Unstable-particle scattering and an analytic quasi-unitary isobar model

J.-L. Basdevant* and E. L. Berger
High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439
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An explicit model is constructed for the scattering of unstable particles such as $\rho\pi \to \rho\pi$, $K^*\pi \to K^*\pi$, and $\rho K \to \rho K$. It has correct analyticity properties and satisfies quasi-two-body unitarity. The essential ingredient of the approach is a Chew-Mandelstam function for unstable particles in which the particle-resonance branch cuts are located in second Riemann sheets of the three-particle complex energy plane. The model shall be applied in future studies of final states such as $\rho\pi$, $K^*\pi$, and ρK.

I. INTRODUCTION

High-quality data are accumulating on hadronic processes in which the final state includes unstable particles. Examples are the $\rho\pi$, $K^*\pi$, ρK, and ωK systems produced in diffractive reactions. The experimental analysis[1] of such final states is often based on the isobar model. In previous work,[2] we developed a unitarized Deck model, or a Deck model corrected with final-state interactions to study the $\rho\pi$ and the $K^*\pi/\rho K$ systems produced in diffractive collisions. The treatment of final-state interactions requires a unitary description of, e.g., the $\rho\pi \to \rho\pi$ scattering amplitude. We constructed this amplitude based on a K-matrix parametrization, treating the resonances (ρ and K^*) as stable. While appealing in various respects, this "unitary isobar model" is seriously deficient in that finite widths of resonances are ignored. Sharp threshold effects, including cusps, are manifest in our theoretical amplitudes, whereas in nature they are softened and smeared out by the effects of finite widths.

To avoid the pitfalls of the isobar model, one may follow one of the many three-body approaches which have been proposed and exploited.[3-7] In practice, however, all require some approximations before becoming tractable. Some also suffer from being heuristic relativistic extensions of the nonrelativistic Faddeev equation (even though the physical meaning of this procedure is understood to some extent).[8,9] More important is the fact that the three-body approach involves the solution of complicated integral equations. Results may easily be obscured by technical difficulties associated with resolving the equations. For phenomenological fits to data, it is also cumbersome to deal with solutions of integral equations; analytic expressions are more desirable.

We are motivated to consider that, at least to first approximation, $\rho\pi$ scattering is the scattering of a quasi-two-body system. This is in the spirit of the quark model in which the ρ is a $\bar{q}q$ bound state which becomes unstable only because of its coupling to less massive states. Within this context, we must devise a technique which permits the inclusion of the finite width of the ρ, but nevertheless retains tne quasi-two-body (isobar) structure of the amplitudes.

In this paper we present a simple method for treating particle-resonance scattering. It combines the simplicity of our previous two-body coupled-channel approach, together with the proper analytic structure of the amplitudes: viz., a right-hand unitarity cut starting at the stable-particle threshold $[s = 9m_\pi^2$ not $s = (m_\rho + m_\pi)^2]$, and a particle-resonance branch cut lying in the second sheet. The desired smearing effect due to finite resonance widths is obtained. The method consists essentially of constructing Chew-Mandelstam functions[10] for unstable-particle scattering.

In Sec. II, we summarize our K-matrix approach for coupled-channel scattering. In Sec. III, we present a derivation of Chew-Mandelstam functions for unstable-particle systems. The properties of these functions and their physical meaning are discussed in Sec. IV. In Sec. V, we present what may be termed an analytic quasiunitary isobar model. Concluding remarks are provided in Sec. VI. A technical treatment of final-state interactions is included in an Appendix.

II. K-MATRIX PARAMETRIZATION OF COUPLED-CHANNEL SCATTERING

We shall employ the standard K-matrix formalism to obtain a unitary parametrization of the $n \times n$ partial-wave T matrix for n two-body channels, with proper analyticity properties. The diagonal phase-space matrix is denoted by $\rho(s)$, with elements

$$\rho_{ij}(s) = \delta_{ij} \frac{2q_i}{\sqrt{s}}\, \theta(s - s_i) . \tag{1}$$

Here s_i is the physical threshold for channel i ($i = 1, \ldots, n$), and q_i is the center-of-mass momentum in channel i. A diagonal matrix $C(s)$ is introduced whose elements are analytic in s except for right-hand (elastic) branch cuts across which the discontinuities are provided by

$$\mathrm{Im}C(s)=\rho(s)\,. \tag{2}$$

For convenience, we choose to define $C(s)$ with a subtraction at $s=0$: $C(0)=0$. The elements of $C(s)$ have the form $C_{ij}=C_i(s)\delta_{ij}$, with

