SU(4) × U(1) gauge theory. I. Muon-number nonconservation

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This is the first in a series of three papers in which we study a unified gauge theory based on the group SU(4) × U(1). In this paper we present the general group-theoretical structure of SU(4) × U(1) and study in particular the muon-number-nonconserving processes which occur in the model. In the subsequent papers we shall study *CP* violation and present a new approach to understanding the Cabibbo angle. The central theme of our work is that all of the phenomena discussed in this series can be understood through the spontaneous-breakdown mechanism. In this paper we construct an explicit Higgs potential which breaks SU(4) × U(1) down to the usual SU(2) × U(1) theory of Weinberg and Salam. Of particular interest in this paper are rare muon-number-nonconserving processes, such as $\mu \to e\gamma$, which are forced to take place in our model once the symmetry has been broken. Our model possesses 12 new heavy intermediate vector bosons, and the available experimental constraints on their masses are presented.

I. INTRODUCTION

This is the first in a series of three papers[1] in which we explore SU(4) × U(1) as a unified gauge theory. The idea of building a unified theory of weak and electromagnetic interactions based on spontaneously broken gauge invariance was suggested by Weinberg and Salam[2] some ten years ago. Since that time many of the general ideas involved have been given experimental support,[3] so that there is now a fair measure of confidence in unified theories, and in particular in their simplest form based on the group SU(2) × U(1). Because of the successes of the standard model, there has been little need to consider more complicated or larger groups, though the idea of other gauge groups has been entertained from time to time. It is nonetheless useful to explore other groups that are able to produce a cohesive picture of weak interactions, particularly if they can provide insight into phenomena which SU(2) × U(1) is not too well suited to discuss.

In this series of papers we will study the group SU(4) × U(1) as a candidate gauge theory. There are three reasons for suggesting this group. First there is the well-known analogy between the familiar four leptons and four quarks.[4] By using the group SU(4) there is then an intimate connection between the strong and weak interactions with the flavor currents of SU(4) now playing a dual role; they are generators of both the strong-interaction $SU(4)_L \times SU(4)_R$ global chiral symmetry and of the weak-interaction $SU(4)_L \times U(1)$ local gauge symmetry. The second reason is that the muon and the electron are placed together in the same multiplet along with their neutrinos, thus unifying the lepton species and abandoning the artificial division made in conventional theories of separate muon and electron sectors. In our model this division arises dynamically and hence sharpens our understanding of weak-interaction symmetries. The final reason is that SU(4) × U(1) contains the usual SU(2) × U(1) as a subroup so that none of the conventional phenomenology is lost, but rather is seen to be only a first approximation to a higher symmetry.

Since SU(4) × U(1) is not even remotely a good symmetry for the usual weak interactions we shall work in a spontaneously broken scheme. We break SU(4) × U(1) down to the conventional SU(2) × U(1) picture so that the 12 new vector bosons which accompany the familiar $W_\pm$, Z, and A acquire superheavy masses. In this respect SU(2) × U(1) is only an approximate residual symmetry which is then itself spontaneously broken. The actual implementation of the complete breaking program is highly nontrivial and constitutes the bulk of our work. Moreover, we shall see that the pattern of breaking which emerges is surprisingly rich and permits discussion of three longstanding theoretical issues, muon-number nonconservation, *CP* violation, and the origin of the Cabibbo angle. Each one of these phenomena can arise through the explicit effects of spontaneous breakdown in the gauge-boson mass matrix since there are adjoint representations of Higgs bosons in our theory which carry off muon number, an odd unit of *CP*, and strangeness into the vacuum. It is particularly interesting that it is not essential to have any mixing in either the lepton or the quark mass matrices in order to understand muon-number nonconservation and the Cabibbo angle, and even more interesting that *CP* violation shares a common origin with these two other phenomena.

In order to bring out the specific physical effects involved in as transparent a way as possible we shall present each of the above three topics in a separate paper, discussing the breaking pattern appropriate to each one by itself rather than discussing one overall breaking pattern. As will be seen, this will involve no loss of generality. In this first paper we shall discuss muon-number nonconservation and also analyze some general properties of the group structure of SU(4). Some aspects of our work on muon-number nonconservation have already been presented in a Letter,[5] and in this paper we give the detailed analysis.

The present paper is organized as follows. In Sec. II we introduce the various sectors of our model, the gauge bosons, the Higgs bosons, the leptons, and the quarks, and discuss their interactions. In Sec. III we study the requirements on the pattern of symmetry breaking imposed by the group-theoretical structure of SU(4) × U(1), explaining what is needed in order to break the group down to the usual Weinberg-Salam theory. In Sec. IV we then study the structure of the Higgs potential and obtain a set of dynamical conditions on the various parameters in the potential which produce the pattern of breaking we require. At this point the model and the breaking pattern are completely specified, so we can examine the phenomenological implications of the theory. We discuss the general implications in Sec. V, and in Sec. VI we study some very interesting processes such as $\mu \to e\gamma$ which are forced to take place in our model. We present our conclusions in Sec. VII.

II. STRUCTURE OF THE UNBROKEN MODEL

In SU(4) × U(1) there are 15 gauge fields W_i transforming like SU(4) generators and a singlet gauge field W_0. Left-handed leptons and quarks are in fundamental representations of SU(4). We denote these multiplets by generic $(\nu_e, e, \mu, \nu_\mu)_L$ and $(u, d, s, c)_L$. Later on we will discuss whether or not these states mix in their mass matrices. For the moment they are just labels to describe the group theory. Right-handed quarks and right-handed massive leptons form SU(4) singlets. The charge operator is given in terms of the SU(4) × U(1) generators $[F_i, (i=1,\ldots,15), F_0]$ by

$$Q = F_3 + \frac{1}{\sqrt{3}}F_8 - (\tfrac{2}{3})^{1/2}F_{15} + F_0 . \tag{2.1}$$

With Y denoting the eigenvalue of F_0 we have for the left-handed leptons $Y = -\frac{1}{2}$, for the left-handed quarks $Y = \frac{1}{6}$, and for all right-handed fermions $Y = Q$.

The most convenient basis for the generators F_i $(i=1,\ldots,15)$ is

$$\begin{aligned}
F_W^{\pm} &= F_1 + F_{13} \pm i(F_2 - F_{14}) ,\\
F_X^{\pm} &= F_1 - F_{13} \pm i(F_2 + F_{14}) ,\\
F_U^{\pm} &= F_4 + F_{11} \pm i(F_5 - F_{12}) ,\\
F_V^{\pm} &= F_4 - F_{11} \pm i(F_5 + F_{12}) ,\\
F_R &= F_3 + \frac{1}{\sqrt{3}}F_8 - (\tfrac{2}{3})^{1/2}F_{15} ,\\
F_S &= \tfrac{1}{2}(-F_3 + \sqrt{3}F_8) ,\\
F_T &= \tfrac{1}{2}\left[F_3 + \frac{1}{\sqrt{3}}F_8 + 2(\tfrac{2}{3})^{1/2}F_{15}\right]
\end{aligned} \tag{2.2}$$

with F_6, F_7, F_9, and F_{10} as before. It is useful to to have the commutation relations for these generators and we have assembled them together in Appendix A. In the same way as for the generators, we define a similar basis for normalized gauge-boson fields:

$$\begin{aligned}
W_\pm &= \tfrac{1}{2}[(W_1 + W_{13}) \mp i(W_2 - W_{14})] ,\\
X_\pm &= \tfrac{1}{2}[(W_1 - W_{13}) \mp i(W_2 + W_{14})] ,\\
U_\pm &= \tfrac{1}{2}[(W_4 + W_{11}) \mp i(W_5 - W_{12})] ,\\
V_\pm &= \tfrac{1}{2}[(W_4 - W_{11}) \mp i(W_5 + W_{12})] ,\\
R &= \frac{1}{\sqrt{2}}\left[W_3 + \frac{1}{\sqrt{3}}W_8 - (\tfrac{2}{3})^{1/2}W_{15}\right] ,\\
S &= \tfrac{1}{2}(-W_3 + \sqrt{3}W_8) ,\\
T &= \frac{1}{2}\left[W_3 + \frac{1}{\sqrt{3}}W_8 + 2(\tfrac{2}{3})^{1/2}W_{15}\right] .
\end{aligned} \tag{2.3}$$

with W_6, W_7, W_9, and W_{10} as before. The interactions of these gauge bosons with the fermions are given by [using $g/\sqrt{2}$ and g' as the SU(4) and U(1) coupling constants]

$$\begin{aligned}
\frac{\sqrt{2}}{g}\mathcal{L}_{\rm int} &= \sum_i W_i^\lambda \bar{\Psi}\lambda_i\gamma_\lambda\Psi_L + \sqrt{2}\frac{g'}{g}W_0^\lambda(\bar{\Psi}Y\gamma_\lambda\Psi_L + \bar{\Psi}Q\gamma_\lambda\Psi_R)\\
&= \sqrt{2}\frac{g'}{g}W_0 J^{\rm em} + \frac{1}{\sqrt{2}}\left(R - \frac{g'}{g}W_0\right)(\bar{\nu}_e\nu_e + \bar{\nu}_\mu\nu_\mu - \bar{e}e - \bar{\mu}\mu + \bar{u}u + \bar{c}c - \bar{d}d - \bar{s}s)\\
&\quad + W_+(\bar{\nu}_e e + \bar{\nu}_\mu\mu + \bar{u}d + \bar{c}s) + X_+(\bar{\nu}_e e - \bar{\nu}_\mu\mu + \bar{u}d - \bar{c}s) + U_+(\bar{\nu}_e\mu + \bar{\nu}_\mu e + \bar{u}s + \bar{c}d) + V_+(\bar{\nu}_e\mu - \bar{\nu}_\mu e + \bar{u}s - \bar{c}d)\\
&\quad + W_6(\bar{e}\mu + \bar{\mu}e + \bar{d}s + \bar{s}d) + iW_7(\bar{\mu}e - \bar{e}\mu + \bar{s}d - \bar{d}s) + W_9(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e + \bar{u}c + \bar{c}u) + iW_{10}(\bar{\nu}_\mu\nu_e - \bar{\nu}_e\nu_\mu + \bar{c}u - \bar{u}c)\\
&\quad + S(\bar{e}e - \bar{\mu}\mu + \bar{d}d - \bar{s}s) + T(\bar{\nu}_e\nu_e - \bar{\nu}_\mu\nu_\mu + \bar{u}u - \bar{c}c)\\
&\quad + \text{H.c. of the charged sector} .
\end{aligned} \tag{2.4}$$

In our notation $\bar{a}b$ denotes $\bar{a}\gamma_\lambda\frac{1}{2}(1-\gamma_5)b$ in the above. Again, the gauge-boson fields are just labels, and their relation to the physical gauge bosons awaits the diagonalization of the gauge-boson mass matrix. The utility of the above basis is, for instance, seen from the fact that it is W_+ which is coupled universally to $\bar{\nu}_e e$ and $\bar{\nu}_\mu' \mu$ rather than the separate W_1+iW_2 and $W_{13}-iW_{14}$.

