PHYSICAL REVIEW D VOLUME 19, NUMBER 1 1 JANUARY 1979

K_{l3} form factors

A. O. Barut
Department of Physics and Astrophysics, The University of Colorado, Boulder, Colorado 80309

Raj Wilson
School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin-4, Ireland
and Department of Theoretical Physics, University of Madras, Madras-600025, India

(Received 28 March 1978)

The decay form factors $f_{\pm}(t)$ for the process $K^{\pm}\to\pi^0 l^{\pm}\nu_l(\bar{\nu}_l)$ have been evaluated in closed form within the framework of the rest-frame relativistic symmetry SO(4,2) ⊗ SU(3) ⊗ SU(3). Symmetry breaking is introduced via Lorentz boosting. The resulting explicit expressions and conclusions are in agreement with experiment and contain, in a unified and general manner, the successful results obtained in various other models based on current algebra, Kemmer fields, S-matrix analysis, and algebraic methods.

I. INTRODUCTION

With recent high-statistics data[1,2] the experimental investigation on the semileptonic kaon decays has come to a temporary conclusion. It is not likely that new experiments will be forthcoming in the near future. It is appropriate therefore to reexamine all the theoretical implications to see if a unified picture emerges. The basic motivation underlying these investigations is to obtain an understanding of the dynamical symmetry breaking and information on the structure of hadrons.[3,4] For example, in the most extensively used model based on chiral algebra [with the usual assumptions of conserved vector current (CVC) and partially conserved axial-vector current (PCAC)] there is a crucial uncertainty, whether the symmetry is broken according to SU(3)⊗SU(3) → SU(2) ⊗SU(2) → SU(2) (strong PCAC), or according to SU(3)⊗ SU(3) → SU(3) → SU(2) (so-called "weak PCAC" or pole dominance of the divergence of the axial-vector current). Gaillard[4,5] has shown that this uncertainty is closely related to the sign of the invariant quantity $\xi(t)$ in the K_{l3} decay $[K^{\pm}\to\pi^0 l^{\pm}\nu_l(\bar{\nu}_l)]$ defined in Sec. II. The experiments seem to favor a negative sign for this quantity, which in turn implies that SU(3) is a better symmetry than SU(2)⊗ SU(2).

Aside from the sign of $\xi(t)$, the theory has to explain the complete form factors in the K_{l3} decays. There are a number of other investigations with some "success": The (3, 3*)⊗ (3*, 3) model,[6] models based on hard-meson techniques,[7] and those based on the Kemmer formalism.[8] In our investigation, these successful results will emerge as part of a general picture, and we shall compare various new and earlier results with experiment and with each other.

The basic idea in our theory of the K_{l3} form factors is the use of the relativistic "wave functions" for the hadrons, which allows us to obtain closed expressions for the complete functional form of the form factors. This method has been successfully applied to the prediction of the proton electromagnetic form factor.[9] In the case of the weak interactions the assumptions of the theory are as follows[10].

(1) Hadron states at rest are like the totality of infinitely many excited states of an atom; specifically, they are given by the concept of an internal dynamical symmetry. We assign them to an irreducible unitary representation of the dynamical symmetry group SO(4, 2).

(2) For moving hadrons the states are obtained by relativistic boost operators. This allows us to define precisely relativistic wave functions.

(3) In the rest frame the symmetry is unbroken, and the weak current is the direct product of a term acting on the internal quantum numbers and a term acting on the space-time quantum numbers. The symmetry breaking arises from the boost of the hadrons to their respective masses essentially, and to a lesser degree, from the so called "tilt" of the "physical states" relative to the "group states".

In fact the theory is a generalization of the standard weak-interaction theory: instead of taking the currents between the Dirac four-dimensional SO(4, 2) spinors, we take them between the infinite-dimensional SO(4, 2) spinors which more accurately describe a hadron and permit automatically the calculation of form factors.

This simple framework gives a good overall picture of all the electromagnetic properties of the proton, and as we shall see here, a good overall picture of the K_{l3} form factors.

In Sec. II (and Appendix I) we derive exact expressions for the form factors $f_{\pm}(t)$, hence for $\xi(t)$ and $f_0(t)$. The whole computation boils down to calculating matrix elements of Lorentz boosts between the SO(4, 2) states which are by now exactly known,[11] hence the form factors can be evaluated in closed form.

In Sec. III we discuss various implications of the results and compare them with other theories and experiment.

II. THE THEORY

A. Decay matrix elements

In the framework of the standard ($V-A$) theory the transition matrix element for the decay process $K^{\pm} \to \pi^0 l^{\pm} \nu_l(\bar{\nu}_l)$ is given by[4]

$$M = \langle \pi^0 | V_\mu^{4\mp i5} | K^{\pm} \rangle \langle l^{\pm} | A^\mu | \nu_l(\bar{\nu}_l) \rangle. \tag{2.1}$$

The $|\Delta Y| = 1$ hadronic matrix element of the vector current V_μ itself can be factorized into an internal symmetry part and a dynamical part

$$\begin{aligned}\langle \pi^0 | V_\mu^{4-i5} | K^+ \rangle &= \frac{G}{\sqrt{2}} \sin\theta \langle \alpha_{\pi^0} p_{\pi^0} | V_\mu | \alpha_{K^+} p_{K^+} \rangle \\ &\quad \times \langle \underline{8}(II_3Y)_{\pi^0} | (V^4 - iV^5) | \underline{8}(II_3Y)_{K^+} \rangle \frac{1}{\sqrt{2}} \\ &= i\frac{G}{2} \sin\theta \langle \alpha_{\pi^0} p_{\pi^0} | V_\mu | \alpha_{K^+} p_{K^+} \rangle. \end{aligned} \tag{2.2}$$