$$\begin{aligned}C_i(s)&\equiv C(s;m,\mu)\\&=-\frac{2}{\pi}\left\{-\frac{1}{s}\,[(m+\mu)^2-s]^{1/2}[(m-\mu)^2-s]^{1/2}\right.\\&\qquad\times\ln\frac{[(m+\mu)^2-s]^{1/2}+[(m-\mu)^2-s]^{1/2}}{2(m\mu)^{1/2}}\\&\qquad\left.+\frac{m^2-\mu^2}{2s}\ln\frac{m}{\mu}-\frac{m^2+\mu^2}{2(m^2-\mu^2)}\ln\frac{m}{\mu}-\frac{1}{2}\right\}.\end{aligned} \tag{3}$$

Here m and μ are the two masses of channel i. For historical reasons, we term $C_i(s)$ the Chew-Mandelstam (CM) function.

Introducing a matrix $K(s)$ which is meromorphic in s, we construct a unitary analytic partial-wave $T(s)$ matrix as follows:

$$T(s)=[1-K(s)C(s)]^{-1}K(s)\,. \tag{4}$$

The $\mathcal{S}$ matrix is

$$\mathcal{S}=1+2i\rho^{1/2}T\rho^{1/2}\,, \tag{5}$$

and the operator S which appears in the Hilbert problem of final-state interactions, or the unitarity relation,[11] is

$$S=(1-KC^+)^{-1}(1-KC^-)\,, \tag{6}$$

where C^+ and C^- are the values of $C(s)$ above and below its cut. The unitarity relation is

$$T^+=ST^-\,. \tag{7}$$

III. CHEW-MANDELSTAM FUNCTION FOR PARTICLE-RESONANCE SCATTERING

In many instances it is of practical interest to investigate the scattering of unstable particles, such as $\rho\pi\to\rho\pi$, $K^*\pi\to K^*\pi$, and $\rho K\to\rho K$. This is done, for example, in the isobar model used in the analysis of multiparticle final states. If the resonances are treated as stable particles, and the formalism of Sec. II is employed, the unitarity discontinuities begin on the real axis. Consequently, threshold effects are more abrupt (e.g., sharp cusps at the opening of inelastic channels) than in the true physical situation where the finite width of the resonance softens the singularity.

In a genuine three-body (or multibody) approach,[3-7] these difficulties do not arise. However, whether approximate or rigorous, the three-body approach requires solving integral equations, a technically complicated and time-consuming task, especially when one is attempting to vary parameters to obtain a fit to data.

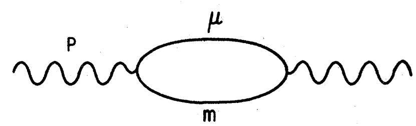

FIG. 1. Feynman graph representing the Chew-Mandelstam function for the scattering of stable particles with masses m and μ.

To circumvent both of these difficulties, and to retain the concept that particle-resonance scattering is approximately a quasi-two-body process, but with the two-body discontinuity beginning in the second Riemann sheet of the three-body complex energy plane, we introduce new Chew-Mandelstam functions for unstable particles.

The standard CM functions may be represented by the Feynman loop graph of Fig. 1, where P is a momentum flow ($P^2=s$). Assuming that there is no momentum dependence in the vertices and making a subtraction at $s=0$, we derive (up to factors of π)

$$C(P^2;m,\mu)=\int\frac{d^4k}{k^2-m^2}\left[\frac{1}{(P-k)^2-\mu^2}-\frac{1}{k^2-\mu^2}\right]. \tag{8}$$

If one of the particles is unstable, we may follow the example of the σ model,[12] and replace the stable (or "bare") propagator for this particle by the dressed propagator

$$\frac{1}{k^2-m^2}\to\frac{1}{k^2-m^2+f^2\Sigma(k^2)}\equiv\frac{1}{d(k^2)}\,. \tag{9}$$

Here f denotes a coupling strength, and $\Sigma(k^2)$ is the mass operator. We assume that m^2 and f^2 are such that Eq. (9) has a pole in the complex plane (not on the real axis) at the resonance position $m^*=m_R-i\Gamma/2$. The function $\Sigma(k^2)$ has a right-hand cut starting at $k^2=(m_1+m_2)^2$, where m_1 and m_2 are the masses of the particles into which the resonance decays. We therefore define an unstable-particle Chew-Mandelstam function

$$\tilde{C}(P^2;m^*,\mu)=\int\frac{d^4k}{k^2-m^2+f^2\Sigma(k^2)}\times\left[\frac{1}{(P-k)^2-\mu^2}-\frac{1}{k^2-\mu^2}\right]. \tag{10}$$

