Coherent electromagnetic wave propagation through randomly distributed dielectric scatterers

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We present a vector multiple-scattering analysis of the coherent wave propagation through an inhomogeneous media consisting of a random distribution of identical, oriented, nonspherical, dielectric scatterers. The single-scattering aspect of the problem is dealt with through application of the transition or T matrix. Configurational averaging techniques are employed to determine the "hole" correction integrals which are subsequently solved to yield the dispersion relations characterizing the bulk or effective properties of the medium. Closed-form solutions in the Rayleigh limit are derived for both spherical and spheroidal scatterer geometries. These solutions, together with the T matrix, form the basis of our computational method for determining the coherent wave phase velocity and attenuation as a function of frequency (ka) and scatterer concentration. Numerical results are presented for spherical and oblate spheroidal geometries over a range of ka values (0.05–2.0) and scatterer concentrations (0.05–0.20).

I. INTRODUCTION

We consider the multiple scattering of vector electromagnetic waves by three-dimensional, oriented, identical, randomly distributed scatterers. The single-scattering problem involving one of the scatterers is treated through the transition or T matrix first introduced by Waterman.[1,2] Our attention focuses on determining the dispersion equations for coherent propagation of vector waves in the random medium using the configurational averaging procedure introduced by Foldy[3] and later by Lax.[4,5] The approach follows the work of Varadan *et al.*,[6] who analyzed multiple scattering of scalar waves by two-dimensional, randomly distributed, arbitrarily shaped scatterers. Previous work which is related to the present analysis includes the work of Vezzetti and Keller,[7] Fikioris and Waterman,[8] Peterson and Ström,[9] and more recently Twersky,[10] who considers the coherent electromagnetic field in a pair-correlated random distribution of aligned scatterers. However, most of the previous computational results are confined to simple geometries, such as circular cylinders and spheres, and often to low frequencies and scatterer concentrations. Our approach, which combines the T matrix of a single scatterer and suitable statistical averaging procedures, yields a powerful computational method for determining the coherent field characteristics over a wide range of frequencies, concentrations, and scatterer geometries.

We consider N number of identical, oriented, randomly distributed homogeneous scatterers with dielectric constant ϵ_r as shown in Fig. 1. We assume that time-harmonic plane polarized waves are incident along the z axis. When N is very large (such that the number of scatterers per unit volume is finite), the T-matrix formalism for a single scatterer is combined with a statistical analysis to provide the basis for solution of the problem. The statistically averaged equations are then solved using Lax's "quasicrystalline" approximation to yield the propagation characteristics of the "average" waves in the medium. Analytical results are obtained for the dispersion relation in the Rayleigh or low-frequency limit for spherical and spheroidal geometry. Computational results are presented for propagation speeds and attenuation at higher frequencies to demonstrate the broad applicability of the method.

II. T-MATRIX FORMULATION OF MULTIPLE SCATTERING

Consider N arbitrarily shaped, oriented, identical scatterers with a smooth surface S in free space which are referred to a coordinate system as shown in Fig. 1. 0_i and 0_j refer to the centers of the ith and jth scatterers. Let ϵ_r be the dielectric constant of the homogeneous scatterers.

We represent an incident plane electromagnetic wave of unit amplitude, $e^{-i\omega t}$ time dependence, and wave vector k by

$$\vec{E}^0(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}\hat{e}, \tag{1}$$

where k is the free-space wave number and $\hat{e}$ is the unit polarization vector. The incident wave is assumed to propagate in the positive z direction, and thus $\vec{k}\cdot\vec{r} = kz$.

We represent the scattered electric field by $\vec{E}^s(\vec{r})$ and the total field inside the scatterers by $\vec{E}^t(\vec{r})$. Both these fields satisfy the vector Helmholtz equation

$$\nabla\times(\nabla\times\vec{E}^s) - k^2\vec{E}^s = 0, \tag{2a}$$

$$\nabla\times[\nabla\times\vec{E}^t(\vec{r})] - (k')^2\vec{E}^t(\vec{r}) = 0, \tag{2b}$$

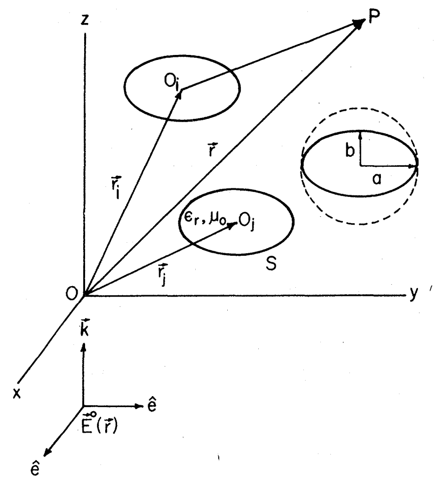

FIG. 1. Geometry of aligned, identical, homogeneous scatterers with dielectric constant ϵ_r excited by an incident plane wave $\vec{E}^0(\vec{r})$. Each scatterer is assumed to have an imaginary circumscribing spherical shell of radius a.

where $k' = k(\epsilon_r)^{1/2}$ is the wave number in the dielectric.

The total electric field at any point outside the scatterers is the sum of the incident field and the fields scattered by all the scatterers. This is written as

$$\vec{E}(\vec{r}) = \vec{E}^0(\vec{r}) + \sum_{i=1}^{N} \vec{E}_i^s(\vec{r} - \vec{r}_i), \tag{3}$$

where $\vec{E}_i^s(\vec{r} - \vec{r}_i)$ is the field scattered by the ith scatterer to the point of observation $\vec{r}$. The field that excites the ith scatterer is the incident field $\vec{E}^0$ plus the fields scattered from all the other dielectrics. The exciting-field term $\vec{E}^e$ is used to distinguish between the field actually incident on a scatterer and the external incident field $\vec{E}^0$ produced by a source at infinity. Thus, at a point $\vec{r}$ in the vicinity of the ith scatterer, we write

$$\vec{E}_i^e(\vec{r}) = \vec{E}^0(\vec{r}) + \sum_{j \neq i}^{N} \vec{E}_j^s(\vec{r} - \vec{r}_j), \quad a \leq |\vec{r} - \vec{r}_i| \leq 2a \tag{4}$$

where a is the radius of the imaginary sphere circumscribing a scatterer (see Fig. 1). In this analysis we have assumed that there is no interpenetration of the imaginary spherical shells of radius a which circumscribe each scatterer.