Here G is the Fermi constant, θ is the Cabibbo angle, and $(\alpha_{\pi^0}, p_{\pi^0})$ is the collection of internal quantum numbers (including spin) and four-momentum for π^0, and (α_{K^+}, p_{K^+}) are the corresponding quantities for K^+. In the SU(3)-octet multiplet $|\underline{8};(II_3Y)_{\pi^0}\rangle = |\underline{8}, 100\rangle$ and $|\underline{8}, (II_3Y)_{K^+}\rangle = |\underline{8}; \frac{1}{2}, -\frac{1}{2}, 1\rangle$, and the SU(3)-reduced matrix element is calculated to be $i/\sqrt{2}$. By virtue of the $|\Delta I| = \frac{1}{2}$ property of the conserved vector current V_μ, one can write

$$\langle \alpha_{\pi^0} p_{\pi^0} | V_\mu | \alpha_{K^+} p_{K^+} \rangle = (4p_{\pi^0}^0 p_{K^+}^0)^{-1/2} [P_\mu f_+(t) + q_\mu f_-(t)], \tag{2.3}$$

with $P_\mu = (p_{K^+} + p_{\pi^0})$, $q_\mu = (p_{K^+} - p_{\pi^0})$, $t = q^2$; $\langle p' | p \rangle = 2p^0 \delta^3(\vec{p} - \vec{p}')$. Equation (2.3) defines the form factors $f_\pm(t)$.

B. Space-time part of the matrix elements

One starts with the set of states $|nlm\rangle$ with quantum numbers n, l, m as in the H atom. These states span a unitary irreducible representation of the dynamical group SO(4, 2).[12] In this space of states the physical states are given by

$$|\alpha\rangle = \frac{1}{N_\alpha} e^{-i\theta_\alpha T} |nlm\rangle, \tag{2.4}$$

where θ_α are the so-called tilt angles, T is the dilation generator of SO(4, 2), and N_α are the normalization factors. The 15 generators of SO(4, 2) are the following: $\vec{J}$ = angular momentum, $\vec{A}$ = the Lenz vector, $\vec{M}$ = the Lorentz boosts, Γ_μ = current four-vector, T = dilation, and S = scalar.

The pion π and kaon K are assigned[12] to be the ground states of two SO(4, 2) towers of states distinguished by the internal quantum numbers, i.e., $|n=1, l=0, m=0\rangle$. Thus

$$|\pi^0\rangle = N_\pi^{-1} e^{-i\theta_\pi T} |100\rangle, \quad |K^+\rangle = N_K^{-1} e^{-i\theta_K T} |100\rangle. \tag{2.5}$$

The physical states $|\alpha\rangle$ are solutions of a wave equation

$$(J_\mu P^\mu + \beta S + \gamma) |\alpha\rangle = 0, \tag{2.6}$$

where β and γ are parameters and S is the scalar generator of SO(4, 2). The tilt operation $\exp(-i\theta_\alpha T)$ in Eq. (2.4) is due to the term βS in the wave equation. Here J_μ is a conserved vector current which we take, as in many previous works, as

$$J_\mu \equiv V_\mu = \alpha_1 \Gamma_\mu + (\alpha_2 + \alpha_3 S) P_\mu, \tag{2.7}$$

where $\alpha_1, \alpha_2, \alpha_3$ are constant parameters and Γ_μ are the vector generators of SO(4, 2). The angles θ_α are related to the parameters α_i, β, γ by the covariant subsidiary conditions

$$\begin{aligned}(\alpha_2 P_\mu P^\mu + \gamma)\cosh\theta_\alpha &= -\alpha_1 (P_\mu P^\mu)^{1/2} \Gamma_0, \\ (\alpha_2 P_\mu P^\mu + \gamma)\sinh\theta_\alpha &= -(\alpha_3 P_\mu P^\mu + \beta)\Gamma_0, \end{aligned} \tag{2.8}$$

so that the theory has four parameters altogether; but then the complete functional forms of the form factors are determined.

With (2.4) and (2.6) the matrix elements become

$$\begin{aligned}F_\mu &\equiv \langle \alpha_{\pi^0} p_{\pi^0} | V_\mu | \alpha_{K^+} p_{K^+} \rangle = P_\mu f_+(t) + q_\mu f_-(t) \\ &= (4p_{\pi^0}^0 p_{K^+}^0)^{-1/2} N_\pi^{-1} N_K^{-1} \langle 100 p_{\pi^0} | e^{i\theta_\pi T} (\alpha_1 \Gamma_\mu + \alpha_2 P_\mu + \alpha_3 P_\mu S) e^{-i\theta_K T} | 100 p_{K^+} \rangle. \end{aligned} \tag{2.9}$$

Hence, in the rest frame of the kaon $[p_{K^+} = (m_K, 0, 0, 0),\ p_{\pi^0} = (p_{\pi^0}^0, 0, 0, -p_{\pi^0}^3)]$,

$$f_-(t) = (q_0 P_3 - q_3 P_0)^{-1} (P_3 F_0 - P_0 F_3), \quad f_+(t) = (q_3 P_0 - q_0 P_3)^{-1} (q_3 F_0 - q_0 F_3). \tag{2.10}$$

Thus the invariant form factors are simply related to the matrix elements F_0 and F_3 which have been explicitly evaluated in Appendix I. Using these we have the final most general results:

$$f_-(t) = N_\pi^{-1}N_K^{-1}\frac{1}{m_K m_\pi}\alpha_1(m_\pi\cosh\theta_K - m_K\cosh\theta_\pi)\frac{1}{[2\cosh^2(\frac{1}{2}\beta)]^2} \tag{2.11}$$

$$f_+(t) = N_\pi^{-1}N_K^{-1}\frac{1}{m_K m_\pi}\left\{\alpha_1(m_\pi\cosh\theta_K + m_K\cosh\theta_\pi)\frac{1}{[2\cosh^2(\frac{1}{2}\beta)]^2}\right.$$
$$\left. + 2\alpha_2 m_\pi m_K\frac{1}{(2\cosh^2\frac{1}{2}\beta)} - 2\alpha_3 m_\pi m_K(\sinh\theta_\pi + \sinh\theta_K)\frac{1}{[2\cosh^2(\frac{1}{2}\beta)]^2}\right\}. \tag{2.12}$$