This function is represented in Fig. 2. Equation (10) may be transformed readily into a one-dimensional integral if we use a dispersion relation for

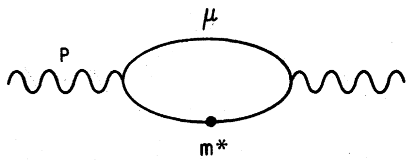

FIG. 2. Feynman graph representing the Chew-Mandelstam function for the scattering of a stable particle with mass μ from a resonance system of mass m^*.

$d^{-1}(k^2)$,

$$\frac{1}{k^2-m^2+f^2\Sigma(k^2)}=\frac{1}{\pi}\int_{(m_1+m_2)^2}^{\infty}\frac{f^2\,\mathrm{Im}\Sigma(s')}{|s'-m^2+f^2\Sigma(s')|^2}\times\frac{ds'}{k^2-s'}. \tag{11}$$

We obtain

$$\tilde{C}(s;m^*,\mu)=\frac{1}{\pi}\int_{(m_1+m_2)^2}^{\infty}ds'\frac{f^2\,\mathrm{Im}\Sigma(s')}{|d(s')|^2}\times C(s;\sqrt{s'},\mu). \tag{12}$$

Note that $\tilde{C}(s)$ is a superposition of "stable-particle" CM functions $C(s;m,\mu)$ with weights $(f^2/\pi)\,\mathrm{Im}\Sigma(m^2)/|d(m^2)|^2$. A similar idea is suggested in Ref. 5 in a somewhat more sophisticated context.

We conclude this section with three comments:

(i) In the limit of a very narrow resonance, i.e., $\Gamma/m \ll 1$, or $f^2\to 0$, the weight

$$\frac{f^2}{\pi}\frac{\mathrm{Im}\Sigma(s')}{|d(s')|^2}\to\delta(s'-m^2)\,, \tag{13}$$

and the usual stable-particle CM formula is recovered from Eq. (12).

(ii) Pinching arguments may be used to show that $\tilde{C}(s;m^*,\mu)$ is analytic in the complex s plane except for a three-body discontinuity from $s=(m_1+m_2+\mu)^2$ to infinity. There is also a particle resonance cut on the second-sheet, from $s=(m_R-i\Gamma/2+\mu)^2$ to ∞. This structure is depicted in Fig. 3. For unstable-particle scattering, the function $\tilde{C}$ is therefore a simple and appealing alternative to the stable-particle CM function. It has the desired analyticity properties.

(iii) When *both* particles are unstable (e.g., $\rho\omega$, ρK^*), it is easy to generalize Eq. (12), and a double integral results.

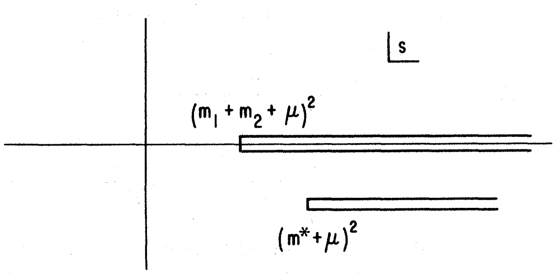

FIG. 3. Singularity structure of $\tilde{C}(s;m^*,\mu)$.

IV. EXPLICIT $\tilde{C}$ FUNCTIONS INVOLVING THE ρ AND THE K^*

We provide parametrizations for the propagators $d^{-1}(s)$ for the ρ and the K^* in order to give a practical illustration of the above formalism and because we shall use the resulting $\tilde{C}$ functions in subsequent papers.

We work in the elastic approximation. A good parametrization for the ρ propagator is

$$d_\rho(s)=s-m_1^2+f_1^2(s-4m_\pi^2)C(s;m_\pi,m_\pi)\,, \tag{14}$$

with $m_1^2=0.575\ \mathrm{GeV}^2$ and $f_1^2=0.196$. These provide a resonance pole at $m_\rho=0.77$ GeV, with $\Gamma_\rho=0.14$ GeV. Similarly, we write

$$d_{K^*}(s)=s-m_2^2+f_2^2[s-(m_K+m_\pi)^2]\,C(s;m_K,m_\pi)\,, \tag{15}$$

with $m_2^2=0.817\ \mathrm{GeV}^2$ and $f_2^2=0.181$. These provide a pole at $m_{K^*}=0.89$ GeV, with $\Gamma_{K^*}=0.05$ GeV.