The T-matrix formulation of scattering we adopt here is based on the extended integral equation approach due to Waterman. The vector fields are expanded in terms of a complete set of basis functions $(\vec{M}_{mn}, \vec{N}_{mn})$ which form solutions to the vector Helmholtz equation and are given by (Stratton[11])

$$\vec{M}_{mn} = \nabla \times [\vec{r} h_n(kr) P_n^m(\cos\theta) e^{im\phi}], \tag{5a}$$

$$\vec{N}_{mn} = (1/k) \nabla \times \vec{M}_{mn}, \tag{5b}$$

where h_n are the spherical Bessel functions of the third kind of order n. Field quantities that are regular at the origin are expanded in terms of the regular (Re) basis set ($\text{Re}\vec{M}_{mn}$, $\text{Re}\vec{N}_{mn}$) obtained by replacing h_n in Eqs. (5a) and (5b) by j_n, the spherical Bessel functions of the first kind of order n. The scattered and exciting fields are expanded as follows:

$$\vec{E}_j^s(\vec{r}) = \sum_{n=0}^{\infty} \sum_{l=-n}^{n} [B_n^l \vec{M}_{nl}^j(\vec{r} - \vec{r}_j) + C_n^l \vec{N}_{nl}^j(\vec{r} - \vec{r}_j)], \tag{6a}$$

$$\vec{E}_i^e(\vec{r}) = \sum_{n=0}^{\infty} \sum_{l=-n}^{n} [b_n^l \text{Re}\vec{M}_{nl}^i(\vec{r} - \vec{r}_i) + c_n^l \vec{N}_{nl}^i(\vec{r} - \vec{r}_i)], \tag{6b}$$

where the superscript i or j on $\vec{M}$ (and $\vec{N}$) refer to expansions with respect to 0_i or 0_j, respectively, and B_n^l, C_n^l and b_n^l, c_n^l are the unknown expansion coefficients. The choice of the basis set in (6a) satisfies the radiation condition at infinity for the scattered field $\vec{E}_j^s$ while the choice in (6b) satisfies the regularity of the exciting field, $\vec{E}_i^e$ in the region $0 < |r - r_i| < 2a$.

It has been shown that if the total field outside a scatterer is the sum of the incident and scattered fields, the unknown scattered-field expansion coefficients can be related to the incident-field expansion coefficients through the transition or T matrix (Waterman[2]). We extend this definition to the present case. Since $(\vec{E}_j^e + \vec{E}_j^s)$ is the total field at any point in free space, the expansion coefficients of the field scattered by the jth scatterer may be formally related to the coefficients of the field exciting the jth scatterer through the T matrix:

$$\begin{pmatrix} B_n^l \\ C_n^l \end{pmatrix} = \begin{pmatrix} (T^{11})_{nm}^{lp} & (T^{12})_{nm}^{lp} \\ (T^{21})_{nm}^{lp} & (T^{22})_{nm}^{lp} \end{pmatrix} \begin{pmatrix} b_m^p \\ c_m^p \end{pmatrix}. \tag{7}$$

The elements of the T matrix involve surface integrals which can be evaluated in closed form for spherical geometry, while for obstacles of arbitrary shape they can only be evaluated numerically. The T matrix for a single scatterer is of the form

$$T = (Q^{-1}) \text{Re}Q, \tag{8}$$

where ReQ and Q are matrices which are functions of the surface S of the scatterer and of the nature of the boundary conditions. A detailed derivation of Eq. (7) can be found in Waterman.[2]

Substituting Eqs. (6a), (6b) in Eq. (4) gives

$$\sum_{\beta=0}^{\infty}\sum_{\alpha=-\beta}^{\beta}[b_\beta^\alpha \operatorname{Re}\vec{M}_{\beta\alpha}^i(\vec{r}-\vec{r}_i)+c_\beta^\alpha \operatorname{Re}\vec{N}_{\beta\alpha}^i(\vec{r}-\vec{r}_i)]=\vec{E}^0(\vec{r})+\sum_{j\neq i}^{N}\sum_{n=0}^{\infty}\sum_{l=-n}^{n}[B_n^l\vec{M}_{nl}^j(\vec{r}-\vec{r}_j)+C_n^l\vec{N}_{nl}^j(\vec{r}-\vec{r}_j)]. \quad (9)$$

Note that the series on the right-hand side of Eq. (9) is expressed with respect to the center of the jth scatterer. The addition theorem for the vector spherical harmonics will be employed later to express these quantities with respect to the center of the ith scatterer. It then remains to expand the incident wave also in the form of a series centered at the ith scatterer. Specifically, we assume $\vec{k}=k\hat{z}$ and $\hat{e}=\hat{x}$ so that $k\hat{z}\cdot\vec{r}=k\hat{z}\cdot(\vec{r}-\vec{r}_i)+k\hat{z}\cdot\vec{r}_i$. Expansions for plane waves in terms of vector spherical harmonics are given by Stratton.[11] Hence, $\vec{E}^0(\vec{r})$ can be expressed as

$$\vec{E}^0(\vec{r})=\frac{1}{2i}e^{ik\xi_i}\sum_{s=1}^{\infty}\sum_{t=-s}^{s}\frac{2s+1}{s(s+1)}i^s\left\{\operatorname{Re}\vec{M}_{st}^i[\delta_{t,1}+s(s+1)\delta_{t,-1}]+\frac{1}{k}\operatorname{Re}\vec{N}_{st}^i[\delta_{t,1}-s(s+1)\delta_{t,-1}]\right\}, \quad (10)$$

where $\hat{z}\cdot\vec{r}_i=\xi_i$ and δ_{mn} is the Kronecker δ. We now translate the basis functions $\{\vec{M}^j(\vec{r}-\vec{r}_j),\ \vec{N}^j(\vec{r}-\vec{r}_j)\}$ to an origin centered at 0_i. This is done by employing the addition theorems for the vector spherical harmonics (Cruzan[12] or Peterson and Ström[9]):

$$\vec{M}_{nl}^j=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(B_{\nu\mu}^{nl}\operatorname{Re}\vec{M}_{\nu\mu}^i+C_{\nu\mu}^{nl}\operatorname{Re}\vec{N}_{\nu\mu}^i), \quad (11a)$$