The t dependence of the form factors is contained entirely in $\cosh^2(\frac{1}{2}\beta)$:

$$2\cosh^2(\tfrac{1}{2}\beta) = (2m_\pi m_K)^{-1}\Delta(1 - \Delta^{-1}\cosh\theta_\pi\cosh\theta_K t), \tag{2.12$'$}$$

where

$$\Delta \equiv (m_\pi^2 + m_K^2)\cosh\theta_\pi\cosh\theta_K + 2m_K m_\pi(1 - \sinh\theta_\pi\sinh\theta_K). \tag{2.12$''$}$$

The normalization factors N_π, N_K for the states are determined from the condition $\langle\alpha|j_0|\alpha\rangle = 2m$ to be

$$\begin{aligned}2m_\pi N_\pi^2 &= \alpha_1\cosh\theta_\pi + 2\alpha_2 m_\pi - 2\alpha_3 m_\pi\sinh\theta_\pi,\\ 2m_K N_K^2 &= \alpha_1\cosh\theta_K + 2\alpha_2 m_K - 2\alpha_3 m_K\sinh\theta_K.\end{aligned} \tag{2.13}$$

We also give the following general expressions:

$$\xi(t) \equiv \frac{f_-(t)}{f_+(t)} = (m_\pi\cosh\theta_K - m_K\cosh\theta_\pi)\left[(m_\pi\cosh\theta_K + m_K\cosh\theta_\pi) - 2\frac{\alpha_3}{\alpha_1}m_\pi m_K(\sinh\theta_\pi + \sinh\theta_K)\right.$$
$$\left. + \frac{2\alpha_2}{\alpha_1}m_\pi m_K[2\cosh^2(\tfrac{1}{2}\beta)]\right]^{-1}, \tag{2.14}$$

$$f_0(t) \equiv f_+(t) + \frac{t}{m_K^2 - m_\pi^2}f_-(t)$$
$$= N_\pi^{-1}N_K^{-1}\frac{1}{m_\pi m_K}\left\{\alpha_1\left[\left(1 + \frac{t}{m_K^2 - m_\pi^2}\right)m_\pi\cosh\theta_K + \left(1 - \frac{t}{m_K^2 - m_\pi^2}\right)m_K\cosh\theta_\pi\right]\frac{1}{[2\cosh^2(\frac{1}{2}\beta)]^2}\right.$$
$$\left. + 2\alpha_2 m_\pi m_K\frac{1}{2\cosh^2(\frac{1}{2}\beta)} - 2\alpha_3 m_\pi m_K(\sinh\theta_\pi + \sinh\theta_K)\frac{1}{[2\cosh^2(\frac{1}{2}\beta)]^2}\right\}. \tag{2.15}$$

III. CONSEQUENCES

A. Exact pole and dipole behavior of form factors

From (2.11) and (2.12) the form factors can be written as

$$\begin{aligned}f_-(t) &= f_-(0)\left(1 - \frac{\lambda_-}{2m_\pi^2}t\right)^{-2},\\ \lambda_- &\equiv 2m_\pi^2\cosh\theta_\pi\cosh\theta_K/\Delta,\\ f_+(t) &= f_+(0)\left(1 - \frac{\lambda_-}{2m_\pi^2}t\right)^{-2}\\ &\quad - \frac{\alpha_2}{N_\pi N_K\Delta'}\frac{\lambda_-}{2m_\pi^2}t\left(1 - \frac{\lambda_-}{2m_\pi^2}t\right)^{-2},\\ \Delta' &\equiv \Delta/4m_\pi m_K.\end{aligned} \tag{3.1}$$

Thus $f_-(t)$ is *exactly* a dipole form factor, while $f_+(t)$ has both pole and dipole terms. Recall that similar behavior is found in the electromagnetic form factors of the proton.[9] In certain current-algebra models, pole and dipole forms are assumed.[13]

B. Expansion of the form factors

For $|(\lambda_-/2m_\pi^2)t| \ll 1$, we can write

$$\begin{aligned}f_-(t) &= f_-(0)\sum_{n=1}^{\infty} n\left(\frac{\lambda_-}{2m_\pi^2}t\right)^{n-1}\\ &\cong f_-(0)\left(1 + \frac{\lambda_-}{m_\pi^2}t\right),\\ f_+(t) &= f_+(0)\sum_{n=1}^{\infty} n\left(\frac{\lambda_-}{2m_\pi^2}t\right)^{n-1}\\ &\quad - \frac{\alpha_2}{N_\pi N_K\Delta'}\sum_{n=1}^{\infty}\left(\frac{\lambda_-}{2m_\pi^2}t\right)^{n}\\ &\cong f_+(0)\left(1 + \frac{\lambda_+}{m_\pi^2}t + \frac{\lambda_+'}{m_\pi^4}t^2\right),\\ \lambda_+ &\equiv \lambda_-[1 - \alpha_2/2N_\pi N_K\Delta' f_+(0)],\\ \lambda_+' &\equiv \tfrac{3}{4}\lambda_-^2[1 - 2\alpha_2/3N_\pi N_K\Delta' f_+(0)].\end{aligned} \tag{3.2}$$

Experimental data suggest $\lambda_+ \neq \lambda_-$. In the present theory the difference $(\lambda_+ - \lambda_-)$ is proportional to $\alpha_2 \neq 0$ which has implications on the mass spectrum of the meson towers (cf. Sec. III G).