As we shall see in Sec. III, our model also needs two scalar adjoint representations of Higgs fields, ϕ_i and ψ_i ($i=1,\ldots,15$), and four fundamentals, χ_i^s ($s=a,b,c,d$ labeling the multiplet, and $i=1,\ldots,4$ the components of each multiplet). These Higgs fields provide the spontaneous breakdown in Sec. III, and we present here only some general properties. Each Higgs adjoint possesses seven neutral components. From the usual minimal coupling of the SU(4) gauge bosons to the neutral adjoint Higgs fields the mass term of the gauge bosons is given by

$$\tfrac{1}{2}g^2W_i^\dagger M_{ij}W_j=\tfrac{1}{2}g^2f_{kil}W_i^\dagger\phi_l^\dagger f_{kjm}W_j\phi_m . \tag{2.5}$$

We have calculated the matrix M_{ij} and display it in Appendix B. We note here only that the gauge boson R is not involved in this piece of the mass matrix. This follows since the adjoint Higgs bosons have $Y=0$, so that for them F_R is the charge operator which commutes with all the neutral adjoint Higgs bosons. Consequently, breaking the symmetry in the adjoint representation will always leave R massless. The couplings of the gauge bosons to the fundamental Higgs fields are obtained analogously and their contribution to the gauge-boson mass matrix is also presented in Appendix B. The couplings of the fundamental Higgs fields to the fermions are given in Appendix C. Because of the left-handed nature of the fermions, the adjoint Higgs bosons do not contribute to the fermion mass matrix.

III. THE BREAKING PATTERN

Our immediate objective is to break SU(4) × U(1) down to SU(2) × U(1). In this section we shall discuss what we require from the breaking pattern by studying the group structure of SU(4), and in Sec. IV we will show how the Higgs potential can produce the breaking we want. We note first that SU(4) possesses many SU(2) subgroups, so that picking the breaking scheme to give the SU(2) group we want is nontrivial. More important, it turns out that there is more than one SU(2) subgroup we can choose which gives the usual weak-interaction phenomenology, since the generator F_R appears in more than one SU(2). While we shall always require F_R to be a member of the SU(2) (so that R can ultimately mix with W_0 in the conventional manner), we are able to choose linear combinations of the four charged generators such that the combinations also close on an SU(2) with F_R. There is thus a freedom in how to identify the usual W bosons from among the charged bosons of Eq. (2.3). In fact we shall exploit this freedom in our other papers as this is precisely what is needed in order to introduce CP violation and the Cabibbo angle. However, in this paper we shall only discuss the simplest case in which the SU(2) subgroup is the usual Weinberg-Salam one generated by the set

$$G=\left\{F_1+F_{13},F_2-F_{14},F_3+\frac{1}{\sqrt{3}}F_8-(\tfrac{2}{3})^{1/2}F_{15}\right\} \tag{3.1}$$

with its associated gauge bosons $W_\pm$ and R, i.e., we identify $W_\pm$ with the usual W bosons.

In trying to break the symmetry we note first that we cannot break SU(4) down all the way to SU(2) with one adjoint alone.[6] As can be seen from the commutators in Appendix A, if we choose the breaking in any one of the canonical directions ϕ_3, ϕ_6, ϕ_7, ϕ_8, ϕ_9, or ϕ_{10} we then break SU(4) down to SU(2) × U(1) × U(1) with all of the charged bosons becoming massive. For instance, choosing $\langle\phi_6\rangle$ to be nonzero leaves W_9, W_{10}, T, W_6, and R massless with the other 10 bosons becoming massive. Furthermore, if we choose the breaking in the remaining canonical direction ϕ_{15}, we break SU(4) down to an SU(3) × U(1) subgroup which does not itself contain the usual SU(2) × U(1) of Weinberg and Salam (nor can it be reached by a rotation of the basis). Consequently, we must break SU(4) down along a direction other than a canonical one. In order to decide which direction, we now note the commutators

$$\begin{aligned}
&[F_W^+,\phi_W^-]=\sqrt{2}\phi_R ,\\
&[F_W^+,\phi_X^-]=\phi_T-\phi_S ,\\
&[F_W^+,\phi_U^-]=\phi_9-\phi_6 ,\\
&[F_W^+,\phi_V^-]=i(\phi_7-\phi_{10}) ,
\end{aligned} \tag{3.2}$$

where we use the same notation for the ϕ fields as given previously for the gauge bosons. Hence for $W_\pm$ to stay massless we require

$$\begin{aligned}
&\langle\phi_R\rangle=0 ,\\
&\langle\phi_T\rangle-\langle\phi_S\rangle=0 ,\\
&\langle\phi_6\rangle-\langle\phi_9\rangle=0 ,\\
&\langle\phi_7\rangle-\langle\phi_{10}\rangle=0
\end{aligned} \tag{3.3}$$

[i.e., $\langle\phi_3\rangle=\langle\phi_8\rangle-\sqrt{2}\,\langle\phi_{15}\rangle=0$]. Thus the breaking must be along some linear combination of $\phi_T + \phi_S$, $\phi_6 + \phi_9$, and $\phi_7 + \phi_{10}$. We choose the breaking so that $\langle\phi_T\rangle = \langle\phi_S\rangle$ [i.e., so that $\langle\phi_8\rangle = \sqrt{2}\,\langle\phi_{15}\rangle \neq 0$] with all other members of the ϕ multiplet having vanishing expectation values. With this choice we observe from the commutators that $U_\pm$, W_6, W_9, $V_\pm$, W_7, and W_{10} acquire masses with $F_W^\pm$, R, $F_X^\pm$, $F_S - F_T$, and $F_S + F_T$ generating a so far unbroken SU(2) × U(2) subgroup.

In order to complete the breaking down to the usual SU(2) the second adjoint, ψ, must also satisfy Eq. (3.3). We choose the nontrival expectation value to be orthogonal to $\langle\psi_T\rangle + \langle\psi_S\rangle$ by setting $\langle\psi_6\rangle = \langle\psi_9\rangle \neq 0$ with all other $\langle\psi_i\rangle = 0$. With this particular choice the gauge-boson mass matrix is now diagonal in the basis of states introduced in Eq. (2.3). In terms of the parameters $\langle\phi_8\rangle$ and $\langle\psi_6\rangle$ the masses of the heavy gauge bosons are given by the mass term

$$\mathcal{L}_{\text{mass}} = \tfrac{3}{8} g^2 \langle\phi_8\rangle^2 (2U_+U_- + W_6{}^2 + W_9{}^2 + 2V_+V_- + W_7{}^2 + W_{10}{}^2) + \tfrac{1}{2} g^2 \langle\psi_6\rangle^2 (2X_+X_- + S^2 + T^2 + 2V_+V_- + W_7{}^2 + W_{10}{}^2)\,. \tag{3.4}$$

The degeneracies of this pattern follow since ($X_\pm$, $S-T$), ($U_\pm$, $W_6 - W_9$), and ($V_\pm$, $W_7 - W_{10}$) form three triplets under the SU(2) of Eq. (3.1). We note that the mass pattern of Eq. (3.4) is not unique even if $W_\pm$ stay massless. By breaking in different orthogonal directions while maintaining Eq. (3.3), we can change the identity and relative masses of the heavy gauge bosons. The particular choice we have adopted is found to be the most useful for the phenomenological applications of this paper.

The breaking of SU(2) × U(1) is then completed using the fundamentals, χ_i^s. This leads to the conventional mixing of W_0 and R into new fields

$$Z = R\cos\theta_W - W_0 \sin\theta_W\,,$$
$$A = -R\sin\theta_W - W_0\cos\theta_W\,, \tag{3.5}$$

where $\tan\theta_W = g'/g$. The masses of $W_\pm$ and Z satisfy the Weinberg relation $M_W = M_Z\cos\theta_W$, and the Fermi coupling constant is given by $G_F/\sqrt{2} = g^2/8M_W{}^2$. For clarity we present the complete mass pattern for the gauge bosons in Fig. 1.

The χ_i^s can also cause quark mixing and lepton mixing in the mass matrix. (We will dispense with this requirement when we discuss an alternative mechanism for introducing the Cabibbo angle in the third paper of this series.) We take $(u', d, s, c')_L$ and $(\nu_1, e', \mu', \nu_2)_L$ as the SU(4) quartets needed for Eq. (2.4), where

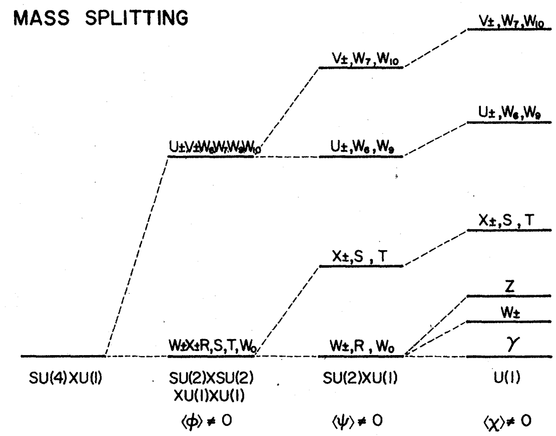

FIG. 1. The mass spectrum of the 16 gauge bosons at various stages of the breaking of SU(4) × U(1). The actual mass values indicated are only illustrative.

$$u' = u\cos\theta_C - c\sin\theta_C\,,$$
$$c' = u\sin\theta_C + c\cos\theta_C\,,$$
$$e' = e\cos\theta_L + \mu\sin\theta_L\,,$$
$$\mu' = -e\sin\theta_L + \mu\cos\theta_L\,, \tag{3.6}$$
$$\nu_1 = \nu_e\cos\theta_L + \nu_\mu\sin\theta_L\,,$$
$$\nu_2 = -\nu_e\sin\theta_L + \nu_\mu\cos\theta_L\,.$$

Here u, c, s, d, e, μ, ν_e, and ν_μ are eigenstates of the mass matrix. The angle θ_C is the Cabibbo angle and θ_L is an analogous lepton angle. Both neutrino and charged lepton fields are rotated through the same angle to maintain the conventional definitions of ν_e and ν_μ. This angle is not observable in the usual SU(2) × U(1) sector since the neutrinos are massless, but becomes observable in the couplings to the heavy bosons and plays an important role in some of the muon-number-violating processes we shall discuss in this paper. (Although θ_C and θ_L are independent when introduced through fermion mass matrix mixing, they become related once the Cabibbo angle arises through mixing in the gauge-boson mass matrix. This will be discussed in paper III.) Further details concerning the fermion mass matrix are given in Appendix C.