$$\vec{N}_{nl}^j=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(C_{\nu\mu}^{nl}\operatorname{Re}\vec{M}_{\nu\mu}^i+B_{\nu\mu}^{nl}\operatorname{Re}\vec{N}_{\nu\mu}^i), \quad (11b)$$

$$B_{\nu\mu}^{nl}=\sum_{q=|n-\nu|}^{n+\nu}(-1)^{\mu}i^{\nu+q-n}a(l,n|-\mu,\nu|q)a(n,\nu,q)h_q(kr_{ij})P_q^{l-\mu}(\cos\theta_{ij})e^{i(l-\mu)\phi_{ij}}, \quad (12a)$$

$$C_{\nu\mu}^{nl}=\sum_{q=|n-\nu|+1}^{n+\nu}(-1)^{\mu+1}i^{\nu+q-n}b(n,\nu,q)a(l,n|-\mu,\nu|q,q-1)h_q(kr_{ij})P_q^{l-\mu}(\cos\theta_{ij})e^{i(l-\mu)\phi_{ij}}. \quad (12b)$$

Expressions for $a(n,\nu,q)$, $b(n,\nu,q)$, $a(l,n|-\mu,\nu|q)$, and $a(l,n|-\mu,\nu|q,q-1)$ are given by Cruzan[12] and Fig. 2 depicts the geometry of the translation between the jth and ith scatterers.

Substituting Eqs. (11a), (11b), and Eq. (10) in Eq. (9) gives

$$\begin{aligned}\sum_{\beta=0}^{\infty}\sum_{\alpha=-\beta}^{\beta}(b_\beta^\alpha\operatorname{Re}\vec{M}_{\beta\alpha}^i+c_\beta^\alpha\operatorname{Re}\vec{N}_{\beta\alpha}^i)&=\frac{1}{2i}e^{ik\xi_i}\sum_{s=1}^{\infty}\sum_{t=-s}^{s}\frac{2s+1}{s(s+1)}i^s\Big\{[\delta_{t,1}+s(s+1)\delta_{t,-1}]\operatorname{Re}\vec{M}_{st}^i\\&\qquad+\frac{1}{k}[\delta_{t,1}-s(s+1)\delta_{t,-1}]\operatorname{Re}\vec{N}_{st}^i\Big\}\\&\quad+\sum_{j\neq i}^{N}\sum_{n=0}^{\infty}\sum_{l=-n}^{n}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}[B_n^l(B_{\nu\mu}^{nl}\operatorname{Re}\vec{M}_{\nu\mu}^i+C_{\nu\mu}^{nl}\operatorname{Re}\vec{N}_{\nu\mu}^i)+C_n^l(C_{\nu\mu}^{nl}\operatorname{Re}\vec{M}_{\nu\mu}^i+B_{\nu\mu}^{nl}\operatorname{Re}\vec{N}_{\nu\mu}^i)].\end{aligned} \quad (13)$$

Equation (13) is but an expansion of Eq. (4) in terms of vector spherical harmonics with respect to an origin 0_i centered at the ith scatterer. A relation between the unknown expansion coefficients $(b_\beta^\alpha, c_\beta^\alpha)$ of the exciting field and the unknown expansion coefficients (B_n^l, C_n^l) of the scattered field can be obtained by using the orthogonality properties of the vector spherical harmonics. It is easily seen that

$$b_m^p=\psi_{mp}^i=\frac{2m+1}{m(m+1)}i^m\frac{e^{ik\xi_i}}{2i}[\delta_{p,1}+m(m+1)\delta_{p,-1}]+\sum_{j=1}^{N}{}'\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}(B_{n_1}^{m_1}B_{mp}^{n_1m_1}+C_{n_1}^{m_1}C_{mp}^{n_1m_1}), \quad (14a)$$

$$c_m^p=\chi_{mp}^i=\frac{2m+1}{m(m+1)}i^m\frac{e^{ik\xi_i}}{2i}[\delta_{p,1}-m(m+1)\delta_{p,-1}]+\sum_{j=1}^{N}{}'\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}(B_{n_1}^{m_1}C_{mp}^{n_1m_1}+C_{n_1}^{m_1}B_{mp}^{n_1m_1}), \quad (14b)$$

where $\sum'$ denotes the sum over all scatterers except the ith.

Substituting the above expressions for (b_m^p, c_m^p) in Eq. (7) yields (with superscripts on ψ, χ referring to the ith scatterer)

$$\begin{bmatrix}B_n^{l(i)}\\C_n^{l(i)}\end{bmatrix}=\begin{bmatrix}(T^{11})_{nm}^{lp} & (T^{12})_{nm}^{lp}\\(T^{21})_{nm}^{lp} & (T^{22})_{nm}^{lp}\end{bmatrix}\begin{bmatrix}\psi_{mp}^i\\\chi_{mp}^i\end{bmatrix}, \quad (15)$$

where summation over the m and p indices is implied by the repeated index convention. Thus we have eliminated the unknown exciting-field expansion coefficients (b_m^p, c_m^p) through the use of the T matrix resulting in a set of equations involving the expansion coefficients (B_n^l, C_n^l) of the scattered field only. The coefficients B_n^l, C_n^l are functions of the positions of all the scatterers.

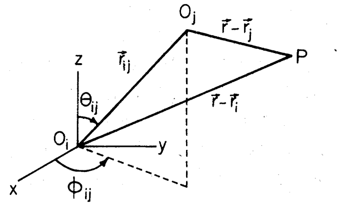

FIG. 2. Translation of the coordinate system of the jth and ith scatterer.

III. STATISTICALLY AVERAGED WAVE FIELDS

In order to average the wave fields over the positions of all the scatterers we define a probability density function of finding the first scatterer at $\vec{r}_1$, the second scatterer at $\vec{r}_2$, and so forth by $p(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$. This probability density function may be written as

$$p(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = p(\vec{r}_i)\,p(\vec{r}_1, \vec{r}_2, \ldots ', \ldots \vec{r}_N/\vec{r}_i)$$
$$= p(\vec{r}_i)\,p(\vec{r}_j)\,p(\vec{r}_1, \vec{r}_2, \ldots ', \ldots ', \ldots \vec{r}_N \,|\, \vec{r}_i, \vec{r}_j)$$
$$= \cdots , \tag{16}$$

where $p(\vec{r}_i)$ denotes the probability density of finding a scatterer at $\vec{r}_i$ while $p(\vec{r}_j | \vec{r}_i)$ denotes the conditional probability of finding a scatterer at $\vec{r}_j$ if a scatterer is known to be at $\vec{r}_i$. A prime in the first expansion of Eq. (16) means $\vec{r}_i$ is absent while two primes in the second of Eq. (16) mean both $\vec{r}_i$ and $\vec{r}_j$ are absent.