C. Exact-SU(3) limit

In the limit of an exact SU(3) symmetry, $m_K = m_\pi = m$ and $\theta_K = \theta_\pi = \theta$; hence, $N_K = N_\pi = N$. Consequently,

$$f_-(0) = 0,$$
$$f_+(t) = \left(1 + \frac{\alpha_2}{N^2}\sinh^2(\tfrac{1}{2}\beta)\right)\frac{1}{\cosh^4(\tfrac{1}{2}\beta)} \neq 0. \tag{3.3}$$

This is, of course, what one would expect as a consequence of current conservation.

D. Glashow-Weinberg limit, κ meson, F_K and F_π, Ademollo-Gatto theorem

At $t = 0$, we find

$$f_-(0) = 0 + O(\epsilon_8)$$
$$= N_\pi^{-1} N_K^{-1}\alpha_1(m_\pi\cosh\theta_K - m_K\cosh\theta_\pi)4m_\pi m_K/\Delta^2,$$
$$f_+(0) = 1 + O(\epsilon_8^2)$$
$$= \frac{1}{2F_\pi F_K}\left[\left(\frac{F_K}{\Delta'}\right)^2 + \left(\frac{F_\pi}{\Delta'}\right)^2 - F_\kappa^2\right], \tag{3.4}$$
$$F_\pi \equiv N_\pi, \quad F_K \equiv N_K, \quad F_\kappa^2 \equiv 2\alpha_2\left[\left(\frac{1}{\Delta'}\right)^2 - \left(\frac{1}{\Delta'}\right)\right],$$

where ϵ_8, as in Ref. 4, is a parameter used to define the strength of symmetry breaking. Here we have identified N_π and N_K, by virtue of the definition of charge (2.13), with the pion and kaon decay constants F_π and F_K, although we have not established here a formal connection between current-algebra formalism and ours. (For such a connection see, however, Ref. 14). By comparing $f_+(0)$ with the Glashow-Weinberg relation[6] we have also factorized the expression for the decay constant F_κ of the so-called Goldstone κ meson. In the exact SU(3) limit $f_-(0) = 0$, $F_\pi = F_K$, $\Delta = 4m^2$, $\Delta' = 1$, $F_\kappa^2 = 0$, $f_+(0) = 1$, as it should be. Also $O(\epsilon_8^2)$ is indeed of "second order" in SU(3)-symmetry-breaking parameters F_κ and $(F_K - F_\pi)$, in agreement with the Ademollo-Gatto theorem.[15] Our expression for $f_+(0)$ is in a sense a generalized version of the Glashow-Weinberg relation. Note also that the term F_κ^2 is again proportional to α_2. Since $\alpha_2 \neq 0$, our model naturally admits a contribution which may be identified with a term due to a Goldstone κ meson, which in the exact-symmetry limit automatically vanishes.

We also find the ratio

$$F_K^2/F_\pi^2 = 1 + O(\epsilon_8) = 1 + [m_K(\alpha_1\cosh\theta_\pi + 2\alpha_2 m_\pi - 2\alpha_3 m_\pi\sinh\theta_\pi)]^{-1}[\alpha_1(m_\pi\cosh\theta_K - m_K\cosh\theta_\pi) - 2\alpha_3 m_\pi m_K(\sinh\theta_K - \sinh\theta_\pi)], \tag{3.5}$$

i.e., first order in the symmetry-breaking parameter, in agreement with Cabibbo theory.[3,4]

E. The ξ parameter

From (2.14) we obtain explicitly

$$\xi(t) = (m_\pi\cosh\theta_K - m_K\cosh\theta_\pi)[(m_\pi\cosh\theta_K + m_K\cosh\theta_\pi) - (2\alpha_3/\alpha_1)m_\pi m_K(\sinh\theta_K + \sinh\theta_\pi) + (\alpha_2/\alpha_1)(\Delta - \cosh\theta_K\cosh\theta_\pi t)]^{-1},$$
$$\frac{1}{\xi(t)} = \frac{1}{\xi(0)} + \frac{1}{\xi(0)}(\lambda_+ - \lambda_-)\frac{t}{m_\pi^2}, \quad \xi(0) = \frac{f_-(0)}{f_+(0)}. \tag{3.6}$$

Thus $1/\xi(t)$ is exactly linear in t. However, the commonly assumed linear relationship $\xi(t) \cong \xi(0) + \xi(0)(\lambda_- - \lambda_+)t/m_\pi^2$ holds approximately, using the expansions (3.2).

In the limit $\theta_\pi = \theta_K = \theta = 0$, we get

$$\xi((m_K + m_\pi)^2) = -\frac{(m_K - m_\pi)}{(m_K + m_\pi)}. \tag{3.7}$$

This is precisely the result obtained by Fishbach *et al.*[8] in Kemmer formalism, and Böhm and Werle[16] in an algebraic framework.

Furthermore, in the soft-pion limit, $m_\pi^2 = 0$, we obtain from (3.6)

$$\xi(m_K^2) = -1 + O(m_\pi^2). \tag{3.8}$$

It has been shown that[17] this relation is also a consequence of the pion-gauge condition, i.e., $\lim M = 0$, as $p_\mu^\pi \to 0$ [see (2.1)].

The general expression for $\xi(t)$ is of the form $a/(b - t)$ and shows only a weak t dependence, in agreement with experiment[2] (Fig. 2).

Note that, instead of the Callan-Treiman relation, $f_+(m_K^2) + f_-(m_K^2) = F_K/F_\pi$, we find

$$f_+(m_K^2) + f_-(m_K^2) = 0 + O(m_\pi^2). \tag{3.9}$$

We discuss this relation further in the Sec. III F.