In terms of the new fermion fields the interaction Lagrangian of Eq. (2.4) can be reexpressed as

$$
\begin{aligned}
\frac{\sqrt{2}}{g}\mathcal{L}_{\rm int} = & -\sqrt{2}\sin\theta_W(A^\lambda+\tan\theta_W Z^\lambda)J_\lambda^{\rm em} \\
& +(\sqrt{2}\cos\theta_W)^{-1}Z^\lambda[(\bar\nu_e\gamma_\lambda\nu_{eL}+\bar\nu_\mu\gamma_\lambda\nu_{\mu L}-\bar e\gamma_\lambda e_L-\bar\mu\gamma_\lambda\mu_L)+(\bar u\gamma_\lambda u_L+\bar c\gamma_\lambda c_L-\bar d\gamma_\lambda d_L-\bar s\gamma_\lambda s_L)] \\
& +W_+^\lambda(\bar\nu_e\gamma_\lambda e_L+\bar\nu_\mu\gamma_\lambda\mu_L+\bar u'\gamma_\lambda d_L+\bar c'\gamma_\lambda s_L) \\
& +X_+^\lambda[\cos2\theta_L(\bar\nu_e\gamma_\lambda e_L-\bar\nu_\mu\gamma_\lambda\mu_L)+\sin2\theta_L(\bar\nu_e\gamma_\lambda\mu_L+\bar\nu_\mu\gamma_\lambda e_L)+\bar u'\gamma_\lambda d_L-\bar c'\gamma_\lambda s_L] \\
& +U_+^\lambda[\cos2\theta_L(\bar\nu_e\gamma_\lambda\mu_L+\bar\nu_\mu\gamma_\lambda e_L)-\sin2\theta_L(\bar\nu_e\gamma_\lambda e_L-\bar\nu_\mu\gamma_\lambda\mu_L)+\bar u'\gamma_\lambda s_L+\bar c'\gamma_\lambda d_L] \\
& +V_+^\lambda(\bar\nu_e\gamma_\lambda\mu_L-\bar\nu_\mu\gamma_\lambda e_L+\bar u'\gamma_\lambda s_L-\bar c'\gamma_\lambda d_L) \\
& +W_6^\lambda[\cos2\theta_L(\bar e\gamma_\lambda\mu_L+\bar\mu\gamma_\lambda e_L)-\sin2\theta_L(\bar e\gamma_\lambda e_L-\bar\mu\gamma_\lambda\mu_L)+\bar d\gamma_\lambda s_L+\bar s\gamma_\lambda d_L] \\
& +iW_7^\lambda(\bar\mu\gamma_\lambda e_L-\bar e\gamma_\lambda\mu_L+\bar s\gamma_\lambda d_L-\bar d\gamma_\lambda s_L) \\
& +W_9^\lambda[\cos2\theta_L(\bar\nu_e\gamma_\lambda\nu_{\mu L}+\bar\nu_\mu\gamma_\lambda\nu_{eL})-\sin2\theta_L(\bar\nu_e\gamma_\lambda\nu_{eL}-\bar\nu_\mu\gamma_\lambda\nu_{\mu L})+\bar u'\gamma_\lambda c_L'+\bar c'\gamma_\lambda u_L'] \\
& +iW_{10}^\lambda(\bar\nu_\mu\gamma_\lambda\nu_{eL}-\bar\nu_e\gamma_\lambda\nu_{\mu L}+\bar c\gamma_\lambda u_L-\bar u\gamma_\lambda c_L) \\
& +S^\lambda[\cos2\theta_L(\bar e\gamma_\lambda e_L-\bar\mu\gamma_\lambda\mu_L)+\sin2\theta_L(\bar e\gamma_\lambda\mu_L+\bar\mu\gamma_\lambda e_L)+\bar d\gamma_\lambda d_L-\bar s\gamma_\lambda s_L] \\
& +T^\lambda[\cos2\theta_L(\bar\nu_e\gamma_\lambda\nu_{eL}-\bar\nu_\mu\gamma_\lambda\nu_{\mu L})+\sin2\theta_L(\bar\nu_e\gamma_\lambda\nu_{\mu L}+\bar\nu_\mu\gamma_\lambda\nu_{eL})+\bar u'\gamma_\lambda u_L'-\bar c'\gamma_\lambda c_L'] \\
& +\text{H.c. of the charged sector}. \qquad (3.7)
\end{aligned}
$$

We have presented Eq. (3.7) mainly as a convenience since there is a small additional effect so far not yet included which arises once the SU(2) × U(1) symmetry is broken. The breaking of the SU(2) × U(1) symmetry automatically causes $W_\pm$ and Z to mix with the heavy gauge bosons. In fact, as we shall see in Sec. VI, this mixing is responsible for processes such as $\mu\to e\gamma$. Rather than consider all possible mixings for the purposes of this paper, we shall only need W, X mixing and Z, S mixing. We define

$$
\begin{aligned}
\bar W_\pm &= W_\pm\cos\xi+X_\pm\sin\xi\,,\\
\bar X_\pm &= -W_\pm\sin\xi+X_\pm\cos\xi\,,\\
\bar Z &= Z\cos\xi'+S\sin\xi'\,,\\
\bar S &= -Z\sin\xi'+S\cos\xi'\,.
\end{aligned}
\qquad (3.8)
$$

We shall leave ξ and ξ' as free parameters for the moment, noting only from Appendix B that the contributions of the adjoints and fundamentals to ξ and ξ' are highly constrained, i.e., setting $\langle\phi_R\rangle(\langle\phi_T\rangle-\langle\phi_S\rangle)$ nonzero will mix W and X alone, while setting $\langle\chi_2^b\rangle^2\neq\langle\chi_3^c\rangle^2$ will simultaneously mix W with X and Z with S.

We have now described what we want from the breaking pattern. We discuss next how the Higgs potential achieves it.

IV. STRUCTURE OF THE HIGGS POTENTIAL

It is convenient to use a tensor basis to describe the adjoint representations. We set

$$
\phi_{ab}=\sum_{i=1}^{15}\lambda_{ab}^i\phi_i=\begin{bmatrix}
\frac{1}{\sqrt2}\phi_R+\phi_T & \phi_1-i\phi_2 & \phi_4-i\phi_5 & \phi_9-i\phi_{10}\\
\phi_1+i\phi_2 & -\frac{1}{\sqrt2}\phi_R+\phi_S & \phi_6-i\phi_7 & \phi_{11}-i\phi_{12}\\
\phi_4+i\phi_5 & \phi_6+i\phi_7 & -\frac{1}{\sqrt2}\phi_R-\phi_S & \phi_{13}-i\phi_{14}\\
\phi_9+i\phi_{10} & \phi_{11}+i\phi_{12} & \phi_{13}+i\phi_{14} & \frac{1}{\sqrt2}\phi_R-\phi_T
\end{bmatrix}
\qquad (4.1)
$$

Since electric charge will remain unbroken, we shall only be interested in the electrically neutral sector of ϕ_{ab} in the following. We parametrize the expectation values in the neutral sector as

$$
\begin{aligned}
\langle\phi_6\rangle &= \eta_0\sin\beta_1\cos\alpha_1\,,\\
\langle\phi_7\rangle &= \eta_0\sin\beta_1\sin\alpha_1\,,\\
\langle\phi_S\rangle &= \eta_0\cos\beta_1\,,\\
\langle\phi_9\rangle &= \eta_0'\sin\beta_1'\cos\alpha_1'\,,\\
\langle\phi_{10}\rangle &= \eta_0'\sin\beta_1'\sin\alpha_1'\,,\\
\langle\phi_T\rangle &= \eta_0'\cos\beta_1'\,,
\end{aligned}
\qquad (4.2)
$$

where

$$\eta_0^2=\langle\phi_6\rangle^2+\langle\phi_7\rangle^2+\langle\phi_S\rangle^2,\qquad \eta_0'^2=\langle\phi_9\rangle^2+\langle\phi_{10}\rangle^2+\langle\phi_T\rangle^2. \tag{4.3}$$

We introduce the Hermitian, unitary matrix U_ϕ,

$$U_\phi=\begin{bmatrix}\cos\frac{1}{2}\beta_1' & 0 & 0 & \sin\frac{1}{2}\beta_1' e^{-i\alpha_1'}\\ 0 & \cos\frac{1}{2}\beta_1 & \sin\frac{1}{2}\beta_1 e^{-i\alpha_1} & 0\\ 0 & \sin\frac{1}{2}\beta_1 e^{i\alpha_1} & -\cos\frac{1}{2}\beta_1 & 0\\ \sin\frac{1}{2}\beta_1' e^{i\alpha_1'} & 0 & 0 & -\cos\frac{1}{2}\beta_1'\end{bmatrix}, \tag{4.4}$$

which diagonalizes $\langle\phi_{ab}\rangle$, so that

$$\langle\phi_{ab}\rangle=U_\phi\begin{bmatrix}\frac{1}{\sqrt{2}}\langle\phi_R\rangle+\eta_0' & 0 & 0 & 0\\ 0 & -\frac{1}{\sqrt{2}}\langle\phi_R\rangle+\eta_0 & 0 & 0\\ 0 & 0 & -\frac{1}{\sqrt{2}}\langle\phi_R\rangle-\eta_0 & 0\\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}\langle\phi_R\rangle-\eta_0'\end{bmatrix}U_\phi. \tag{4.5}$$

Thus $\langle\phi_{ab}\rangle$ has been written in a very compact form.

The most general renormalizable SU(4)-invariant potential for a single adjoint Higgs multiplet that is also for simplicity invariant under the reflection $\phi\to-\phi$ is

$$V(\phi)=-\mu_1^2\operatorname{Tr}(\phi^2)+\lambda_1[\operatorname{Tr}(\phi^2)]^2+\lambda_2\operatorname{Tr}(\phi^4). \tag{4.6}$$

We now minimize the neutral sector

$$V(\langle\phi\rangle)=-2\mu_1^2(\eta_0^2+\eta_0'^2+\langle\phi_R\rangle^2)+4\lambda_1(\eta_0^2+\eta_0'^2+\langle\phi_R\rangle^2)^2+\lambda_2(2\eta_0^4+2\eta_0'^4+\langle\phi_R\rangle^4+6\eta_0^2\langle\phi_R\rangle^2+6\eta_0'^2\langle\phi_R\rangle^2) \tag{4.7}$$

as a function of its three independent variables η_0, η_0', and $\langle\phi_R\rangle$. After minimizing we find that the location of the minimum of the potential depends on the sign of λ_2. (We take μ_1^2 and λ_1 to be positive.) For $\lambda_2<0$, the minimum surface is given by solutions in which the eigenvalues have a typical form $\sim(1,1,1,-3)$. This breaks SU(4) down to SU(3) × U(1), a result previously noted by Li.[6] On the other hand, the choice $\lambda_2>0$ leads to a minimum surface in which the eigenvalues are all equal to each other up to overall signs. An example is

$$\langle U_\phi\phi_{ab}U_\phi\rangle=\begin{bmatrix}\kappa & 0 & 0 & 0\\ 0 & \kappa & 0 & 0\\ 0 & 0 & -\kappa & 0\\ 0 & 0 & 0 & -\kappa\end{bmatrix}, \tag{4.8}$$

where the parameter κ is the solution to a cubic equation. This choice breaks SU(4) down to SU(2) × U(2) (Ref. 6), and is the one we want. Different assignments of relative signs among the eigenvalues in Eq. (4.8) then lead to different specific SU(2) × U(2) subgroups within the SU(4). Without loss of generality we shall choose the solution actually given in Eq. (4.8), where explicitly

$$\langle\phi_R\rangle=0,\quad \eta_0=\eta_0'. \tag{4.9}$$

As noted in Sec. III, we obtain the SU(2) × U(2) subgroup that we are specifically interested in if we set

$$\langle\phi_R\rangle=0,\qquad \langle\phi_S\rangle=\langle\phi_T\rangle\neq0,\qquad \langle\phi_6\rangle=\langle\phi_7\rangle=\langle\phi_9\rangle=\langle\phi_{10}\rangle=0. \tag{4.10}$$

This choice is then a particular solution to Eq. (4.9); it is, moreover, the simplest one since in it $\langle\phi_{ab}\rangle$ is already diagonal. This assignment of expectation values is not unique since it is possible, for instance, to have $\langle\phi_S\rangle$, $\langle\phi_T\rangle$, $\langle\phi_6\rangle$, and $\langle\phi_9\rangle$

all nonzero in the same multiplet. As we will show in paper III, this latter assignment leads to a nonzero Cabibbo angle through gauge-boson mixing. It is further possible to satisfy Eq. (4.9) by allowing the fields with opposite *CP* properties, i.e., ϕ_6 and ϕ_7 (and also ϕ_9 and ϕ_{10}) to simultaneously have nonvanishing expectation values. This leads to a theory of *CP* violation which will be fully discussed in paper II. However, for the purposes of this paper we shall only consider the simplest solution given above in Eq. (4.10).