Owing to the centrifugal barrier threshold factors $(s-4m_\pi^2)$ and $[s-(m_K+m_\pi)^2]$ in Eqs. (14) and (15), the term $\mathrm{Im}\Sigma(s')$ in Eq. (12) grows linearly with s' at infinity. This causes no problem since the function $C(s;m,\mu)$ behaves as m^{-2} for fixed s and μ as $m\to\infty$; the integral in Eq. (12) is convergent. For higher partial waves ($l\geq 2$), however, form factors of the type $[(s-s_{\mathrm{thr}})/(s+s_0)]^l$ must be introduced; these act as natural cutoffs and provide extra parameters enabling one to represent the phase shifts more accurately over a larger energy interval. For our P-wave situations (ρ, K^*), adding a cutoff does not change any results appreciably.

In constructing the above "propagators" for the ρ and K^*, we have relied on the fact that the P wave $I=1$ $\pi\pi$ and $I=1/2$ $K\pi$ amplitudes are fitted to very good accuracy by simple one-pole formulas such as[13]

$$t_{\pi\pi}^{I=l=1}(s)=\frac{-f_1^2(s-4m_\pi^2)}{d_\rho(s)}. \tag{16}$$

The fact that a propagator $d_\rho^{-1}(s)$ can be constructed which has precisely the $I=l=1$ phase is necessary for the consistency and unitarity properties of the isobar model. In this respect it is important that many hadron-hadron amplitudes at low energy (except, perhaps S waves) are dominated by a small number of relatively narrow resonances. We remark that our propagators should be regarded as practical approximations to the Omnès function.[14] For example,

$$d_\rho(s)=\lambda\exp\left(\frac{-s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\delta_{\pi\pi}^{I=l=1}(s')ds'}{s'(s'-s)}\right), \tag{17}$$

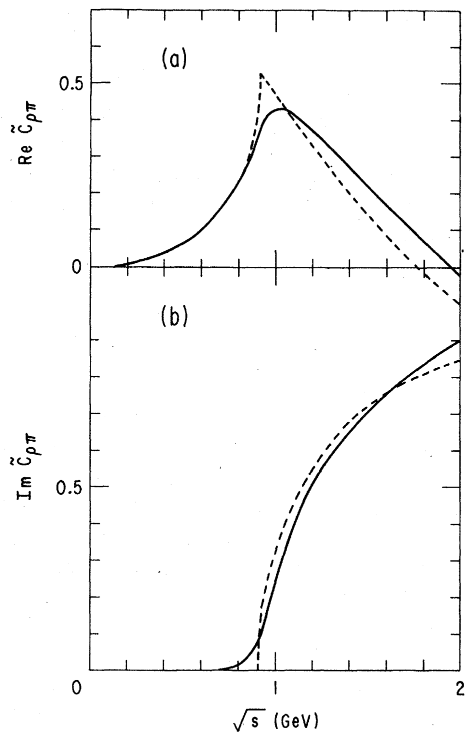

FIG. 4. The function $\tilde{C}_{\rho\pi}(s)$ (solid line) is compared with the stable particle Chew-Mandelstam function (dashed line): (a) real parts, (b) imaginary parts.

where λ is a normalization constant such that Eq. (13) holds in the limit of a narrow resonance. The normalization condition fixing λ can be written more generally as

$$\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{-\mathrm{Im}\,d_\rho(s')}{|d_\rho(s')|^2}ds'=1\,. \tag{18}$$

In order to illustrate the softening or smearing effect of our procedure, we compare the function $\tilde{C}_{\rho\pi}(s)$ from Eq. (12) with the Chew-Mandelstam function obtained if the ρ is treated as stable. This comparison is presented in Fig. 4; the smearing effect is obvious. The imaginary part $\mathrm{Im}\tilde{C}(s)$ represents the "smeared" phase-space factor $2\tilde{q}/\sqrt{s}$ which enters in the specification of unstable-particle cross sections. We may make restricted selections ("cuts") in the $\pi\pi$ or $K\pi$ mass and evaluate the phase-space factor which results from an integral of ImC over a limited s' region in Eq. (12) about the resonance peak. This is analogous to the procedure followed experimentally.