F. Zeros of $f_0(t)$ and the relation to S-matrix analysis

The scalar form factor $f_0(t)$,

$$f_0(t) = f_0(0)\left(1 - \frac{\lambda_-}{2m_\pi^2}\right)^{-2} + g(0)\frac{\lambda_-}{2m_\pi^2}t\left(1 - \frac{\lambda_-}{2m_\pi^2}t\right)^{-2},$$

where

$$f_0(0)=f_+(0), \quad g(0)\equiv f_-(0)\,\frac{2m_\pi^2}{\lambda_-}(m_K^2-m_\pi^2)^{-1}-\alpha_2/N_\pi N_K\Delta',$$

can be expanded as

$$f_0(t)=f_0(0)\sum_{n=1}^{\infty}n\left(\frac{\lambda_-}{2m_\pi^2}t\right)^{n-1}+g(0)\sum_{n=1}^{\infty}n\left(\frac{\lambda_-}{2m_\pi^2}t\right)^{n}, \quad \left|\frac{\lambda_-}{2m_\pi^2}t\right|\ll 1,$$

$$\cong f_0(0)\left(1+\frac{\lambda_0}{m_\pi^2}t+\frac{\lambda_0'}{m_\pi^4}t^2\right),$$

with

$$\lambda_0\equiv\lambda_+ +\left(\frac{m_\pi^2}{m_K^2-m_\pi^2}\right)\xi(0), \quad \lambda_0'\equiv\lambda_+'+\lambda_-\left(\frac{m_\pi^2}{m_K^2-m_\pi^2}\right)\xi(0). \tag{3.10}$$

We see that $f_0(t)$ has a zero at

$$\begin{aligned} t=\{&\alpha_1(m_\pi\cosh\theta_K+m_K\cosh\theta_\pi)-2\alpha_3 m_\pi m_K(m_K^2-m_\pi^2)(\sinh\theta_K+\sinh\theta_\pi)\\ &+\alpha_2(m_K^2-m_\pi^2)[(m_K^2+m_\pi^2)\cosh\theta_K\cosh\theta_\pi+2m_\pi m_K(1-\sinh\theta_\pi\sinh\theta_K)]\}\\ &\times[\alpha_1(m_K\cosh\theta_\pi-m_\pi\cosh\theta_K)+\alpha_2(m_K^2-m_\pi^2)\cosh\theta_\pi\cosh\theta_K]^{-1}. \end{aligned} \tag{3.11}$$

According to an S matrix analysis due to Kang,[18] the zero of $f_0(t)$ must lie between $t_-=(m_K-m_\pi)^2$ and $t_+=(m_K+m_\pi)^2$ in order to satisfy the result $\xi(m_K^2)\cong -1$. In our case, indeed these two limiting values are obtained by the limiting values of the tilt angles: $(\theta_\pi,\theta_K)=0$ gives from (3.11) $t\cong t_+$, and $(\theta_\pi,\theta_K)=\infty$ gives $t\cong t_-$. Further, the zero point of $f_0(t)$ between t_- and t_+ implies[18] that the Callan-Treiman relation which holds only for the soft-pion limit would not extrapolate smoothly to the physical points. However, the relation (3.9) does smoothly extrapolate to the physical points.

G. The roles of the α_2 and α_3 terms in the current

Turning to the basic form of the current V_μ, Eq. (2.7), we know from previous studies of hadron structure that the α_2 and α_3 terms express the composite structure of hadrons or atoms, besides the algebraic term $\alpha_1\Gamma_\mu$ (which is analogous to the Dirac current γ_μ). The terms $\alpha_1\Gamma_\mu$ and $\alpha_2 P_\mu$ give rise to a linear mass spectrum, whereas the α_3 term implies a saturation of the mass spectrum ($m\to m_{\mathrm{sat}}$, as $n\to\infty$) as in the H atom. The larger the factor α_3/α_2, the earlier sets the saturation.[12]

The effect of a nonzero α_2 term can be seen from the conclusions:

(i) $\xi(t)\neq\xi(0)=$ constant; i.e., the t dependence is due to the α_2 term.

(ii) $\lambda_+-\lambda_-\neq 0$; i.e., $(\lambda_+-\lambda_-)$ is proportional to the α_2 term.

(iii) The decay constant of the κ meson F_κ is proportional to the α_2 term.

The mass spectra of the excited states of the meson and kaon towers, as well as the electromagnetic properties of π^0 and K^+, could determine the parameters $\alpha_1,\alpha_2,\alpha_3$ as well as θ_π and θ_K. But this information is so far lacking. However, at low masses, a linear mass spectrum is a good approximation; hence, $\alpha_3=0$ is an adequate simplifying assumption.

H. Comparison with experiment

The experimental conclusions on the decay form factors are unfortunately not quite conclusive, and in many respects, somewhat controversial. While the data[1,2] for $f_+(t')$, $t'=-t/m_\pi^2$, appear to be free from any typical t-dependent input, the data for $\xi(t')$ and $f_0(t')$ are generally obtained by using linear fits for $f_+(t')=f_+(0)(1+\lambda_+t')$ as input, and the extracted data for $\xi(t')$ and $f_0(t')$ are fitted generally to linear expressions. In our theory such linear relations will be satisfied only for small values of t, and hence the constant parameters $(\theta_\pi,\theta_K,\alpha_1,\alpha_2)$ in our model may not be exactly determined with the help of the existing experimental data. However, we find that, even by approximate values of these parameters, our final conclusions are in good agreement with the experimental analysis.