In order to complete the breaking down to SU(2) we introduce the second adjoint, ψ. We parametrize its neutral sector as

$$\langle\psi_{ab}\rangle = U_\psi \begin{bmatrix} \frac{1}{\sqrt{2}}\langle\psi_R\rangle+\zeta_0' & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}}\langle\psi_R\rangle+\zeta_0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}}\langle\psi_R\rangle-\zeta_0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}\langle\psi_R\rangle-\zeta_0' \end{bmatrix} U_\psi , \tag{4.11}$$

where

$$\begin{aligned} \zeta_0^{\,2} &= \langle\psi_6\rangle^2+\langle\psi_7\rangle^2+\langle\psi_S\rangle^2 , \\ \zeta_0'^2 &= \langle\psi_9\rangle^2+\langle\psi_{10}\rangle^2+\langle\psi_T\rangle^2 \end{aligned} \tag{4.12}$$

and

$$U_\psi = \begin{bmatrix} \cos\frac{1}{2}\beta_2' & 0 & 0 & \sin\frac{1}{2}\beta_2' e^{-i\alpha_2'} \\ 0 & \cos\frac{1}{2}\beta_2 & \sin\frac{1}{2}\beta_2 e^{-i\alpha_2} & 0 \\ 0 & \sin\frac{1}{2}\beta_2 e^{i\alpha_2} & -\cos\frac{1}{2}\beta_2 & 0 \\ \sin\frac{1}{2}\beta_2' e^{i\alpha_2'} & 0 & 0 & -\cos\frac{1}{2}\beta_2' \end{bmatrix} . \tag{4.13}$$

The most general renormalizable SU(4)-invariant, reflection-invariant potential for the two adjoints is

$$\begin{aligned} V_1(\phi,\psi) = &-\mu_1^{\,2}\operatorname{Tr}(\phi^2)+\lambda_1[\operatorname{Tr}(\phi^2)]^2+\lambda_2\operatorname{Tr}(\phi^4) \\ &-\mu_2^{\,2}\operatorname{Tr}(\psi^2)+\lambda_3[\operatorname{Tr}(\psi^2)]^2+\lambda_4\operatorname{Tr}(\psi^4) \\ &+g_1\operatorname{Tr}(\phi^2\psi^2)+g_2\operatorname{Tr}(\phi^2)\operatorname{Tr}(\psi^2) \\ &+g_3[\operatorname{Tr}(\phi\psi)]^2+g_4\operatorname{Tr}(\phi\psi\phi\psi) , \end{aligned} \tag{4.14}$$

which we must now minimize afresh. Since $V_1(\phi,\psi)$ breaks SU(4) down beyond SU(2) × U(2) for arbitrary values of its parameters, in general we cannot maintain Eqs. (4.10). Thus we shall seek a specific range of values of the parameters for which $\langle\phi\rangle$ retains its previous structure while $\langle\psi\rangle$ completes the breaking of SU(4) down to SU(2), and no further. In our parametrization we find

$$\begin{aligned} V_1(\langle\phi\rangle,\langle\psi\rangle) = &-2\mu_1^{\,2}(\eta_0^{\,2}+\eta_0'^2+\langle\phi_R\rangle^2)+4\lambda_1(\eta_0^{\,2}+\eta_0'^2+\langle\phi_R\rangle^2)^2+\lambda_2(2\eta_0^{\,4}+2\eta_0'^4+\langle\phi_R\rangle^4+6\eta_0^{\,2}\langle\phi_R\rangle^2+6\eta_0'^2\langle\phi_R\rangle^2) \\ &-2\mu_2^{\,2}(\zeta_0^{\,2}+\zeta_0'^2+\langle\psi_R\rangle^2)+4\lambda_3(\zeta_0^{\,2}+\zeta_0'^2+\langle\psi_R\rangle^2)^2+\lambda_4(2\zeta_0^{\,4}+2\zeta_0'^4+\langle\psi_R\rangle^4+6\zeta_0^{\,2}\langle\psi_R\rangle^2+6\zeta_0'^2\langle\psi_R\rangle^2) \\ &+g_1[\langle\phi_R\rangle^2\langle\psi_R\rangle^2+\langle\phi_R\rangle^2(\zeta_0^{\,2}+\zeta_0'^2)+\langle\psi_R\rangle^2(\eta_0^{\,2}+\eta_0'^2)+2\eta_0^{\,2}\zeta_0^2+2\eta_0'^2\zeta_0'^2+4\langle\phi_R\rangle\langle\psi_R\rangle(\eta_0\zeta_0\cos\lambda+\eta_0'\zeta_0'\cos\lambda')] \\ &+4g_2(\eta_0^{\,2}+\eta_0'^2+\langle\phi_R\rangle^2)(\zeta_0^{\,2}+\zeta_0'^2+\langle\psi_R\rangle^2)+4g_3(\langle\phi_R\rangle\langle\psi_R\rangle+\eta_0\zeta_0\cos\lambda+\eta_0'\zeta_0'\cos\lambda')^2 \\ &+g_4[\langle\phi_R\rangle^2\langle\psi_R\rangle^2+\langle\phi_R\rangle^2(\zeta_0^{\,2}+\zeta_0'^2)+\langle\psi_R\rangle^2(\eta_0^{\,2}+\eta_0'^2)-2\eta_0^{\,2}\zeta_0^{\,2}-2\eta_0'^2\zeta_0'^2 \\ &\quad+4\eta_0^{\,2}\zeta_0^{\,2}\cos^2\lambda+4\eta_0'^2\zeta_0'^2\cos^2\lambda'+4\langle\phi_R\rangle\langle\psi_R\rangle(\eta_0\zeta_0\cos\lambda+\eta_0'\zeta_0'\cos\lambda')] , \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} \cos\lambda &= \cos\beta_1\cos\beta_2+\sin\beta_1\sin\beta_2\cos(\alpha_1-\alpha_2) , \\ \cos\lambda' &= \cos\beta_1'\cos\beta_2'+\sin\beta_1'\sin\beta_2'\cos(\alpha_1'-\alpha_2') . \end{aligned} \tag{4.16}$$

Here λ and λ′ are the "angles" between the two adjoints. The full potential $V_1(\langle\phi\rangle, \langle\psi\rangle)$ is thus a function of eight variables. We find that the solution

$$\eta_0=\eta_0', \quad \zeta_0=\zeta_0', \quad \langle\phi_R\rangle=\langle\psi_R\rangle=0, \quad \cos\lambda=\cos\lambda'=0 \tag{4.17}$$

satisfies the stationarity conditions. Moreover, by evaluating the eigenvalues of the eight-dimensional matrix V_1'', it can be established that the above solution is a local minimum of V_1 if

$$\text{all } g_i>0, \quad \text{all } \lambda_i \gg \text{all } g_i. \tag{4.18}$$

The set of conditions of Eqs. (4.17) is found to break SU(4) down to SU(2). In particular, we can satisfy Eqs. (4.17) by Eqs. (4.10) and by

$$\langle\psi_R\rangle=0, \quad \langle\psi_6\rangle=\langle\psi_9\rangle\neq 0, \quad \langle\psi_7\rangle=\langle\psi_{10}\rangle=\langle\psi_S\rangle=\langle\psi_T\rangle=0. \tag{4.19}$$

As noted in Sec. III, this then yields as unbroken the usual SU(2) that we want. The set of conditions of Eqs. (4.18) thus constitute the key dynamical assumptions of our work since they break SU(4) × U(1) down to the Weinberg-Salam theory.

It will be noted that not all of the parameters of $\langle\phi_{ab}\rangle$ and $\langle\psi_{ab}\rangle$ have been specified by Eqs. (4.17). The potential $V_1(\phi,\psi)$ fixes the relative orientations of the two adjoints, but does not fix their overall orientation with respect to the quark basis. In our theory the quarks get their masses through the fundamentals χ_i^s. Consequently, the remaining parameters of $\langle\phi_{ab}\rangle$ and $\langle\psi_{ab}\rangle$ and also the orientations of the gauge-boson mass eigenstates relative to the quark mass eigenstates can only be determined by introducing ϕ, ψ, χ, cross terms into the Higgs potential. Since these terms lead to a theory of the Cabibbo angle and of *CP* violation through gauge-boson mixing, we shall defer discussion of this point until paper III, and shall study first the phenomenology which ensues when we make very simple choices for the so far undetermined angles and parameters.

V. PROPERTIES OF THE SPONTANEOUSLY BROKEN THEORY

After having constructed the breaking pattern given above, we turn now to discuss its implications. The initial unbroken SU(4) × U(1) theory possesses the exact global symmetries of isospin, strangeness and charm, and their leptonic analogs. In the symmetry limit the only allowed processes are those which do not involve any quantum-number exchange. Thus processes such as $\bar{u}+d\to\bar{c}+s$, $\bar{u}+d\to\bar{\nu}_\mu+\mu$, and $\bar{\nu}_\mu+\mu\to\bar{\nu}_e+e$ are forbidden in the symmetry limit. [Since the quarks and leptons share SU(4) flavors, processes such as $\bar{u}+d\to\bar{\nu}_e+e$ are allowed.] In order for processes such as the usual muon β decay to take place, it is therefore necessary to break the SU(4) global symmetries spontaneously, and indeed it is the choice of nonvanishing $\langle\psi_6\rangle$ and $\langle\psi_9\rangle$ which takes the necessary quantum numbers into the vacuum. Thus the theory has to generate Goldstone bosons which are subsequently absorbed by the gauge bosons after the extension to a local gauge. This is to be contrasted with the situation in the Weinberg-Salam model. There muon β decay is allowed in the symmetry limit and is a long-ranged process, with the Higgs bosons' main role being to make the process short-ranged. In conventional theories the Goldstone bosons are needed to give masses to gauge bosons, whereas in our theory the gauge bosons are needed to remove the Goldstone bosons, without which there would be no muon-β-decay matrix element in the first place.

