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Cancellation of infrared divergences in massive-quark potential scattering

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We present a proof of the cancellation of infrared divergences in the potential scattering of a massive quark when a sum is performed over soft-gluon emission. The proof applies to quantum chromodynamics and works to all orders in perturbation theory.

Several authors have verified the cancellation of infrared (IR) divergences coming from real and virtual soft gluons in quantum chromodynamics (QCD) to low orders.[1] Here we present a simple proof that IR divergences cancel to all orders in perturbation theory for the scattering of a massive quark by a singlet potential when a sum is performed over soft-gluon emission.

The process we consider is illustrated in the diagram of Fig. 1. We average over the initial spin and color f_i of the single fermion, and sum over all gluon emissions which take away less than some resolution energy $\epsilon \ll (p_i)_0$. This is equivalent to summing over a range in the spacelike momentum transfer q.

Since we only sum over final-state gluons we cannot immediately apply the Lee-Nauenberg-Kinoshita theorems[2] to prove finiteness, as these theorems require a sum over both initial and final degenerate states.

Our argument can be carried out in any standard gauge: The result is gauge invariant. It proceeds in three steps: (a) First, we identify regions in momentum space which can give rise to mass divergence. Such divergences can roughly be characterized as either truly IR, involving the vanishing of gluon momenta, or collinear, involving the coupling of parallel moving, on-shell, finite-energy gluons. (b) Next, we sum over the cuts of individual Feynman graphs such as Fig. 1. This sum is seen to eliminate all collinear divergence. Divergences left after the sum over cuts of a given graph are truly IR. (c) Finally, we sum over sets of graphs whose gluon sectors are topologically the same, but differ in their attachment to the fermion lines.

As in Refs. 3 and 4, we identify possible divergences by looking for *"pinch singular points"* (PSP's) of the Feynman integrals, that is, points where momentum-space contour integrals are trapped. To each such point corresponds a reduced diagram where all off-shell lines have been contracted. As discussed by Coleman and Norton,[5] the resulting graphs represent physically realizable processes with vertices reinterpreted as space-time points. Power counting[3,4] then determines which pinch singular points actually give rise to logarithmic divergences.

In our case, the physical processes associated with mass divergences must be of the type illustrated in Fig. 2, where we have indicated the possibility of a single jet J being produced in addition to the final quark line p_f. In general, soft gluons attach to all quark and gluon jet lines at the pinch singular point.

We choose to sum over all final states with the restriction that the final-state gluons have energies E_i, where $\sum_i E_i \le E_{\max}$. Then we sum over all such cuts of our reduced diagrams. Since all the gluons external to subdiagram S must be simultaneously soft, one can integrate over its entire range of internal momenta without violating the phase-space condition.

The sum over cuts gives the discontinuity of a Feynman integral for the reduced diagram (now an amplitude) in the neighborhood of the singular point in question, but integrated over all en-

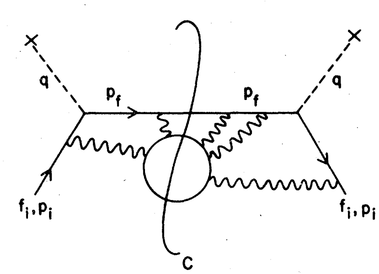

FIG. 1. Contribution to cross section for massive-quark potential scattering with soft-gluon emission. An average (trace) is taken over the initial quark color f_i. p_i and p_f denote the initial and final quark momenta.