If the scatterers are randomly distributed, the positions of all scatterers are equally probable within the volume V accessible to the scatterers, and hence

$$p(\vec{r}_i) = \begin{cases} n_0/N, & \vec{r}_i \in V \\ 0, & \vec{r}_i \notin V \end{cases} \tag{17}$$

where n_0 is the uniform number density of the scatterers and N is the total number of scatterers. In addition, for nonoverlap of the imaginary spherical shells circumscribing each scatterer we approximate the conditional density as follows:

$$p(\vec{r}_j | \vec{r}_i) = \begin{cases} n_0/N, & |\vec{r}_i - \vec{r}_j| > 2a \\ 0, & |\vec{r}_i - \vec{r}_j| < 2a \end{cases} \tag{18}$$

where a is the radius of the circumscribing shell (see Fig. 1). A suitable correlation in the position may also be added to (18) but is omitted here for simplicity. The form of the pair correlation in Eq. (18) describes the usual radially symmetric distribution function with an exclusion surface or "hole" corresponding to a sphere of radius $2a$. Other types of exclusion surfaces are discussed in a recent paper by Twersky.[10]

We denote the configurational average of a statistical quantity f as

$$\langle f \rangle_i = \int_V \cdots \int_V f p(\vec{r}_1, \vec{r}_2, \ldots ', \ldots, \vec{r}_N | \vec{r}_i) \times d\vec{r}_1 d\vec{r}_2 \ldots ', \ldots, d\vec{r}_N , \tag{19a}$$

$$\langle f \rangle_{ij} = \int_V \cdots \int_V f p(\vec{r}_1, \vec{r}_2, \ldots ', \ldots ', \ldots, \vec{r}_N | \vec{r}_i, \vec{r}_j) \times d\vec{r}_1 d\vec{r}_2 \ldots ', \ldots ', \ldots, d\vec{r}_N , \tag{19b}$$

where in Eq. (19a) we have averaged over all scatterers except the ith and in Eq. (19b), over all scatterers except the ith and jth, and so on.

Multiplying both sides of Eq. (15) by the probability density given by Eq. (16) and using Eqs. (17)–(19), we obtain the configurational average of B_n^l, C_n^l:

$$\begin{bmatrix} \langle B_n^{l(i)} \rangle_i \\ \langle C_n^{l(i)} \rangle_i \end{bmatrix} = \begin{bmatrix} (T^{11})^{lp}_{nm} & (T^{12})^{lp}_{nm} \\ (T^{21})^{lp}_{nm} & (T^{22})^{lp}_{nm} \end{bmatrix} \begin{bmatrix} \langle \psi^i_{mp} \rangle \\ \langle \chi^i_{mp} \rangle \end{bmatrix} , \tag{20}$$

where

$$\langle \psi^i_{mp} \rangle = \frac{2m+1}{m(m+1)}\, i^m \,\frac{e^{ik\zeta_i}}{2i}\, [\delta_{p,1} + m(m+1)\delta_{p,-1}] + \frac{1}{V} {\sum_{j=1}^{N}}' \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} \left[\int_{V'} (\langle B_{n_1}^{m_1(j)} \rangle_{ij} B_{mp}^{n_1 m_1} + \langle C_{n_1}^{m_1(j)} \rangle_{ij} C_{mp}^{n_1 m_1}) d\vec{r}_j \right] , \tag{21a}$$

$$\langle \chi^i_{mp} \rangle = \frac{2m+1}{m(m+1)}\, i^m \,\frac{e^{ik\zeta_i}}{2i}\, [\delta_{p,1} - m(m+1)\delta_{p,-1}] + \frac{1}{V} {\sum_{j=1}^{N}}' \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} \left[\int_{V'} (\langle B_{n_1}^{m_1(j)} \rangle_{ij} C_{mp}^{n_1 m_1} + \langle C_{n_1}^{m_1(j)} \rangle_{ij} B_{mp}^{n_1 m_1}) d\vec{r}_j \right] , \tag{21b}$$

and V' denotes the volume of the medium excluding the volume of a sphere of radius $2a$. For identical scatterers, $\sum_{j\neq i}^{N}$ can be replaced by $(N-1)$. Equation (20) indicates that the conditional average with one scatterer fixed, viz., $[\langle B_n^l\rangle_i, \langle C_n^l\rangle_i]$ is given in terms of the conditional average with two scatterers fixed, viz., $[\langle B_{n_1}^{m_1}\rangle_{ij}, \langle C_{n_1}^{m_1}\rangle_{ij}]$. However, Lax[5] has suggested a quasicrystalline approximation to close the system:

$$\left.\begin{aligned}\langle B_n^{m(j)}\rangle_{ij} &\simeq \langle B_n^{m(j)}\rangle_j\\ \langle C_n^{m(j)}\rangle_{ij} &\simeq \langle C_n^{m(j)}\rangle_j\end{aligned}\right\}\ i\neq j\,. \tag{22}$$

We use this approximation which implies that there is no correlation between the ith and jth scatterers other than there should be no interpenetration of any two scatterers.