It is of interest to discuss which processes are allowed and which are forbidden at each stage of the breaking. After the breaking down to SU(2) × U(1) but before SU(2) × U(1) itself is broken we note that our theory already permits certain muon-number-violating processes to occur. As can be seen from Eq. (3.7) the "wrong" β decay, $\mu^-\to e^-+\nu_e+\bar{\nu}_\mu$, is allowed (even with $\theta_L=0$) through the exchange of the new heavy gauge bosons because of the lack of degeneracy of U and V exhibited in Eq. (3.4). Moreover, the usual β decay, $\mu^-\to e^-+\bar{\nu}_e+\nu_\mu$, is also a fermion number-violating process in the sense that in our theory the muon and its neutrino initially have separate quantum numbers. However, because of the degeneracy of the massless neutrinos, we were able to redefine new neutrino states in Eq. (3.6) so that within the subsector of the theory which only involves the $W_\pm$ and R bosons the muon would always appear in combination with its neutrino, and likewise for the electron. It is this latter property which is conventionally referred to as muon or electron number. Thus the usual β decay violates the SU(4) × U(1) quantum number associated with the muon itself, but not that associated with the muon and muon-neutrino combination. Further, if the neutrinos were to possess masses in our theory and mix through an angle different from that through which the electron and muon mix, then the couplings of $W_\pm$ would not even exhibit conservation of the quantum number associated with the combined muon and muon-neutrino system. In this respect the fact that β decay does exhibit this feature is thus a dynamical consequence of the masslessness of the two neutrinos.

It is important to note that while the theory also permits processes such as $\mu \to 3e$ and the μ-capture process $\mu + N \to e + N$ to take place, it is not until the $SU(2) \times U(1)$ symmetry is broken that the most interesting process, namely $\mu \to e\gamma$, can take place.[7] We will discuss the phenomenology of all these processes together in Sec. VI, but make the remark now to indicate that there are different aspects to muon-number nonconservation depending on how much the $SU(4) \times U(1)$ symmetry is broken.

It is possible to get some information on the masses of the new bosons from limits on rare decay processes. The most sensitive information comes from strangeness-changing processes involving the neutral kaons. The W_7 boson mediates the direct transition $K_L \to \mu e$. (In fact, this process is not forbidden in the unbroken theory since the μ, e system is equivalent to the s, d system.) The amplitude A for the process is given in a straightforward manner by allowing K_L to couple to the seventh axial-vector current, i.e.,

$$A(K_L \to \mu^+ e^-) = G_F \left(\frac{M_W}{M_7}\right)^2 \sqrt{2}\, F_K k^\mu l_\mu \,, \tag{5.1}$$

where F_K is the kaon decay constant ($F_K \approx 1.28\, F_\pi \approx 120$ MeV), $l_\mu = \bar{u}_e \gamma_\mu (1-\gamma_5) v_\mu$, and normalization factors are suppressed. The amplitude for the usual $K^+ \to \mu^+ \nu_\mu$ decay is given by

$$A(K^+ \to \mu^+ \nu_\mu) = G_F \sin\theta_C F_K k^\mu l_\mu \,. \tag{5.2}$$

Neglecting the mass of the electron, we find for the ratio of the widths

$$\frac{\Gamma(K_L \to \mu^+ e^-)}{\Gamma(K^+ \to \mu^+ \nu_\mu)} = \frac{2}{\sin^2\theta_C}\left(\frac{M_W}{M_7}\right)^4 . \tag{5.3}$$

From the experimental upper limit $\Gamma(K_L \to \mu^+ e^-)/\Gamma(K_L \to \text{all}) < 6 \times 10^{-9}$ we conclude that

$$M_7 > 450 M_W \,. \tag{5.4}$$

Another tight constraint can be derived from the magnitude of the $K_L - K_S$ mass difference. From our Lagrangian it can be seen that W_6 and W_7 respectively contribute to the self-energies of K_S and K_L. The contributions are easily calculated to be

$$\frac{m_L - m_S}{m_K} = \sqrt{2}\, G_F F_K^2 \left(\frac{M_W^2}{M_6^2} - \frac{M_W^2}{M_7^2}\right). \tag{5.5}$$

From the experimental value of $(m_L - m_S)/m_K \sim 7 \times 10^{-15}$, we conclude that

$$|M_7^{-2} - M_6^{-2}| < 3 \times 10^{-8} M_W^{-2} \,. \tag{5.6}$$

We have also considered other decay processes such as $K_S \to \mu^+\mu^-$ and $K^+ \to \pi^+\mu^+e^-$; however, they do not give any constaints stronger than Eqs. (5.4) and (5.6). These equations imply either that $M_6 \approx M_7 > 450 M_W$ or that both M_6 and M_7 are greater than $10^4 M_W$. From the mass term of Eq. (3.4) we obtain the mass sum rule

$$M_S^2 = M_7^2 - M_6^2 \,. \tag{5.7}$$

Consequently for M_7 and M_6 close enough the value for M_S could be substantially lower than 450 M_W.

An important feature of the particular formulation of our model given in Eq. (3.7) is that there is no $d-s$ mixing but rather the Cabibbo angle has been introduced in the $u-c$ sector of the mass matrix. While immaterial for the $SU(2) \times U(1)$ sector of the theory, we note that this choice leads to observable differences for the other interactions. Strictly speaking, the Cabibbo angle as measured in conventional weak interactions is only the difference between the amount of mixing in the $d-s$ sector and that in the $u-c$ sector. In the presence of the gauge bosons in our theory, each mixing angle becomes observable itself and not merely the difference. Thus we see that there are now two directions in which there is a mismatch between the strong and weak interactions, and not just one as in conventional theories. (In fact there are two further observable angles in general and their properties will be analyzed in paper III). With our choice of Cabibbo mixing in the $u-c$ sector the gauge boson S does not participate in the strangeness-changing processes, and is therefore free to be much lighter than the minimum $450 M_W$ already found in Eq. (5.4). Further information on the mass of S will be provided below when we study the $\mu \to e\gamma$ decay and related processes.

VI. $\mu \to e\gamma$ AND RELATED DECAYS

We turn our attention now to the $\mu \to e\gamma$ decay, a process which has generated a lot of current interest.[8] Since the discovery of the muon some 30 years ago and the gradual realization that it was decoupled from the electron sector by possessing its own quantum number, the whole question of the significance of the muon has been a mystery. Searches for nonconservation of muon number have repeatedly produced negative results. At the present time of writing there is no definitive evidence of any violation. Nonetheless, the subject of muon number is of deep theoretical interest since it concerns the nature and origin of symmetries in particle physics. In fact one of the more interesting aspects of the theory described in this series of papers, $SU(4) \times U(1)$, is that an understanding of muon number provides simultaneously some understanding of strangeness. Moreover, we have already indicated in Sec. V some of the consequences that follow once muon-number conservation is spontaneously broken in the effective interaction of the leptons with the gauge bosons. In this section we study the muon-number-violating processes that arise after the residual SU(2) × U(1) symmetry is broken. As we shall see, the $\mu \to e\gamma$ decay occurs, while the $\mu \to 3e$ decay and the μ-capture process receive significant new contributions. We thus discuss all three processes together.

For the applications of this section only the bosons of the SU(2) × U(2) × U(1) subgroup generated by $F_W^\pm$, F_R, $F_X^\pm$, $F_S - F_T$, $F_S + F_T$, and F_0 play any significant role as long as they are substantially lighter than $U_\pm$, W_6, W_9, $V_\pm$, W_7, and W_{10}. Following the mixings given in Eq. (3.8) which occur after the SU(2) × U(1) symmetry is broken, the interactions of the gauge bosons of the above subgroup with the mixed fermions are given from Eq. (3.7) by

$$\begin{aligned}\frac{\sqrt{2}}{g}\mathcal{L}_{\text{int}} = & -\sqrt{2}\sin\theta_W[A + \tan\theta_W(\cos\xi'\overline{Z} - \sin\xi'\overline{S})]J^{\text{em}} \\ & + \overline{Z}\{\cos\xi'(\sqrt{2}\cos\theta_W)^{-1}(\overline{\nu}_e\nu_e + \overline{\nu}_\mu\nu_\mu - \overline{e}e - \overline{\mu}\mu + \overline{u}u + \overline{c}c - \overline{d}d - \overline{s}s) \\ & \quad + \sin\xi'[\cos 2\theta_L(\overline{e}e - \overline{\mu}\mu) + \sin 2\theta_L(\overline{e}\mu + \overline{\mu}e) + \overline{d}d - \overline{s}s]\} \\ & + \overline{S}\{\cos\xi'[\cos 2\theta_L(\overline{e}e - \overline{\mu}\mu) + \sin 2\theta_L(\overline{e}\mu + \overline{\mu}e) + \overline{d}d - \overline{s}s] \\ & \quad - \sin\xi'(\sqrt{2}\cos\theta_W)^{-1}(\overline{\nu}_e\nu_e + \overline{\nu}_\mu\nu_\mu - \overline{e}e - \overline{\mu}\mu + \overline{u}u + \overline{c}c - \overline{d}d - \overline{s}s)\} \\ & + \overline{W}_+\{\cos\xi(\overline{\nu}_e e + \overline{\nu}_\mu\mu + \overline{u}'d + \overline{c}'s) \\ & \quad + \sin\xi[\cos 2\theta_L(\overline{\nu}_e e - \overline{\nu}_\mu\mu) + \sin 2\theta_L(\overline{\nu}_e\mu + \overline{\nu}_\mu e) + \overline{u}'d - \overline{c}'s]\} \\ & + \overline{X}_+\{\cos\xi[\cos 2\theta_L(\overline{\nu}_e e - \overline{\nu}_\mu\mu) + \sin 2\theta_L(\overline{\nu}_e\mu + \overline{\nu}_\mu e) + \overline{u}'d - \overline{c}'s] \\ & \quad - \sin\xi(\overline{\nu}_e e + \overline{\nu}_\mu\mu + \overline{u}'d + \overline{c}'s)\} \\ & + T[\cos 2\theta_L(\overline{\nu}_e\nu_e - \overline{\nu}_\mu\nu_\mu) + \sin 2\theta_L(\overline{\nu}_e\nu_\mu + \overline{\nu}_\mu\nu_e) + \overline{u}'u' - \overline{c}'c'] \\ & + \cdots + \text{H.c. of the charged sector}.\end{aligned} \tag{6.1}$$

The gauge bosons of Eq. (6.1) are the exact eigenstates of the mass matrix, so we can analyze the implications of Eq. (6.1) directly.