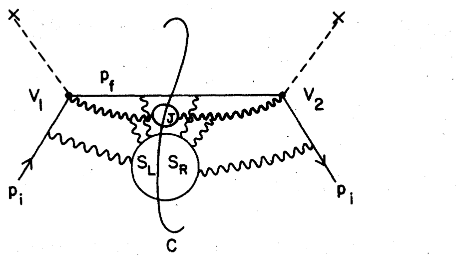

FIG. 2. A typical reduced diagram of the scattering at a pinch singular point. V_1 and V_2 denote hard vertices, the heavy wavy line to J a gluon jet, $S_{L,R}$ the (cut) soft-gluon Green's function. The light wavy lines are soft.

ergies.[4] One then reapplies the Coleman-Norton criterion[5] to the uncut integral. First, note there is no way that both the on-shell quark line p_f and the possible finite-energy (but less than $E_{\max}$) gluonic jet J can arrive at the space-time point corresponding to V_2 after free propagation from V_1. Therefore, collinear divergences due to gluonic jets cancel in the sum, and we only need to show that divergences due to soft gluons cancel.

By power counting,[3,4] these soft gluons may only couple to the fermion at three-point vertices. We organize into a "skeleton" expansion those graphs which give rise to such three-point vertices at the pinch singular point. Then we find that, because of Lorentz invariance, the soft gluons may only couple through the effective vertex $\gamma_\mu T$ (where T is the relevant color coupling matrix from the Lagrangian.)

A typical logarithmically divergent reduced diagram with n soft gluons divided among the lines p on either side of the (forward) scattering amplitude, and with an arbitrary number attached to the intermediate on-shell line p_f, is shown in Fig. 3.

We now show that cancellation occurs among the different eikonal fermion denominators which arise from attaching a given soft gluonic Green's function S in all ways.

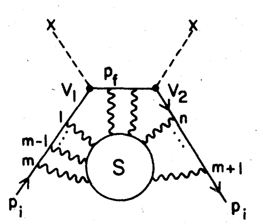

FIG. 3. Reduced diagram after the sum over cuts, 1 through m of the gluons are sequentially attached to the left quark line and $m+1$ through n are sequentially attached to the right. For all m, the color factor [Eq. (1)] is the same.

We will begin with the case of reduced graphs that are two-particle irreducible (2PI) in the vertical channel. Then at least one soft line is attached to p_f. In the sum over spins and colors a constant numerator matrix factor

$$(T_n \times T_{n-1} \times \cdots \times T_{m+1} \times T_m \times \cdots \times T_1) \cdot \prod_{i=1}^{n} (\not p \gamma_{\mu_i}) \not p \tag{1}$$

multiplies the spin-color factor occurring between hard vertices V_1 and V_2. The trace over the color and spin is taken. As mentioned before, the cancellation occurs between the different eikonal factors associated with (1) without any need to consider the factors between V_1 and V_2.

In (1) we have only exhibited the $\not p$ factors, which are responsible for the divergences when the singular point is approached. All other terms are at least of order k_i and so are nondivergent. Similarly, we only need consider the eikonal parts of the fermion denominators. The sum over all m of the eikonal factors associated with a fixed color group factor (1) in Fig. 3 is

$$D = \sum_{m=0}^{n} \frac{1}{a_m a_{m,m-1} \cdots a_{m,m-1,\ldots,1}} \times \frac{(-1)^{m-n}}{a_{m+1} a_{m+1,m+2} \cdots a_{m+1,\ldots,n}},$$

where

$$a_{i,j,\ldots,k} = 2p \cdot (k_i + k_j + \cdots + k_k). \tag{2}$$

Note that we have chosen a convention where the k_i always flow out of S.

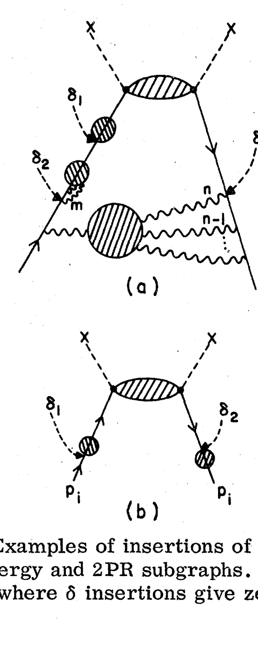

FIG. 4. (a) Examples of insertions of "fake momenta" δ_i into self-energy and 2PR subgraphs. (b) Example of self-energies where δ insertions give zero identically.