IV. PROPAGATION CHARACTERISTICS OF THE AVERAGE WAVES IN THE MEDIUM

We now find a plane-wave solution with an effective wave number K characterizing the bulk medium:

$$\begin{aligned}\langle B_n^{l(i)}\rangle_i &= i^l Y_n^l e^{i\vec{K}\cdot\vec{r}_i}\,,\\ \langle C_n^{l(i)}\rangle_i &= i^l Z_n^l e^{i\vec{K}\cdot\vec{r}_i}\,.\end{aligned} \tag{23}$$

The effective wave vector $\vec{K}$ is assumed to be parallel to that of the incident wave which in the present case is along the z axis. In Eq. (23), Y_n^l and Z_n^l are unknown constants. Substituting Eq. (23) in Eqs. (21a) and (21b) gives

$$\begin{aligned}i^l Y_n^l e^{i\vec{K}\cdot\vec{r}_i} =& \sum_{m=0}^{\infty}\sum_{p=-m}^{m}(T^{11})_{nm}^{lp}\frac{2m+1}{m(m+1)}i^m\frac{e^{i\vec{k}\cdot\vec{r}_i}}{2i}[\delta_{p,1}+m(m+1)\delta_{p,-1}]\\ &+\sum_{m=0}^{\infty}\sum_{p=-m}^{m}(T^{12})_{nm}^{lp}\frac{2m+1}{m(m+1)}i^m\frac{e^{i\vec{k}\cdot\vec{r}_i}}{2i}[\delta_{p,1}-m(m+1)\delta_{p,-1}]\\ &+\frac{N-1}{V}\sum_{m=0}^{\infty}\sum_{p=-m}^{m}\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}i^{m_1}\left[(T^{11})_{nm}^{lp}\int_{V'}(Y_{n_1}^{m_1}B_{mp}^{n_1m_1}+Z_{n_1}^{m_1}C_{mp}^{n_1m_1})e^{i\vec{K}\cdot\vec{r}_j}dr_j\right.\\ &\left.+(T^{12})_{nm}^{lp}\int_{V'}(Y_{n_1}^{m_1}C_{mp}^{n_1m_1}+Z_{n_1}^{m_1}B_{mp}^{n_1m_1})e^{i\vec{K}\cdot\vec{r}_j}d\vec{r}_j\right],\end{aligned} \tag{24a}$$

$$i^l Z_n^l e^{i\vec{K}\cdot\vec{r}_i} = \ldots . \tag{24b}$$

The expression for Z_n^l in Eq. (24b) can be obtained by replacing T^{11}, T^{12} by T^{21}, T^{22} in Eq. (24a), respectively.

Consider the following volume integral which appears in Eq. (24a):

$$\begin{aligned}\int_{V'}Y_{n_1}^{m_1}B_{mp}^{n_1m_1}e^{i\vec{K}\cdot\vec{r}_j}d\vec{r}_j =& \sum_{q=|n_1-m|}^{n_1+m}(-1)^p i^{m+q-n_1}a(m_1,n_1|-p,m|q)a(n_1,m,q)Y_{n_1}^{m_1}\\ &\times\int_{|\vec{r}_i-\vec{r}_j|>2a}e^{i\vec{K}\cdot\vec{r}_j}h_q(kr_{ij})P_q^{m_1-p}(\cos\theta_{ij})e^{i(m_1-p)\phi_{ij}}\,d\vec{r}_j,\end{aligned} \tag{25}$$

where the expansion in Eq. (12a) has been used for $B_{mp}^{n_1m}$. Again, consider the integral in Eq. (25) and let

$$I_A = \int_{|\vec{r}_i-\vec{r}_j|>2a}e^{i\vec{K}\cdot\vec{r}_j}h_q(kr_{ij})P_q^{m_1-p}(\cos\theta_{ij})\,e^{i(m_1-p)\phi_{ij}}\,d\vec{r}_j\,. \tag{26}$$

To evaluate I_A we observe first that the integration over ϕ_{ij} can be performed and that it vanishes for $m_1\neq p$. We then observe that the two terms in the integrand, viz., $e^{i\vec{K}\cdot\vec{r}_j}$ and the product $(h_q P_q^{m_1-p})$ satisfy the scalar wave equation with wave numbers K and k, respectively. Thus we can rewrite the integral in the following form:

$$I_A = \frac{2\pi\delta_{m_1p}}{k^2-K^2}\int_{|\vec{r}_i-\vec{r}_j|>2a}\{\nabla^2(e^{i\vec{K}\cdot\vec{r}_j})h_q(kr_{ij})P_q(\cos\theta_{ij}) - e^{i\vec{K}\cdot\vec{r}_j}\nabla^2[h_q(kr_{ij})P_q(\cos\theta_{ij})]\}d\vec{r}_j\,. \tag{27}$$

Using Green's theorem gives

$$I_A = \frac{2\pi\delta_{m_1p}}{k^2-K^2}\int_{S_\infty-S_{2a}}\left\{\frac{\partial}{\partial n}(e^{i\vec{K}\cdot\vec{r}_j})h_q(kr_{ij})P_q(\cos\theta_{ij}) - e^{i\vec{K}\cdot\vec{r}_j}\frac{\partial}{\partial n}[P_q h_q(r_{ij})]\right\}dS\,, \tag{28}$$

where S_∞ refers to a sphere of large radius at infinity and S_{2a} refers to the surface of the sphere of radius $2a$ circumscribing the scatterer. Following the method of Varadan *et al.*,[6] we can evaluate the integral on S_{2a} in closed form by expanding the plane-wave term in the integrand in terms of the basis functions to yield

$$I_A = I_\infty + \frac{8\pi a^2}{k^2 - K^2}\,\delta_{m_1 p}(-i)^q e^{i\vec{K}\cdot\vec{r}_i}\left\{ j_q(2Ka)\frac{\partial}{\partial a}h_q(2ka) - h_q(2ka)\frac{\partial}{\partial a}j_q(2Ka)\right\}, \tag{29}$$

where I_∞ represents the surface integral on S_∞ and is proportional to $e^{i\vec{K}\cdot\vec{r}_i}$ and is obtained by using the asymptotic expansion for $h_q(kr_{ij})$ as $r_{ij}\to\infty$ on S_∞.

