde{W}_\pm$, $\tilde{V}_\pm$, W_7, W_{10}, R, and W_0 of (8.10). In terms of the angles β and β' introduced previously we have

$$\beta - \beta' = 2\theta_C \,, \quad -\beta - \beta' = 2\theta_X \,. \tag{8.42}$$

From (8.14) we find that couplings of $\tilde{X}_\pm$ and $\tilde{U}_\pm$ to the quarks are given by the term

$$\mathcal{L} \sim \tilde{X}_+[\cos\theta_X(\bar{u}d - \bar{c}s) + \sin\theta_X(\bar{u}s + \bar{c}d)] + \tilde{U}_+[\cos\theta_X(\bar{u}s + \bar{c}d) - \sin\theta_X(\bar{u}d - \bar{c}s)] + \text{H.c.} \tag{8.43}$$

Thus in the same way as

$$\frac{A(\tilde{W}_+ \to \bar{u}s)}{A(\tilde{W}_+ \to \bar{u}d)} = \tan\theta_C \tag{8.44}$$

we see that

$$\frac{A(\tilde{X}_+ \to \bar{u}s)}{A(\tilde{X}_+ \to \bar{u}d)} = \tan\theta_X \,. \tag{8.45}$$

Thus θ_X is an effective Cabibbo angle for the heavy-boson sector. In the presence of this new mixing we can recalculate $R_{e\gamma}$. The result of the calculation is analogous to (8.31) except that the factor $\sin^2 2\theta_C$ is replaced by $\sin^2 2(\theta_C - \theta_X)$, so that the $\mu \to e\gamma$ decay is sensitive to the new orientation of the vacuum.

Finally, introducing *CP* violation leads to yet further mixings among the bosons (such as the mixings of $U_\pm$, W_6, and W_9 with $V_\pm$, W_7, and W_{10} which were already described in paper II), together with a reidentification of the generators of the Weinberg-Salam theory. The two additional angles which are responsible for *CP* violation can be given a phenomenological interpretation similar to (8.45).

IX. CONCLUSIONS

By mixing the gauge bosons in their mass matrix, we have developed a theory in which the nonconservations of strangeness, *CP*, and muon number are unified; in fact, they are inevitable. Their origins are all rooted in the spontaneous breakdown of the SU(4)×U(1) gauge symmetry. The phenomenological parameters describing the nonconservation processes can all be related to the parameters of the Higgs potential. All of the mixing angles are natural in the sense that they need not be miniscule. In our theory the parameters of smallness are determined by the mass scales of the superheavy bosons, and are characteristic of the low rates associated with such processes as *CP* violation and muon-number nonconservation.

Because the theory is a badly broken higher symmetry, its experimentally testable predictions all pertain to rare processes that are difficult to observe. The capture of muons on heavy nuclei should be most susceptible for detection. Observation of the $\mu \to 3e$ decay before $\mu \to e\gamma$ would be a highly favorable signal for the physical relevance of our theory.

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[3] P. Dittner and S. Eliezer, Phys. Rev. D 8, 1929 (1973).

[4] With the definition of $Y(\theta)$ given in (4.6), we note that $R(Y(\theta)) = R_y(-\theta)$.

[5] The orthogonality of the fundamentals ensures the absence of Higgs mediation of processes such as $\bar{s}d \to \bar{d}s$ and $\bar{s}d \to \bar{\mu}\mu$. However, in our theory the quarks and leptons share common flavors and thus the Higgs scalers can mediate $\bar{s}d \to \bar{\mu}e$. This process is not SU(4)-flavor changing, though it changes strong-interaction strangeness. This process can be prevented by making the Higgs scalars heavy enough (as was done to the W

bosons that mediate the same process), or by introducing separate quark and lepton symmetries.

[6]This aspect of the SU(4) × U(1) theory has also been studied by S. Pakvasa, H. Sugawara, and M. Suzuki, Phys. Lett. 69B, 461 (1977); however, the classification of fermion states which they have considered is different from that used in this paper.

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