For convenience we write our final results (with the simplifying assumption $\alpha_3=0$) in terms of $t'=-t/m_\pi^2$ as follows:

$$\frac{f_+(t')}{f_+(0)}=\frac{1+\kappa\frac12\lambda_-t'}{(1+\frac12\lambda_-t')^2}, \quad \kappa\equiv\alpha_2/[N_\pi N_K\Delta' f_+(0)],$$

$$\frac{1}{\xi(t')}=\frac{1}{\xi(0)}[1-(\lambda_+-\lambda_-)t'], \tag{3.12}$$

$$f_0(t')=\frac{f_+(0)+g(0)\frac12\lambda_-t'}{(1+\frac12\lambda_-t')^2}.$$

In order to obtain some idea about the parameter λ_-, we first assume $\theta_\pi=\theta_K=0$, which gives λ_-

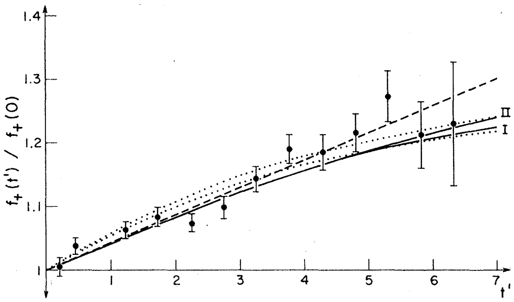

FIG. 1. The normalized form factor $f_+(t')$ against $t' = -t/M_\pi^2$, Eq. (3.12). The dotted lines are for $K = 3.5$ (upper curve), $K=3.4$, and $\lambda_- = 0.095$. The curves I and II are, respectively, for $K=4.1$, $\lambda_- = 0.04$ and $K=4.9$, $\lambda_- = 0.03$. The dashed line is the UCLA-SLAC-JH fit: $f_+(t') = 1 + 0.044t'$; $f_+(0) = 1$. Result: $\lambda_+ = 0.045$ (I) and $\lambda_+ = 0.043$ (II).

$=0.095$. For this value, the effective (or leading) pole of $f_\pm(t)$ occurs at $m_K \equiv (2/\lambda_-)^{1/2} m_\pi \cong 4.59 m_\pi = 633$ MeV, which is remarkably close to the $K\pi^-$ threshold, 630 MeV, and, if m_π is identified with the Goldstone κ meson, then this value is consistent with the inequality $m_\kappa \leq 670$ MeV.[6] This shows that any pole dominance much above the threshold can be obtained only by taking $\theta_\pi \neq \theta_K$ or $(\theta_\pi, \theta_K) \neq 0$—purely as a consequence of this dynamical theory.

Next, with the value of $\lambda_- = 0.095$, we fix the parameter K in (3.12) using a point (preferably the point through which the experimental fit passes) from the UCLA-SLAC-JH data[2] as input. In Fig. 1 we show two such fits (dotted lines) for $K = 3.5$ (upper curve) and $K = 3.4$. The fitted curves, as we expect, are not linear, and the function $f_+(t')/f_+(0)$ reaches its maximum value approximately at $t' \approx 9$ and then decreases asymptotically to zero as $t' \to \infty$. From these two fits we come to know that, although the curves accommodate the data points reasonably well, the value $\lambda_- = 0.095$ is too large to bring the curves much closer to the straight line (experimental fit[2]) for reasonably small values of t'. This indicates that λ_- may be much smaller than 0.095, and, consequently, we should get away from the crude assumption $\theta_\pi \cong \theta_K \cong 0$. Or, in other words, the (leading) pole dominance should occur much higher than the $K\pi^-$ threshold. This conclusion is consistent with certain chiral algebraic models in which, for example, the K^* pole ($M_{K^*} \cong 891$ MeV, hence $\lambda_{K^*} \cong 0.048$) and the K pole ($M_\kappa \simeq 1200$ MeV, hence $\lambda_K \simeq 0.026$) dominances are assumed. Furthermore, smaller values of λ_- in our case make the higher-order (in t') terms relatively negligible so that the comparison with the experimental analysis could be much more justified.

In view of the above observations, we choose the

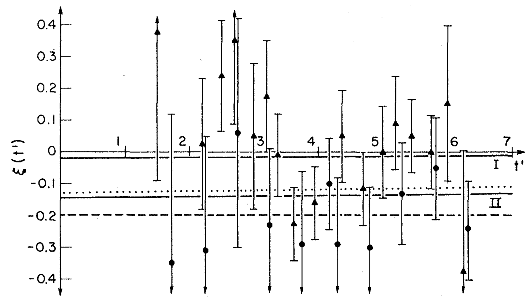

FIG. 2. The form-factor ratio $\xi(t')$ against t' Eq. (3.12). ○ are UCLA-SLAC-JH data points, Δ are SLAC-UCSC data points (approximately). Both are taken in the range $-0.45 \leq \xi(t') \leq +0.45$. The curves I and II are, respectively, for $\xi(0) = -0.018$ and $\xi(0) = -0.14$. The dotted line is drawn using SLAC-UCSC values: $\lambda_+ = 0.03$, $\lambda_0 = 0.019$, and their Eq. (10). The broken line is the constant fit of UCLA-SLAC-JH. Result: $\lambda_0 = 0.043$ (I) and $\lambda_0 = 0.031$ (II).

following two sets of values:

I: $\lambda_- = 0.04$, $\kappa = 4.1$,

$f_+(0) = 0.95$, $\xi(0) = -0.018$,

II: $\lambda_- = 0.03$, $\kappa = 4.9$,

$f_+(0) = 0.90$, $\xi(0) = -0.14$.

These four parameters, in principle, replace the four parameters θ_π, θ_K, α_1, and α_2, that we have in this theory. The choice of the two different values of $\xi(0)$ is motivated by the unparametrized and two-parameter fits of the SLAC-UCSC data.[11] The predicted curves are shown in Figs. (1–3), and they show reasonably good agreement with the data.