Similarly, the various integrals over V' in Eqs. (24a) and (24b) can be evaluated. To derive the dispersion relation, we apply the extinction theorem to the two sets of terms in Eq. (24) (each obeying the wave equation with wave numbers K and k) as discussed in detail by Twersky[10] and Varadan *et al.*[6] This results in an infinite system of equations for the unknown:

$$\begin{aligned} i^l Y_n^l = \frac{6c}{(ka)^2 - (Ka)^2} \sum_{q=|n_1-m|}^{n_1+m} \sum_{m=0}^{\infty} \sum_{p=-m}^{m} \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} & (-1)^p (-i)^q i^{m_1+m+q-n_1} \\ & \times \delta_{m_1 p}(JH)_q \{ Y_{n_1}^{m_1}[(T^{11})_{nm}^{lp} a(n_1,m,q)\, a(m_1,n_1|-p,m|q) \\ & \quad - (T^{12})_{nm}^{lp} b(n_1,m,q)\, a(m_1,n_1|-p,m|q,q-1)] \\ & + Z_{n_1}^{m_1}[(T^{12})_{nm}^{lp} a(n_1,m,q) a(m_1,n_1|-p,m|q) \\ & \quad - (T^{11})_{nm}^{lp} b(n_1,m,q) a(m_1,n_1|-p,m|q,q-1)]\}, \end{aligned} \tag{30a}$$

$$i^l Z_n^l = \cdots, \tag{30b}$$

where Eq. (30b) can be obtained from Eq. (30a) by replacing T^{11}, T^{12} by T^{21}, T^{22}, respectively. We define c as the effective "spherical" concentration of the scatterers per unit volume which equals $4\pi a^3 n_0/3$. The term $(JH)_q$ is now defined by

$$(JH)_q = 2ka\, j_q(2Ka) h_q'(2ka) - 2Ka\, h_q(2ka) j_q'(2Ka). \tag{31}$$

The set of equations given by (30a) and (30b) form an infinite homogeneous system of linear equations in the unknowns Y_n^l, Z_n^l. For a nontrivial solution, we require that the determinant of the truncated coefficient matrix vanish, which yields a relation for the effective wave number K in terms of k and the T matrix of the scatterer. This is the dispersion relation for the scatterer filled medium. Equations (30a) and (30b) form a general expression valid for any arbitrary scatterer, since the T matrix is the only factor that contains information about the exact shape and boundary conditions at the scatterers. In particular, we can use the T matrix corresponding to a perfectly conducting, dielectric, or two-layered scatterer (Peterson and Ström[13] and Bringi and Seliga).[14,15]

V. RAYLEIGH LIMIT SOLUTIONS

In the Rayleigh or low-frequency limit, the size of the scatterer is considered to be very small compared to the incident wavelength. It is then sufficient to take only the lowest-order coefficient in the expansion of the fields. In this limit, the elements of the T matrix can be obtained in closed form for simple shapes, such as the sphere and spheroid:

sphere:

$$(T^{22})_{11}^{11} = \tfrac{2}{3} i(ka)^3\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) + O(k^5 a^5), \tag{32}$$

spheroid:

$$(T^{22})_{11}^{11} \simeq \frac{i(ka)^3(\epsilon_r - 1)[f_2(e) + f_3(e)]}{2(\epsilon_r + 2) - 3(\epsilon_r - 1) f_1(e)}, \tag{33}$$

where e is the eccentricity defined by $e = [(a/b)^2 - 1]^{1/2}$ for the oblate spheroid (a and b being the semimajor and semiminor axes, respectively). The functions f_1, f_2, and f_3 are defined by

$$f_1(e) = e - \tan^{-1}e - \tfrac{1}{3} + \frac{1}{e^2} - \frac{\tan^{-1}e}{e^3}, \tag{34a}$$

$$f_2(e) = \frac{1}{e^2(1+e^2)^{1/2}}\left\{ e^2 - 1 + \frac{(1+e^2)^{1/2}}{e} \times \ln[e + (1+e^2)^{1/2}]\right\}, \tag{34b}$$

$$f_3(e) = \frac{1}{(1+e^2)^{1/2}}\left\{ \tfrac{1}{3} + \frac{1}{e^2} - \frac{(1+e^2)^{1/2}}{e^3} \times \ln[e + (1+e^2)^{1/2}]\right\}. \tag{34c}$$

In view of Eqs. (32) and (33) we obtain the following equation for the unknown Z_1^1 from (30b):

$$iZ_1^1 = \frac{6c}{(ka)^2-(Ka)^2}\left[i(T^{22})_{11}^{11}(JH)_0 + \tfrac{1}{2}i(T^{22})_{11}^{11}Z_1^1(JH)_2\right]. \qquad (35)$$

The dispersion relations are obtained from Eq. (35) using the leading terms in the expansions for the Bessel and Hankel functions composing $(JH)_0$ and $(JH)_2$:

sphere:

$$\left(\frac{K}{k}\right)^2 = \frac{1+2c\,\dfrac{\epsilon_r-1}{\epsilon_r+2}}{1-c\,\dfrac{\epsilon_r-1}{\epsilon_r+2}}, \qquad (36)$$

spheroid:

$$\left(\frac{K}{k}\right)^2 = \left[1+\frac{\left(\dfrac{3c}{2}\right)\left(\dfrac{\epsilon_r-1}{\epsilon_r+2}\right)(f_2+f_3)}{1-\dfrac{3}{2}\left(\dfrac{\epsilon_r-1}{\epsilon_r+2}\right)f_1}\right] \times\left[1-\frac{\left(\dfrac{3c}{4}\right)\left(\dfrac{\epsilon_r-1}{\epsilon_r+2}\right)(f_2+f_3)}{1-\dfrac{3}{2}\left(\dfrac{\epsilon_r-1}{\epsilon_r+2}\right)f_1}\right]^{-1}. \qquad (37)$$

Equation (36) is recognized as the dispersion relation of the Clausius-Mossotti form. The dispersion relation given by Eq. (37) appears to be new; it reduces to Eq. (36) in the limit as $e\to 0$. If the concentration $c \ll 1$, the dispersion relations simplify to

sphere:

$$\frac{K}{k} = 1+\tfrac{3}{2}c\,\frac{\epsilon_r-1}{\epsilon_r+2}, \qquad (38)$$

spheroid:

$$\frac{K}{k} = 1+\frac{\dfrac{9c}{8}\left(\dfrac{\epsilon_r-1}{\epsilon_r+2}\right)(f_2+f_3)}{1-\dfrac{3}{2}\left(\dfrac{\epsilon_r-1}{\epsilon_r+2}\right)f_1}. \qquad (39)$$

VI. DISPERSION AT HIGHER FREQUENCIES

To study the response at resonant and higher frequencies, we must consider higher powers of ka, and this implies that a larger number of terms (Y_n^l, Z_n^l) must be kept in the expansion of the average field. This is best done numerically. For a given value of ka the T matrix for the scatterer is computed (see, e.g., Barber and Yeh[16]). Next, the coefficient matrix C corresponding to Eqs. (30a) and (30b) is formed. The complex determinant of the coefficient matrix is computed using standard Gauss elimination techniques. For a given ka, the root of the equation $\det C = 0$ is searched in the complex K plane (K_1+iK_2) using Muller's method. Good initial guesses were provided by Eqs. (36) and (37) at low values of ka and these could be used systematically to obtain convergence of roots at increasingly higher values of ka. The real part K_1 determines the phase velocity while the imaginary part K_2 determines the attenuation. We define the normalized phase velocity as $v_p/c_0 = k/K_1$, where c_0 is the phase velocity in free space and the attenuation coefficient $\alpha = 4\pi K_2/K_1$ so as to make it dimensionless.