It is easy to see that the quantity D vanishes identically. Stated physically, unaccelerated charges (in this case, the fermion p_i) do not radiate.

The proof that $D=0$ follows immediately from

$$2\pi i D=\int_{-\infty}^{\infty} dx\,\frac{1}{x+i\epsilon}\prod_{i=1}^{n}\frac{1}{(x+a_{i,i-1,\ldots,1}+i\epsilon)}=0\,. \tag{3}$$

This eliminates soft-gluon divergences from PSP's whose reduced diagrams are 2PI. The two-particle reducible cases present a slight difficulty because eikonal denominators vanish identically for self-energies attached to the external lines.

This difficulty can be removed by introducing "fake" momenta δ_i into the highest-numbered gluon vertex of each of the 2PR or self-energy diagrams as illustrated in Fig. 4(a). The δ_i momenta are taken to route in the same direction as p. With these δ_i, the cancellation still works, and the physical result is recovered in the limit $\delta_i\to 0$.

For certain terms, finite δ's give identically zero without any sum over graphs. For example, in Fig. 4(b), the insertion of a δ before a final self-energy gives

$$\frac{1}{\not p+\not\delta}\,\Sigma(p)u(p)=0 \text{ when } p^2=0\,. \tag{4}$$

Thus, outgoing self-energy terms need not be included in the sum over graphs. The effect of such outgoing self-energies is just to cancel the wave-function renormalization factor $(Z^{1/2})^{-2}$ in the definition of the cross section.

To illustrate how the "δ mechanism" works, consider a self-energy graph with n vectors attaching to the fermion line as in Fig. 5(a). Near the SP, the integral is approximately equal to

$$\int \prod_\alpha d^4 l_\alpha F(l_\alpha)\,\frac{1}{\not p+\not\delta-m}\prod_{i=1}^{n-1}(\Gamma_i\not p)\,\frac{1}{2p\cdot\left(\sum_{j=1}^{i} l_j+\delta\right)}\times\delta\left(\sum_\alpha l_\alpha\right). \tag{5}$$

Since we integrate over *all* the internal momenta of $F(l_\alpha)$ at the PSP, $F(l_\alpha)$ is respectively even or odd if the number of its external lines is even or odd. Thus, when δ is strictly zero, the loop integral (5) is always odd in the l_i and therefore zero. Now we examine the neighborhood of the SP by letting the δ be finite.

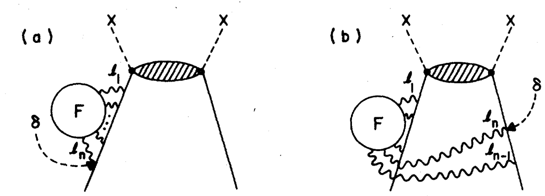

FIG. 5. (a) and (b) Two δ inserted 2PR graphs with the same color factor.

If we denote the $2p\cdot\sum_{j=1}^{i} l_j=C_i$, and $2p\cdot\delta=\Delta$, the denominators that come from the different signs of the l_j (the neighborhood of $l_j=0$ in the integration) give a combination of denominators equal to

$$\frac{1}{\Delta}\left(\frac{1}{\prod_{i=1}^{n-1}(C_i-\Delta)}-\frac{1}{\prod_{i=1}^{n-1}(C_i+\Delta)}\right)=\left[2\left(\sum_{i=1}^{n-1}\prod_{j\neq i}C_j\right)\Big/\prod_{i=1}^{n-1}C_i^{\,2}\right]+O(\Delta)\,. \tag{6}$$

Again, the sum of the self-energy and all possible 2PR graphs with all possible subsets of the lines l_j reattached [with the *same* color matrix factor—see Fig. 5(b)] to the outgoing line will cancel, leaving only the self-energy on the final line to be absorbed in renormalization.

In conclusion, we comment that our method of proof applies to QED as well, and is simpler than the usual proof of infrared cancellation. However, our method does not enable us to exhibit the exponentiation of the divergences or the ϵ dependence of the finite remainder.

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