The curves for $f_+(t')/f_+(0)$ are very close to the linear fit of UCLA-SLAC-JH for small values of t', and then they deviate from the straight line as t' increases. The function $f_+(t')/f_+(0)$ increases to its maximum value at $t' \simeq 25.6$ (I) and 39.5 (II) and

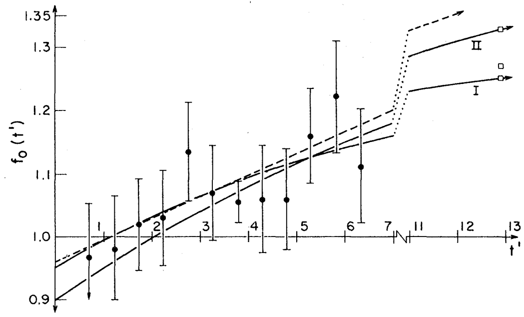

FIG. 3. The scalar form factor $f_0(t')$ against t' Eq. (3.12). The curves I and II are, respectively, for $\lambda_0 = 0.043$ and $\lambda_0 = 0.031$. The broken line is the linear fit of UCLA-SLAC-JH. □ denotes Callan-Treiman point.

then decreases asymptotically to zero as $t' \to \infty$. The predicted curves for $\xi(t')$ show that this ratio is almost a constant, which is compatible with different experimental fits. The scalar form factor $f_0(t')$ peaks at $t' \simeq 27.2$ (I) and 43.3 (II) and decreases to zero as $t' \to \infty$. At $t' = M_K^2/M_\pi^2 \simeq 12.9$, $f_0(t') \simeq 1.25$ (I) and 1.33 (II), which is very close to the Callan-Treiman prediction that $f_0(M_K^2/M_\pi^2) = 1.27$ and the experimental value 1.38.[2] Also, at $t' = (M_K^2 - M_\pi^2)/M_\pi^2 \simeq 11.9$ we get $f_0(t') \cong 1.24$ (I) and 1.3 (II), which is in good agreement with the SLAC-UCSC value 1.22.[1]

Finally, using the values of λ_-, K, $f_+(0)$, and $\xi(0)$, we determine the remaining parameters (Note that, according to our definition of the λ parameters [(3.1), (3.2), (3.10)], they differ from the experimental slope parameters by a negative sign. We have included this sign below):

I: $\lambda_+ = 0.045$, $\lambda_0 = 0.043$, $f_-(0) = -0.0166$,

II: $\lambda_+ = 0.043$, $\lambda_0 = 0.031$, $f_-(0) = -0.129$.

The values of λ_+ and λ_0 are in good agreement with the UCLA-SLAC-JH data[2]: $\lambda_+ = 0.044$, $\lambda_0 = 0.032$, and they are slightly higher with respect to the SLAC-UCSC data[1]: $\lambda_+ = 0.03$ and $\lambda_0 = 0.019$.

In conclusion, we have given a complete description of the functional forms of the K_{l3} decay form factors with a relativistic wave-function formalism which further relates these decay parameters to the mass spectrum and electromagnetic form factors of the mesons and their excited states.

APPENDIX I. EVALUATION OF THE MATRIX ELEMENTS

From the rest-frame states $|\alpha\rangle$,(2.4), the boosted states $|\alpha, p\rangle$, as defined by the wave equation, are given by

$$|\alpha_\pi, p\rangle = \exp(-i\xi_\pi M_3)|\alpha\rangle, \tag{A1}$$

where ξ_π is the boost angle such that $p_\pi^0 = m_\pi \cosh\xi_\pi$, $p_\pi^3 = m_\pi \sinh\xi_\pi$, and M_3 is the Lorentz generator in SO(4,2).

Thus the matrix element F_0 in (2.9) is given by

$$\begin{aligned}
F_0 &= N_\pi^{-1} N_K^{-1} \langle 100 | e^{i\theta_\pi T} e^{i\xi_\pi M_3} (\alpha_1 \Gamma_0 + \alpha_2 P_0 + \alpha_3 P_0 S) e^{-i\theta_K T} | 100\rangle \\
&= N_\pi^{-1} N_K^{-1} \{[\alpha_1 \cosh\theta_K - \alpha_3(m_K + m_\pi \cosh\xi_\pi)\sinh\theta_K + \alpha_2(m_K + m_\pi\cosh\xi_\pi)]\langle 100|G|100\rangle \\
&\qquad - (1/\sqrt{2})[\alpha_1 \sinh\theta_K - \alpha_3(m_K + m_\pi\cosh\xi_\pi)\cosh\theta_K]\langle 100|G|200\rangle\}, \\
G &\equiv e^{i\theta_\pi T} e^{i\xi_\pi M_3} e^{-i\theta_K T}.
\end{aligned} \tag{A2}$$

Here we used various Lie-algebra relations such as $\exp(i\theta T)\Gamma_0(\exp(-i\theta T) = \Gamma_0\cosh\theta - S\sinh\theta$, and $\Gamma_0|100\rangle = |100\rangle$, $S|100\rangle = (1/\sqrt{2})|200\rangle$, $\Gamma_3|100\rangle = (i/\sqrt{2})|210\rangle$, fully described elsewhere. The matrix elements of G between the group states is now known generally[11]:

$$\langle n'l'm'|G|nlm\rangle = \sum_{L=0}^{\min\{n'-1,\,n-1\}} \delta_{m'm} \mathfrak{D}^{[n'-1,\,0]}_{l'mL}(\alpha) V^{L+1}_{n'n}(\beta) \mathfrak{D}^{[n-1,\,0]}_{Lml}(\gamma),$$

where

$$\begin{aligned}
&2\cosh^2(\tfrac{1}{2}\beta) = \cosh\theta_K \cosh\theta_\pi \cosh\xi_\pi - \sinh\theta_K \sinh\theta_\pi + 1, \\
&2\sinh^2(\tfrac{1}{2}\beta) = \cosh\theta_K \cosh\theta_\pi \cosh\xi_\pi - \sinh\theta_K \sinh\theta_\pi - 1, \\
&\sin\alpha \sinh\beta = \cosh\theta_K \sinh\xi_\pi, \quad \sin\gamma \sinh\beta = -\cos\theta_\pi \sinh\xi_\pi, \\
&\cos\alpha \sinh\beta = \sinh\theta_K \cosh\theta_\pi - \cosh\theta_K \sinh\theta_\pi \cosh\xi_\pi, \quad \cos\gamma \sinh\beta = \sinh\theta_K \cosh\theta_\pi \cosh\xi_\pi - \cosh\theta_K \sinh\theta_\pi.
\end{aligned} \tag{A3}$$