In Fig. 3 we show the attentuation coefficient α as a function of ka for three different concentration values, $c=0.05$, 0.10, and 0.20. These calculations were performed for spherical scatterers $(a/b=1.0)$ having dielectric constant $\epsilon_r = 3.17+i0$. (The sample value of ϵ_r for all calculations was chosen to correspond to the dielectric constant of ice at microwave frequencies.) The general tendency of α is to increase smoothly with ka at $c=0.05$ and 0.10 while the curve for $c=0.20$ shows a sharp decrease at $ka \simeq 0.75$ and then continues to increase with ka. Corresponding phase velocity v_p curves as a function of ka are depicted in Fig. 4. These curves gently decrease with ka up to about $ka\simeq 1.6$ and then show a small increase up to $ka = 2.0$.

In Figs. 5 and 6 we show calculations of α as a function of ka for oblate spheroidal scatterers with axial ratios $a/b = 1.25$ and 2.0, respectively. The spheroids are assumed to be oriented with their axis of revolution along the z axis (see Fig. 1). Since the incident wave is also assumed to propa-

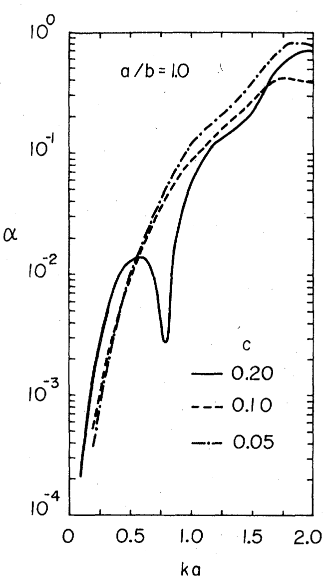

FIG. 3. Attenuation coefficient α as a function of ka for spherical dielectric scatterers at concentration values of 0.05, 0.10, and 0.20.

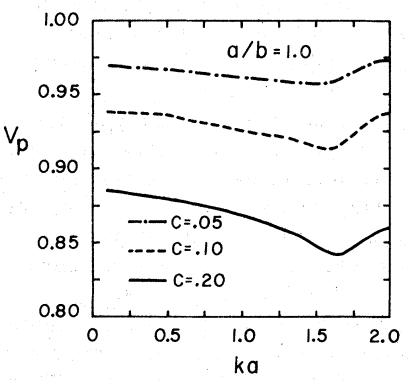

FIG. 4. Normalized phase velocity vs ka for spherical dielectric scatterers at concentration values of 0.05, 0.10, and 0.20.

gate along the z axis, the bulk medium is not anistropic and is characterized by a single wave number K. The significant feature in Figs. 5 and 6 is the occurrence of a sharp null in the attenuation coefficient α at $ka \simeq 0.75$ for the $c = 0.2$ case. In the low-frequency regime up to $ka = 0.5$, the α curves increase rapidly with ka. Also, beyond the null value, the attenuation increases smoothly up to $ka = 2.0$. Calculations for ka values larger than 2.0 were not performed due to lack of adequate computer resources. For the severest cases studied, i.e.,

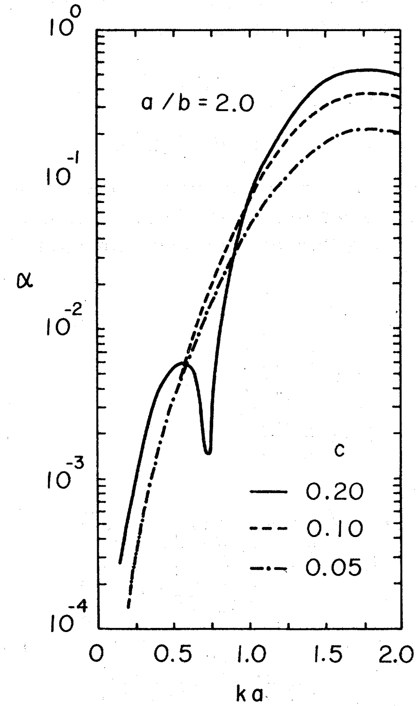

FIG. 6. Attenuation coefficient α vs ka for oblate spheroidal scatterers ($a/b = 2.0$) at concentration values of 0.05, 0.10, and 0.2.

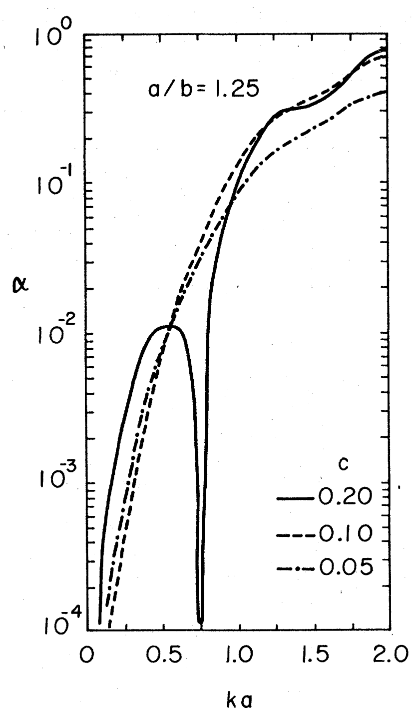

FIG. 5. Attenuation coefficient α vs ka for oblate spheroidal scatterers ($a/b = 1.25$) at concentration values of 0.05, 0.10, and 0.20.