For $\mu \to e\gamma$ the effective interaction of the process is

$$\frac{e}{2m_\mu}\overline{u}_e(p)\sigma_{\alpha\beta}q^\beta(F_V + F_A\gamma_5)u_\mu(p+q)\epsilon^\alpha(q)\,, \tag{6.2}$$

its decay rate is given by

$$\Gamma(\mu \to e\gamma) = \tfrac{1}{8}\alpha(F_V{}^2 + F_A{}^2)m_\mu\,, \tag{6.3}$$

and the branching ratio is given by

$$\begin{aligned}R_{e\gamma} &= \frac{\Gamma(\mu \to e\gamma)}{\Gamma(\mu \to \text{all})} \\ &= \frac{24\pi^3\alpha(F_V{}^2 + F_A{}^2)}{G_F{}^2 m_\mu{}^4}.\end{aligned} \tag{6.4}$$

The form factors F_V and F_A receive contributions $F_{V,A}^C$ from the exchange of charged bosons, and $F_{V,A}^N$ from the exchange of neutral bosons. We show typical lowest-order contributions to $F_{V,A}^C$ and $F_{V,A}^N$ in Figs. 2 and 3, respectively. Note that each of the graphs contains two separate contributions coming from each of two leptons. Further, no matter what the exchange, as long as the SU(2) × U(1) symmetry remains unbroken, the leptons always mutually cancel each other in both $F_{V,A}^C$ and $F_{V,A}^N$. However, once the SU(2) × U(1) symmetry is broken, there are many noncanceling contributions, with $\overline{W}$ exchange and $\overline{Z}$ exchange being the dominant ones since they are the lowest mass states. We have calculated their contributions in the 't Hooft-Feynman gauge using the Feynman rules given in Ref. 9. We give here only the results of the calculation. For $\overline{W}$ exchange Fig. 2 yields the values

$$F_V^C = F_A^C = \frac{G_F}{\sqrt{2}}\frac{5m_\mu{}^2}{48\pi^2}\sin 2\theta_L \sin 2\xi\,, \tag{6.5}$$

while for $\overline{Z}$ exchange Fig. 3 yields the values

$$F_V^N = F_A^N = G_F\cos\theta_W\frac{(5C_V - 1)m_\mu{}^2}{48\pi^2}\sin 2\theta_L \sin 2\xi'\,, \tag{6.6}$$

where $C_V = 1 - 4\sin^2\theta_W \approx -0.2$. We then have to add

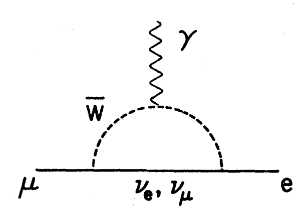

FIG. 2. Contribution to the $\mu \to e\gamma$ decay due to the exchange of a charged vector boson $\overline{W}$. Each of ν_e and ν_μ appears separately as an intermediate lepton.

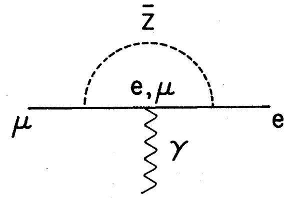

FIG. 3. Contribution to the $\mu \to e\gamma$ decay due to the exchange of a neutral vector boson $\bar{Z}$. Each of e and μ appears separately as an intermediate lepton.

these contributions to get the final values for F_V and F_A.

For $\mu \to 3e$ there are also two main contributions, namely direct $\bar{Z}$ and $\bar{S}$ exchange tree graphs, yielding

$$R_{3e} = \frac{\Gamma(\mu \to 3e)}{\Gamma(\mu \to \text{all})} = \tfrac{1}{2}(|A|^2 + |B|^2), \tag{6.7}$$

where

$$A = \frac{1}{2}\left(\frac{M_W}{M_S}\right)^2 \sin 4\theta_L - \frac{1}{2\sqrt{2}} \cos\theta_W (1 - 2\sin^2\theta_W) \sin 2\theta_L \sin 2\xi' \tag{6.8}$$

and

$$B = \frac{1}{\sqrt{2}} \cos\theta_W \sin^2\theta_W \sin 2\theta_L \sin 2\xi'. \tag{6.9}$$

For the μ-capture process again direct $\bar{Z}$ and $\bar{S}$ exchange tree graphs dominate yielding

$$R_{eN} = \frac{\sigma(\mu + N \to e + N)}{\sigma(\mu + N \to \nu + N')} = C\left(\frac{M_W}{M_S}\right)^4 \sin^2 2\theta_L (1+D)^2, \tag{6.10}$$

where

$$D = \frac{(C_V Z - N)}{2\sqrt{2}(Z + 2N)} \left(\frac{M_S}{M_W}\right)^2 \cos\theta_W \sin 2\xi'. \tag{6.11}$$

Here C is an enhancement factor which consists of a coherence factor $\frac{1}{2}(2N+Z)^2/(2Z+N)$ to be divided by a Pauli suppression factor $\sim 0.25(3Z - A)/A$.[10] For copper, for which there is currently some experimental information on R_{eN}, $C \sim 500$. (C also includes other effects such as form factors which we have not included.)

It is not possible to fix the values of M_S $(=M_X)$, ξ, ξ', and θ_L from measurements of $R_{e\gamma}$, R_{3e}, and R_{eN} alone. The extra constraint would come from processes such as $\nu_\mu + N \to \nu_e + N$ and $D_0 \to \bar{\nu}_e \nu_\mu$ which are mediated by the T boson (also degenerate with S and X). However, these processes are too impractical to be useful, so instead we will have to make further theoretical assumptions in order to extract out any useful information.

The simplest assumption is to keep $\xi' = 0$. We will explain below how this is achieved, but discuss first its implications. Though we now have three equations for three unknowns, we note that the system is in fact constrained. While we anticipate that the mixing of W and X is small, we note that the existence of such a mixing causes a shift in the mass of the W in the $W-X$ mass matrix given by

$$\sin^2\xi = \frac{2M_W \Delta M_W}{M_X{}^2}. \tag{6.12}$$

Since $\xi' = 0$ there is no corresponding shift in the Z mass, thus effectively decreasing the strength of neutral currents. The experimental uncertainty in M_Z/M_W requires the shift in M_W not to differ by more than 5% from the Weinberg-Salam theory.[11] This one piece of information constrains the mixing angle and leads to the inequalities

$$\rho > 2.2 \times 10^7 \sin^2\xi > 10^{10} R_{e\gamma}, \quad \rho^{1/2} > 1500 M_W/M_S, \tag{6.13}$$

where $\rho = R_{eN}/R_{e\gamma}$ and $N=$ copper. From the current upper bound[12] on R_{eN} of 1.6×10^{-8} we find that our model requires $R_{e\gamma}$ to be $< 10^{-9}$. While the preceding analysis is general in that it has not specified the actual mixing mechanism, a tighter constraint is obtained if we achieve the W-X mixing through the adjoint representation as was explained in Sec. III. From the structure of Table I we find that with such a mixing mechanism $\sin\xi < (M_W/M_X)^2$. This condition then leads to a very strong constraint, namely $\rho > 10^5$, so that $R_{e\gamma} < 10^{-14}$, a miserably small number.

Having discussed one simplifying assumption we now examine an alternative possibility. This time we allow both ξ and ξ' to be nonzero with the mixings being given entirely through the fundamentals. This is in fact the most natural situation since the fundamentals are responsible for SU(2) × U(1) breaking. We introduce

$$p = \tfrac{1}{4}g^2(\langle\chi_1^a\rangle^2 + \langle\chi_2^b\rangle^2 + \langle\chi_3^c\rangle^2 + \langle\chi_4^d\rangle^2), \quad q = \tfrac{1}{4}g^2(\langle\chi_2^b\rangle^2 - \langle\chi_3^c\rangle^2), \tag{6.14}$$

and set $\langle\chi_1^a\rangle^2 = \langle\chi_4^d\rangle^2$ so as to ensure no Z, T mixing. In terms of p and q we obtain the following mass matrices from Appendix B which determine ξ and ξ' respectively:

$$M_{WX} = \begin{pmatrix} p & q \\ q & M_X{}^2 \end{pmatrix} \tag{6.15}$$

and

$$M_{ZS}=\begin{pmatrix} p\sec^2\theta_W & -\sqrt{2}q\sec\theta_W \\ -\sqrt{2}q\sec\theta_W & M_S^2 \end{pmatrix}. \quad (6.16)$$

The only difference between Eqs. (6.15) and (6.16) is through the appearance of the Weinberg angle, which leads to the relation

$$\sin\xi'=-\frac{\sqrt{2}}{\cos\theta_W}\sin\xi. \quad (6.17)$$

We also note that

$$\sin\xi=x\left(\frac{M_W}{M_X}\right)^2<\left(\frac{M_W}{M_X}\right)^2, \quad (6.18)$$

where $x=q/p$. From these relations we obtain

$$R_{e\gamma}=\frac{\alpha}{24\pi}(7-10C_V)^2x^2\sin^2 2\theta_L\left(\frac{M_W}{M_S}\right)^4, \quad (6.19)$$

$$R_{3e}=\tfrac{1}{4}\sin^2 2\theta_L\left(\frac{M_W}{M_S}\right)^4 \times[2\cos^2 2\theta_L+2\cos 2\theta_L(1+C_V)x+(1+C_V^2)x^2], \quad (6.20)$$

$$R_{eN}=C\sin^2 2\theta_L\left[1+\frac{(N-C_VZ)x}{(2N+Z)}\right]^2\left(\frac{M_W}{M_S}\right)^4, \quad (6.21)$$

which are three equations for the unknowns θ_L, M_S, and x.

Since the above three equations are all dependent on $\sin^2 2\theta_L$ we are unable to obtain any limits on the masses of the S and $X_\pm$ bosons from them. Further information can be obtained from the charged sector of Eq. (6.1). The $\overline{X}_+$ boson contributes to the ratio of $K^-\to\mu^-\overline{\nu}_\mu$ to $K^-\to e^-\overline{\nu}_e$, a ratio which is explained completely by $\overline{W}_+$ exchange up to radiative corrections of $O(\alpha)$ in the amplitudes. In order that the $\overline{X}_+$ exchange will itself not be larger than the usual radiative corrections we require

$$\left|\left(\frac{1}{M_W^2}-\frac{\cos 2\theta_L}{M_X^2}\right)\Big/\left(\frac{1}{M_W^2}+\frac{\cos 2\theta_L}{M_X^2}\right)-1\right| < O(10^{-2}), \quad (6.22)$$

i.e.,

$$M_X^2>200M_W^2\cos 2\theta_L. \quad (6.23)$$

Noting that $|x|<1$ we can use the upper bound on R_{eN} to obtain the following relation from Eq. (6.21):

$$M_X^2>2\times10^5M_W^2\sin 2\theta_L. \quad (6.24)$$

Hence

$$M_X>14M_W, \quad (6.25)$$

which confirms that the mixing angle ξ is small. [The bound of Eq. (6.25) also ensures that processes such as $\pi\to\mu+\nu_e$ are kept within experimental limits.]

We can also extract out some theoretical inequalities relating $R_{e\gamma}$, R_{eN}, and R_{3e}. Since $R_{3e}>\frac{1}{2}|B|^2$ we obtain the inequality

$$\rho'=\frac{R_{3e}}{R_{e\gamma}}>\frac{48\pi\sin^4\theta_W}{\alpha(7-10C_V)^2}\sim 23, \quad (6.26)$$

and since $|x|<1$ we obtain

$$\rho=\frac{R_{eN}}{R_{e\gamma}}>\frac{24\pi C}{\alpha(7-10C_V)^2}. \quad (6.27)$$

Hence $\rho>7\times10^4$ for copper, so again $R_{e\gamma}<10^{-14}$, making its experimental observation rather remote. The relation in Eq. (6.26) is interesting in that ρ' is not of order α which would be expected if $\mu\to3e$ proceeds through photon conversion. Because our model possesses bosons which mediate $\mu\to3e$ directly, ρ' is three orders of magnitude larger than naively expected. This unexpectedly large value for ρ' is therefore one of the characteristic signals of our model.