Here, $\mathfrak{D}$'s are the Dolginov-Biedenharn functions for SO(4) rotations and the V's are the Bargmann functions for SO(2,1) rotations. We need only the following special cases:

$$\mathfrak{D}^{[00]}_{000}(\alpha) = 1, \quad \mathfrak{D}^{[10]}_{000}(\gamma) = \cos\gamma, \quad \mathfrak{D}^{[10]}_{001}(\gamma) = i\sin\gamma, \quad V^1_{11}(\beta) = \cosh^{-2}(\tfrac{1}{2}\beta), \quad V^1_{12}(\beta) = \sqrt{2}\,\frac{\tanh(\frac{1}{2}\beta)}{\cosh^2(\frac{1}{2}\beta)}. \tag{A4}$$

Therefore, (A2) becomes after substitutions

$$\begin{aligned}
F_0 = N_\pi^{-1} N_K^{-1} \Big\{ &[\alpha_1 \cosh\theta_K - \alpha_3(m_K + m_\pi\cosh\xi_\pi)\sinh\theta_K + \alpha_2(m_K + m_\pi\cosh\xi_\pi)] \frac{1}{\cosh^2(\frac{1}{2}\beta)} \\
&- \tfrac{1}{2}[\alpha_1 \sinh\theta_K - \alpha_3(m_K + m_\pi \cosh\xi_\pi)\cosh\theta_K] \\
&\times \frac{1}{\cosh^4(\frac{1}{2}\beta)} (\sinh\theta_K \cosh\theta_\pi \cosh\xi_\pi - \cosh\theta_K \sinh\theta_\pi) \Big\}.
\end{aligned} \tag{A5}$$

Similarly,

$$F_3 = N_\pi^{-1} N_K^{-1} \left\{ \frac{\alpha_1}{2} \frac{1}{\cosh^4(\frac{1}{2}\beta)} \cosh\theta_\pi \sinh\xi_\pi + (\alpha_2 - \alpha_3 \sinh\theta_K) m_\pi \sinh\xi_\pi \frac{1}{\cosh^2(\frac{1}{2}\beta)} \right.$$

$$\left. + \tfrac{1}{2}\alpha_3 m_\pi \sinh\xi_\pi \cosh\theta_K \frac{1}{\cosh^4(\frac{1}{2}\beta)} (\sinh\theta_K \cosh\theta_\pi \cosh\xi_\pi - \cosh\theta_K \sinh\theta_\pi) \right\}. \tag{A6}$$

---

[1]G. Donaldson *et al.*, Phys. Rev. D 9, 2960 (1974) (SLAC-UCSC experiment).

[2]C. D. Buchanan *et al.*, Phys. Rev. D 11, 457 (1975) (UCLA–SLAC–Johns Hopkins experiment).

[3]See the reviews: H. Pagels, Phys. Rep. 16C, 220 (1975).

[4]L.-M. Chounet, J.-M. Gaillard, and M. K. Gaillard, Phys. Rep. 4C, 200 (1972).

[5]M. K. Gaillard, Nuovo Cimento 61A, 499 (1969); R. A. Brandt and G. Preparata, Lett. Nuovo Cimento 4, 80 (1970).

[6]S. L. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968).

[7]D. H. Boal and R. H. Graham, Phys. Rev. D 15, 1878 (1977).

[8]W. B. Zeleny and A. O. Barut, Phys. Rev. 121, 908 (1961); E. Fischbach, F. Iachello, A. Lande , M. M. Nieto, and C. K. Scott, Phys. Rev. Lett. 26, 1200 (1971); E. Fischbach, M. M. Nieto, H. Primakoff, C. K. Scott, and J. Smith, Phys. Rev. Lett. 27, 1403 (1971); Z. Z. Aydin and A. O. Barut, Phys. Rev. D 7, 3522 (1973).

[9]A. O. Barut, H. Kleinert, and D. Corrigan, Phys. Rev. Lett. 20, 167 (1968).

[10]A. O. Barut and S. Malin, Nucl. Phys. B9, 194 (1969); A. O. Barut and K. C. Tripathy, Phys. Rev. 178, 2278 (1969).

[11]A. O. Barut and R. Wilson, J. Math. Phys. 17, 900 (1976).

[12]A. O. Barut, Nucl. Phys. B4, 455 (1968).

[13]S. C. Chhajlany, L. K. Pandit, and G. Rajasekaran, Phys. Lett. 35B, 166 (1971).

[14]A. O. Barut and J. Komen, Phys. Rev. D 1, 418 (1970).

[15]M. Ademollo and R. Gatto, Phys. Rev. Lett. 13, 264 (1964).

[16]A. Böhm, Phys. Rev. D 13, 2110 (1976); A. Böhm and J. Werle, Nucl. Phys. B106, 165 (1976).

[17]A. O. Barut and Z. Z. Aydin, Phys. Rev. D 6, 3340 (1972).

[18]K. Kang, Phys. Rev. Lett. 25, 414 (1970).