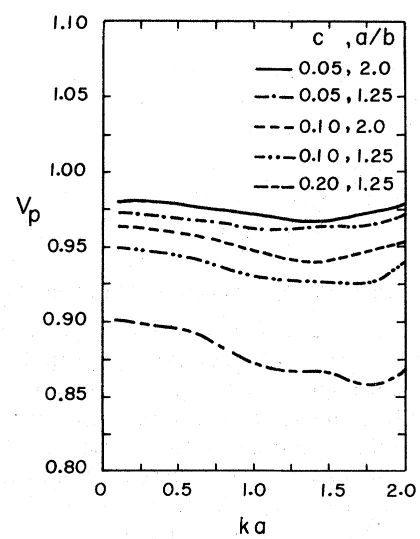

FIG. 7. Phase velocity v_p vs ka for oblate spheroidal scatterers ($a/b = 1.25, 2.0$) at concentrations of 0.05, 0.10, and 0.2.

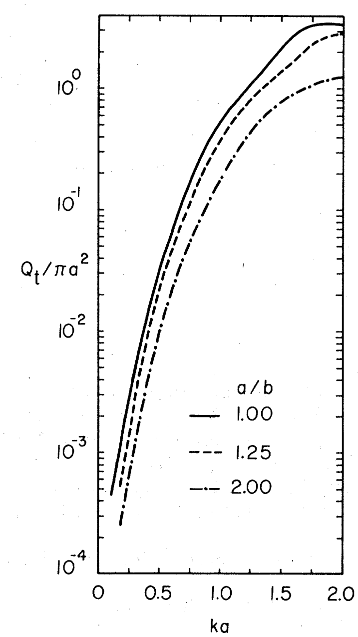

FIG. 8. Normalized attenuation (total) cross section vs ka for a single spherical ($a/b=1.0$) and an oblate spheroidal scatterer ($a/b=1.25, 2.0$).

$a/b=2.0$, $ka=2$, the required size of the coefficient matrix C was 12×12 with a corresponding T-matrix size of 20×20. Currently, our computer code can handle matrix sizes up to 40×40. In Fig. 7 we depict the phase velocity v_p as a function of ka for the parameter values corresponding to Figs. 5 and 6. Again the curves decrease gently with ka up to about $ka\simeq 1.75$ and then show a small increase with ka. Note that the sharp nulls in the attenuation curves are not reflected in the corresponding phase velocity curves.

In Fig. 8 we show the normalized attenuation (total) cross section $Qt/\pi a^2$ for a single dielectric scatterer with $a/b=1.0$, 1.25, and 2.0 as a function of ka (these calculations were performed using the corresponding T matrix). It can be assumed that these curves depict the dispersion for a random assembly of identical scatterers in the single scattering approximation. Comparing these curves with the curves marked $c=0.05$ and $c=0.10$ in Figs. 3, 5, and 6 we note that the general features are similar with no null appearing in the range of ka values considered. However, for all other cases considered the significant difference between Figs. 3, 5, and 6 and the single scattering curves of Fig. 8 is the presence of a sharp dip. This is not unexpected since the concentration values we consider are relatively high. However, in the Rayleigh region and up to $ka\simeq 0.5$ the attenuation curves of Figs. 3, 5, and 6 generally follow the single scattering curves of Fig. 8.

The general form of the dispersion curves, as a function of ka, are similar to the ones obtained by Varadan *et al.*[6] However, their analysis considers dispersion due to elastic wave scattering by two-dimensional circular and elliptical cylinders. Examination of the various dispersion curves (Figs. 3, 5, and 6) suggests that higher modes of propagation may be possible. The frequency at which α exhibits a sharp null is usually referred to as the first passband. This occurs at $ka \simeq 0.75$ for the cases considered in this paper and depends on both the concentration and spheroidal properties of the scatterers. Note that since ϵ_r is chosen to have zero loss, the attenuation is due only to geometric dispersion. There is some experimental evidence in the elastic wave case suggesting that at critical values of α the energy shifts to higher modes of propagation (Varadan *et al.*[6]). Note also that more information than obtained here is contained in the dispersion relation, since there exist many values of K for a given value of ka. We have computed only the lowest root while the transcendental form of the dispersion relation allows infinitely many roots. We are currently investigating these and related problems.

VII. CONCLUSIONS

We have presented a vector multiple-scattering formalism for coherent wave propagation of electromagnetic waves through an inhomogeneous medium composed of a random distribution of identical, three dimensional scatterers. An important advantage is realized through the use of the T matrix to characterize the scattering properties of any one scatterer. Dispersion relations applicable to the effective medium are then obtained through the use of suitable statistical averaging procedures. Closed-form solutions are derived in the Rayleigh limit for spherical and spheroidal geometry. These solutions, together with the T matrix for a single scatterer, form the basis of our computational methodology. Sample numerical results in the form of phase velocity and attenuation coefficient as a function of frequency (or ka) are presented for both spherical and oblate spheroidal dielectric scatterers. Examination of the dispersion curves suggests the possibility of higher-order modes of propagation.

The methodology we adopt is general and can be used for multiple-scattering analyses of acoustic, electromagnetic, and elastic waves emphasizing the inherent unity of the approach. We are currently extending the method to include such effects as (a) anisotropy, (b) scatterer size distribution, and (c) random scatterer orientation.

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[1] P. C. Waterman, Proc. IEEE 53, 805 (1965).

[2] P. C. Waterman, Phys. Rev. D 3, 825 (1971).

[3] L. L. Foldy, Phys. Rev. 67, 107 (1945).

[4] M. Lax, Rev. Mod. Phys. 23, 287 (1951).

[5] M. Lax, Phys. Rev. 85, 621 (1952).

[6] V. K. Varadan, V. V. Varadan, and Y. H. Pao, J. Acoust. Soc. Am. 63, 1310 (1978).

[7] D. J. Vezzetti and J. B. Keller, J. Math. Phys. 8, 1861 (1967).

[8] J. G. Fikioris and P. C. Waterman, J. Math. Phys. 5, 1413 (1964).

[9] B. Peterson and S. Ström, Phys. Rev. D 8, 3661 (1973).

[10] V. Twersky, J. Math. Phys. 19, 215 (1978).

[11] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), pp. 615.

[12] O. R. Cruzan, Q. Appl. Math. 20, 33 (1962).

[13] B. Peterson and S. Ström, Phys. Rev. D 10, 2670 (1974).

[14] V. N. Bringi and T. A. Seliga, Trans. IEEE AP-25, 575 (1977).

[15] V. N. Bringi and T. A. Seliga, Ann. Télécom. 32, 392 (1977).

[16] P. Barber and C. Yeh, Appl. Optics 14, 2684 (1975).