Of the above quantities it is clear that R_{eN} is the most worthwhile to investigate experimentally. The large value obtained for ρ in Eq. (6.27) arises through two effects. First, in our model μ capture on a quark is larger than $\mu\to e\gamma$, and second, there is a large enhancement factor in going from a quark to a heavy nucleus. Thus it appears that μ capture on a nucleus should provide the best chance of seeing a muon-number-nonconserving process in the near future.

VII. CONCLUSIONS

It is useful to compare our work on muon-number nonconservation with other recent studies. Suggestions made thus far fall predominantly into two categories which involve either new leptons[13] or Higgs bosons.[14] In our approach we have enlarged the gauge group, a proposal also made in Ref. 15 where a gauge group other than SU(4) × U(1) was considered. What distinguishes our work from the other proposals in the main is that there is an explicit spontaneous breakdown of muon-number conservation in the couplings of the leptons to the gauge bosons. In the absence of any experimental information, it is not possible to distinguish between the various models, though some of the signatures of our model such as the large values for the ratios $R_{eN}/R_{e\gamma}$ and R_{eN}/R_{3e} given in Sec. VI are not to be typically expected in the other models considered in the literature. Indeed, because of the large value obtained for $R_{eN}/R_{e\gamma}$ we recommend that the μ-capture process be experimentally explored in the near future; it (and also possibly $K_L\to\mu e$) appears to be the most promising way of detecting any muon-number nonconservation. Though one of the main points of our study is that (unlike the proposals in Ref. 13) it is not necessary to require the existence of new leptons in order to understand processes such as $\mu \to e\gamma$, it will be very interesting in the future to see whether new leptons can be experimentally incorporated into a symmetry scheme which is far more restrictive than the Weinberg-Salam model.

When a quantum number (such as muon number) is found to exist experimentally, it is not sufficient merely to inquire how well the quantum number is conserved, or whether it is possible to construct theories in which it is weakly violated. A physical theory should also examine how the existence of the quantum number relates to other phenomena and what its violation can teach us about the structure of the theory. This is the viewpoint we have adopted here and in this series of papers, and it is our belief that rather than something to be avoided, muon-number nonconservation is a highly desirable feature for possible theories of the weak interaction.

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APPENDIX A

In the canonical basis the algebra of SU(4) is given as

$$[F_i, F_j] = if_{ijk}F_k\,, \quad i,j,k = 1, \ldots, 15 \tag{A1}$$

where the structure constants are given, for example, in Ref. 16. A four-dimensional representation of Eq. (A1) is the λ matrices which are given in Ref. 17. From Eq. (A1) we derive the following commutation relations among the generators introduced in Eq. (2.2):

$$\begin{aligned}
&[F_W^+, F_W^-] = 2F_R, && [F_W^+, F_X^-] = 2(F_T - F_S),\\
&[F_W^+, F_U^-] = 2(F_9 - F_6), && [F_W^+, F_V^-] = -2i(F_{10} - F_7),\\
&[F_X^+, F_X^-] = 2F_R, && [F_X^+, F_U^-] = 2i(F_{10} + F_7),\\
&[F_X^+, F_V^-] = -2(F_9 + F_6), && [F_U^+, F_U^-] = 2F_R,\\
&[F_U^+, F_V^-] = 2(F_T + F_S), && [F_V^+, F_V^-] = 2F_R,\\
&[F_R, F_W^+] = F_W^+, && [F_T - F_S, F_W^+] = F_X^+,\\
&[F_9 - F_6, F_W^+] = F_U^+, && [F_{10} - F_7, F_W^+] = -iF_V^+,\\
&[F_R, F_X^+] = F_X^+, && [F_T - F_S, F_X^+] = F_W^+,\\
&[F_9 + F_6, F_X^+] = -F_V^+, && [F_{10} + F_7, F_X^+] = iF_U^+.\\
&[F_R, F_U^+] = F_U^+, && [F_T + F_S, F_U^+] = F_V^+,\\
&[F_9 - F_6, F_U^+] = F_W^+, && [F_{10} + F_7, F_U^+] = -iF_X^+,\\
&[F_R, F_V^+] = F_V^+, && [F_T + F_S, F_V^+] = F_U^+,\\
&[F_9 + F_6, F_V^+] = -F_X^+, && [F_{10} - F_7, F_V^+] = iF_W^+,\\
&[F_T - F_S, F_9 - F_6] = i(F_{10} + F_7),\\
&[F_T - F_S, F_9 + F_6] = i(F_{10} - F_7),\\
&[F_T - F_S, F_{10} - F_7] = -i(F_9 + F_6),\\
&[F_T - F_S, F_{10} + F_7] = -i(F_9 - F_6)\\
&[F_T + F_S, F_9 - F_6] = i(F_{10} - F_7),\\
&[F_T + F_S, F_9 + F_6] = i(F_{10} + F_7),\\
&[F_T + F_S, F_{10} - F_7] = -i(F_9 - F_6),\\
&[F_T + F_S, F_{10} + F_7] = -i(F_9 + F_6),\\
&[F_9 - F_6, F_{10} - F_7] = i(F_T + F_S),\\
&[F_9 - F_6, F_{10} + F_7] = i(F_T - F_S),\\
&[F_9 + F_6, F_{10} - F_7] = i(F_T - F_S),\\
&[F_9 + F_6, F_{10} + F_7] = i(F_T + F_S).
\end{aligned} \tag{A2}$$

Apart from the relations which follow from Hermitian conjugation all other commutators vanish.

APPENDIX B

Using the structure constants of Appendix A we can calculate the couplings of the gauge bosons to an adjoint representation of Higgs fields, ϕ_i say.

TABLE I. Mass matrix of charged gauge bosons.

	W_-	X_-	U_-	V_-
W_+	$\frac{1}{2}(\phi_R^2 + \phi_A^2 + \phi_C^2 + \phi_G^2)$	$\phi_R\phi_A + \frac{1}{2}i(\phi_C\phi_H - \phi_D\phi_G)$	$\phi_R\phi_C + \frac{1}{2}i(\phi_B\phi_G - \phi_A\phi_H)$	$\frac{1}{2}i(\phi_B\phi_C - \phi_A\phi_D) + i\phi_R\phi_G$
X_+		$\frac{1}{2}(\phi_R^2 + \phi_A^2 + \phi_D^2 + \phi_H^2)$	$\frac{1}{2}(\phi_A\phi_C - \phi_B\phi_D) - i\phi_R\phi_H$	$-\phi_R\phi_D + \frac{1}{2}i(\phi_A\phi_G - \phi_B\phi_H)$
U_+	Hermitian conjugate		$\frac{1}{2}(\phi_R^2 + \phi_B^2 + \phi_C^2 + \phi_H^2)$	$\phi_R\phi_B + \frac{1}{2}i(\phi_C\phi_G - \phi_D\phi_H)$
V_+				$\frac{1}{2}(\phi_R^2 + \phi_B^2 + \phi_D^2 + \phi_G^2)$

The Hermitian mass matrix $M_{ij}(\phi)$ of Eq. (2.5) contains two pieces which involve the neutral Higgs fields, one for the charged gauge bosons and the other for the neutral bosons. The mass matrix for the charged gauge bosons is given in Table I in terms of the convenient notation

$$\phi_A=\frac{1}{\sqrt{2}}(\phi_T-\phi_S),\quad \phi_B=\frac{1}{\sqrt{2}}(\phi_T+\phi_S),$$
$$\phi_C=\frac{1}{\sqrt{2}}(\phi_9-\phi_6),\quad \phi_D=\frac{1}{\sqrt{2}}(\phi_9+\phi_6), \tag{B1}$$
$$\phi_G=\frac{1}{\sqrt{2}}(\phi_{10}-\phi_7),\quad \phi_H=\frac{1}{\sqrt{2}}(\phi_{10}+\phi_7),$$

with

$$\phi_R=\frac{1}{\sqrt{2}}\left[\phi_3+\frac{1}{\sqrt{3}}\phi_8-(\tfrac{2}{3})^{1/2}\phi_{15}\right],$$
$$\phi_S=\tfrac{1}{2}(-\phi_3+\sqrt{3}\,\phi_8), \tag{B2}$$
$$\phi_T=\frac{1}{2}\left[\phi_3+\frac{1}{\sqrt{3}}\phi_8+2(\tfrac{2}{3})^{1/2}\phi_{15}\right].$$

The mass matrix for the neutral gauge bosons is given in Table II which uses the convenient notation

$$W_A=\frac{1}{\sqrt{2}}(T-S),\quad W_B=\frac{1}{\sqrt{2}}(T+S),$$
$$W_C=\frac{1}{\sqrt{2}}(W_9-W_6),\quad W_D=\frac{1}{\sqrt{2}}(W_9+W_6), \tag{B3}$$
$$W_G=\frac{1}{\sqrt{2}}(W_{10}-W_7),\quad W_H=\frac{1}{\sqrt{2}}(W_{10}+W_7).$$

The couplings of the fundamental Higgs fields to the gauge bosons contribute the following term to the Lagrangian:

$$\left(\frac{g}{\sqrt{2}}\lambda^i_{kl}W_i\chi^s_l+g'Y_sW_0\chi^s_k\right)^\dagger\left(\frac{g}{\sqrt{2}}\lambda^j_{km}W_j\chi^s_m+g'Y_sW_0\chi^s_k\right), \tag{B4}$$

where i,j are summed from $1,\ldots,15$; k,l,m from $1,\ldots,4$; s from $a,\ldots,d$. Here Y_s are eigenvalues of F_0 for the fundamentals χ^s, and are given by $-Y_a=Y_b=Y_c=-Y_d=\frac{1}{2}$. As explained in Appendix C we can choose the potential such that the nonvanishing expectation values in the fundamentals are

$$\langle\chi^a_1\rangle,\quad \langle\chi^b_2\rangle,\quad \langle\chi^c_3\rangle,\quad \langle\chi^d_4\rangle.$$

With this choice the mass term is

$$\begin{aligned}\frac{2}{g^2}\mathcal{L}_{\text{mass}}=&[\langle\chi^a_1\rangle^2+\langle\chi^b_2\rangle^2+\langle\chi^c_3\rangle^2+\langle\chi^d_4\rangle^2](2W_+W_-+2X_+X_-+2U_+U_-+2V_+V_-+Z^2\sec^2\theta_W)\\&+[\langle\chi^a_1\rangle^2+\langle\chi^d_4\rangle^2](W_9{}^2+W_{10}{}^2+T^2)+[\langle\chi^b_2\rangle^2+\langle\chi^c_3\rangle^2](W_6{}^2+W_7{}^2+S^2)\\&+[\langle\chi^a_1\rangle^2+\langle\chi^b_2\rangle^2-\langle\chi^c_3\rangle^2-\langle\chi^d_4\rangle^2]\left[W_+X_-+W_-X_++\frac{1}{\sqrt{2}}(T-S)Z\sec\theta_W\right]\\&+[\langle\chi^a_1\rangle^2-\langle\chi^b_2\rangle^2+\langle\chi^c_3\rangle^2-\langle\chi^d_4\rangle^2]\left[V_+U_-+V_-U_++\frac{1}{\sqrt{2}}(T+S)Z\sec\theta_W\right].\end{aligned} \tag{B5}$$

APPENDIX C

The most general SU(4) × U(1)-invariant coupling of the fermions to the fundamental Higgs fields is

$$\begin{aligned}\mathcal{L}_{f\chi}=&(g_{u,a}\bar{u}_R+g_{c,a}\bar{c}_R)(\chi^{a\dagger}_1,\chi^{a\dagger}_2,\chi^{a\dagger}_3,\chi^{a\dagger}_4)q_L+(g_{d,b}\bar{d}_R+g_{s,b}\bar{s}_R)(\chi^{b\dagger}_1,\chi^{b\dagger}_2,\chi^{b\dagger}_3,\chi^{b\dagger}_4)q_L\\&+(g_{s,c}\bar{s}_R+g_{d,c}\bar{d}_R)(\chi^{c\dagger}_1,\chi^{c\dagger}_2,\chi^{c\dagger}_3,\chi^{c\dagger}_4)q_L+(g_{c,d}\bar{c}_R+g_{u,d}\bar{u}_R)(\chi^{d\dagger}_1,\chi^{d\dagger}_2,\chi^{d\dagger}_3,\chi^{d\dagger}_4)q_L\\&+(g_{e,b}\bar{e}_R+g_{\mu,b}\bar{\mu}_R)(\chi^{b\dagger}_1,\chi^{b\dagger}_2,\chi^{b\dagger}_3,\chi^{b\dagger}_4)l_L+(g_{\mu,c}\bar{\mu}_R+g_{e,c}\bar{e}_R)(\chi^{c\dagger}_1,\chi^{c\dagger}_2,\chi^{c\dagger}_3,\chi^{c\dagger}_4)l_L+\text{H.c.},\end{aligned} \tag{C1}$$

where

$$q_L=\begin{bmatrix}u\\d\\s\\c\end{bmatrix}_L,\quad l_L=\begin{bmatrix}\nu_e\\e\\\mu\\\nu_\mu\end{bmatrix}_L. \tag{C2}$$

There are eight neutral Higgs fields which can in principle acquire vacuum expectation values: χ^a_1, χ^a_4, χ^b_2, χ^b_3, χ^c_2, χ^c_3, χ^d_1, χ^d_4. The most general SU(4) × U(1)-invariant potential for the Higgs fields is

$$V_2(\chi)=\sum_{s=a,b,c,d}(-\mu_s{}^2\chi^{s\dagger}\chi^s)-\mu_{ad}{}^2\chi^{a\dagger}\chi^d-\mu_{bc}{}^2\chi^{b\dagger}\chi^c+\sum_{s,t,u,v}\lambda^{st}_{uv}(\chi^{s\dagger}\chi^t)(\chi^{u\dagger}\chi^v)+\kappa\det(\chi^a\chi^b\chi^c\chi^d)+\text{H.c.} \tag{C3}$$

TABLE II. Mass matrix of neutral gauge bosons.

	W_A	W_B	W_C	W_D	W_G	W_H
W_A	$\frac{1}{2}(\phi_C^2+\phi_D^2+\phi_G^2+\phi_H^2)$	$\phi_C\phi_D+\phi_G\phi_H$	$-\frac{1}{2}(\phi_A\phi_C+\phi_B\phi_D)$	$-\frac{1}{2}(\phi_A\phi_D+\phi_B\phi_C)$	$-\frac{1}{2}(\phi_A\phi_G+\phi_B\phi_H)$	$-\frac{1}{2}(\phi_A\phi_H+\phi_B\phi_G)$
W_B		$\frac{1}{2}(\phi_C^2+\phi_D^2+\phi_G^2+\phi_H^2)$	$-\frac{1}{2}(\phi_A\phi_D+\phi_B\phi_C)$	$-\frac{1}{2}(\phi_A\phi_C+\phi_B\phi_D)$	$-\frac{1}{2}(\phi_A\phi_H+\phi_B\phi_G)$	$-\frac{1}{2}(\phi_A\phi_G+\phi_B\phi_H)$
W_C			$\frac{1}{2}(\phi_A^2+\phi_B^2+\phi_G^2+\phi_H^2)$	$\phi_A\phi_B+\phi_G\phi_H$	$-\frac{1}{2}(\phi_C\phi_G+\phi_D\phi_H)$	$-\frac{1}{2}(\phi_C\phi_H+\phi_D\phi_G)$
W_D				$\frac{1}{2}(\phi_A^2+\phi_B^2+\phi_G^2+\phi_H^2)$	$-\frac{1}{2}(\phi_C\phi_H+\phi_D\phi_G)$	$-\frac{1}{2}(\phi_C\phi_G+\phi_D\phi_H)$
W_G		Hermitian conjugate			$\frac{1}{2}(\phi_A^2+\phi_B^2+\phi_C^2+\phi_D^2)$	$\phi_A\phi_B+\phi_C\phi_D$
W_H						$\frac{1}{2}(\phi_A^2+\phi_B^2+\phi_C^2+\phi_D^2)$

Here the coefficients λ_{uv}^{st} are summed over all $\{a, b, c, d\}$ and possess some obvious restrictions coming from conservation of the $U(1)$ quantum number Y. (The $\mu_{ad}{}^2$ and $\mu_{bc}{}^2$ terms will play no role in the following since they only lead to a mixing of the Higgs fields which we will assume has already been made.)

In order to simplify the Higgs potential we shall impose two separate additional discrete symmetries on it, viz.,

$$\{\chi^b \to e^{i(\pi/4)}\chi^b,\ \chi^c \to e^{-i(\pi/4)}\chi^c\} \tag{C4}$$

and

$$\{\chi^a \to e^{i(\pi/4)}\chi^a,\ \chi^d \to e^{-i(\pi/4)}\chi^d\}. \tag{C5}$$

We can always choose the basis for the χ_i^s so that

$$\langle\chi^a\rangle = \chi_0^a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\qquad \langle\chi^b\rangle = \chi_0^b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{C6}$$

In that basis the most general structures for $\langle\chi^c\rangle$ and $\langle\chi^d\rangle$ are

$$\langle\chi^c\rangle = \chi_0^c e^{i\psi_1} \begin{bmatrix} 0 \\ \sin\theta_1 e^{i\phi_1} \\ \cos\theta_1 \\ 0 \end{bmatrix},\qquad \langle\chi^d\rangle = \chi_0^d e^{i\psi_2} \begin{bmatrix} \sin\theta_2 e^{i\phi_2} \\ 0 \\ 0 \\ \cos\theta_2 \end{bmatrix}, \tag{C7}$$

where χ_0^s ($s = a, b, c, d$) is the normalization of each expectation value. In the above basis we find that

$$\begin{aligned} V_2(\langle\chi\rangle) = {} & \sum_s (-\mu_s^2\chi_0^{s\,2}) \\ & + \sum_s \lambda_{ss}^{ss}\chi_0^{s\,4} + \sum_{s,t} \lambda_{tt}^{ss}\chi_0^{s\,2}\chi_0^{t\,2} \\ & + \lambda_{da}^{ad}\chi_0^{a^2}\chi_0^{d^2}\sin\theta_2 \\ & + \lambda_{cb}^{bc}\chi_0^{b\,2}\chi_0^{c\,2}\sin^2\theta_1 \\ & + \kappa\chi_0^a\chi_0^b\chi_0^c\chi_0^d\cos\theta_1\cos\theta_2\cos(\psi_1+\psi_2)\,. \end{aligned} \tag{C8}$$

A minimum of $V_2(\langle\chi\rangle)$ is found in which

$$\theta_1=\theta_2=0\,,\quad \psi_1+\psi_2=0 \tag{C9}$$

if

$$\text{all } \lambda_{ss}^{ss} \gg \text{all other } \lambda_{uv}^{st} \geq -\kappa > 0\,. \tag{C10}$$

With this choice the minimization of $V_2(\chi)$ provides the orthogonal set of vacuum values used in Appendix B. (The phases ψ_1 and ψ_2 are not observable in the couplings of the Higgs fields to the gauge bosons.) Moreover, explicit counting shows that $V_2(\chi)$ now generates the 15 Goldstone bosons which are gauged in Eq. (B5).

In passing we also remark that without the additional discrete symmetries, the parameters in Eq. (C3) can be chosen to give the same breaking pattern as given in Eq. (C9). Further, the potential generates the same set of Goldstone bosons as before but causes a rediagonalization of the massive Higgs fields that remain following the gauging of the massless fields. It is this rediagonalization which can lead to flavor-changing processes mediated by the massive Higgs fields, and the discrete symmetries of Eqs. (C4) and (C5) have been introduced specifically to prevent this.

With the above choice of $\langle\chi^s\rangle$, Eq. (C1) leads to quark mixing in the conventional manner, while also allowing for μ-e mixing through the lepton angle θ_L. However, we would like to have mixing in the u-c sector but not in the d-s sector. This requires

$$g_{s,b}=g_{d,c}=0\,, \tag{C11}$$

which we shall impose phenomenologically by extending the discrete symmetry of Eq. (C4) to the right-handed d and s quarks, viz.,

$$\{\chi^b \to e^{i(\pi/4)}\chi^b,\ \chi^c\to e^{-i(\pi/4)}\chi^c,$$
$$d_R\to e^{-i(\pi/4)}d_R,\ s_R\to e^{i(\pi/4)}s_R\}\,. \tag{C12}$$

This leads to Eq. (C11) and there is thus no d-s mixing in the mass matrix. It should be noted that though χ_3^b does not acquire an expectation value in our basis, it still couples to $\bar{d}_R s_L$, so there are strangeness-changing vertices. However, χ_3^b has to be contracted back with an appropriate Higgs scalar in order to give a strangeness-changing process, and our discrete symmetries have been chosen specifically to prevent this. For instance, the K_S-K_L mass difference is propagated by a term $\chi_3^b\chi_2^{c\dagger}$ while our mass matrix only contains terms such as $\chi_3^b\chi_2^c$. Thus there are no strangeness-changing processes mediated by Higgs scalars in our theory, though there are of course charm-changing ones through the presence of the $g_{c,a}$ and $g_{u,d}$ terms.[18] The rates of such processes when available will constrain the masses of the Higgs bosons.

The meaning of the basis of Eqs. (C6) and (C7) relative to the basis of the gauge-boson eigenstates will be discussed in more detail in paper III, where the discrete symmetries will be extended to all the quarks ($g_{c,a}=g_{u,d}=0$ also) and thus become exact properties of the theory to all orders. In such a situation the theory is completely flavor conserving and the Cabibbo angle will be introduced by a totally different procedure from the conventional one employed in this paper.

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[3]See, for example, C. H. Albright, C. Quigg, R. E. Shrock, and J. Smith, Phys. Rev. D 14, 1780 (1976) and references therein for a recent analysis of the Weinberg-Salam model; see also G. Goldhaber, in *Weak Interactions at High Energy and the Production of New Particles*, proceedings of SLAC Summer Institute on Particle Physics, 1976, edited by M. C. Zipf (SLAC, Stanford, 1977), p. 379.

[4]We are aware of the possibility of the existence of new quarks and leptons; such particles can be incorporated into other representations of the SU(4)× U(1) symmetry.

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