V. ANALYTIC, QUASIUNITARY ISOBAR MODEL

A. Skeleton amplitudes and full amplitudes

An isobar or skeleton amplitude for particle resonance scattering may be constructed by the same procedure used in Sec. II for the stable-particle situation. We introduce a matrix $\tilde{C}(s)$ analogous to $C(s)$ of Eqs. (2) and (3), except that we replace the usual CM function by its generalization, Eq. (12), whenever channel involves an unstable particle. Using a meromorphic matrix $K(s)$, we again form an amplitude

$$\tilde{T}(s)=[1-K(s)\tilde{C}(s)]^{-1}K(s)\,. \tag{19}$$

The matrix $\tilde{T}(s)$ depends only on the total invariant energy variable s; it does not depend on the subenergies. It expresses the entire s behavior of the final amplitude.

We now construct the full amplitude for specific stable decay products of the isobars. It is obtained upon multiplying the appropriate element of $\tilde{T}$ by the corresponding isobar propagators and their couplings to the decay particles. We use α to label the initial-state isobar in channel i, and β for the final-state isobar in channel j. Their propagators are $d_\alpha^{-1}(s_{12})$ and $d_\beta^{-1}(s'_{12})$, and their couplings are f_i and f_j. We obtain

$$T_{ij}(s,s_{12},s'_{12})=\frac{f_\alpha(s_{12})}{d_\alpha(s_{12})}\tilde{T}_{ij}(s)\frac{f_\alpha(s'_{12})}{d_\beta(s'_{12})}\,. \tag{20}$$

The scattering process is sketched in Fig. 5. We have assumed for simplicity that we are dealing with nonidentical particles, and only with orbital S-wave states in the system s.

In Eq. (20), we separate completely the dependence on subenergies from the dependence on the total energy. Since the propagators $d_\alpha^{-1}(s)$ have the correct analytic structure and proper phase of the corresponding decay amplitude, Eq. (20) ensures that T_{ij} satisfies the correct discontinuity relation in the subenergies s_{12} and s'_{12}.

B. Unitarity and analyticity properties

As constructed, our matrix $\tilde{T}(s)$ possesses proper analytic structure in the complex s plane, in the sense that the particle-resonance scattering cuts lie in second sheets. Our full amplitude $T(s,s_\alpha,s_\beta)$, Eq. (20), does *not* exactly satisfy the full three-body (or n-body) unitarity relation

$$\Delta T=T(s+i\epsilon,s_\alpha,s_\beta)-T(s-i\epsilon,s_\alpha,s_\beta)$$
$$=2i\int T d\Omega T^\dagger\,, \tag{21}$$

where $\int d\Omega$ represents the integration over all intermediate variables. However, owing to the definition of our functions $\tilde{C}(s;m^*,\mu)$ of Sec. III, it is apparent that all the quasi-two-body unitarity contributions represented in Fig. 6(a) are included. Omitted are the rearrangement contributions of Fig. 6(b). In this sense, our full amplitude satisfies only quasi-two-body unitarity in the spirit of

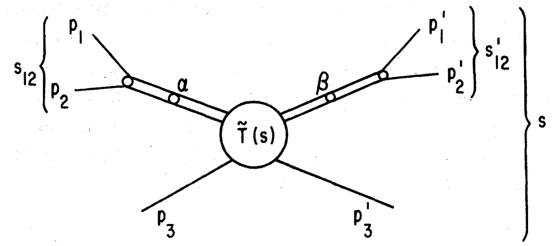

FIG. 5. Diagram representing our full isobar amplitude.

the isobar model.

Various arguments suggest that the missing unitarity contributions of Fig. 6(b) play a minor role and can safely be neglected. First, it is borne out in many calculations[4,5,8] that, provided the resonances are not dynamical effects, but, rather, are "elementary" objects (as in the quark model) which are unstable because of their coupling to less massive states, the rearrangement effect of Fig. 6(b) or, equivalently, the rescattering series of Fig. 7 contributes only in a negligible way to the full three-body amplitude. Second, we are aware that the analytic structure in s of our full amplitude is incomplete in that, for example, the Peierls singularities[15] are absent. These singularities are present in the series of Fig. 7, or alternatively, in diagrams of the type represented in Fig. 8, and are found at positive values of Res. However, they lie on Riemann sheets which are distant from the physical region.[9,16] While they have an influence on the structure of Dalitz plots,[17] these singularities contribute smoothly, if at all significantly, to the behavior of the full amplitude as a function of s. Finally, insofar as s behavior is concerned, which is our main point of interest,

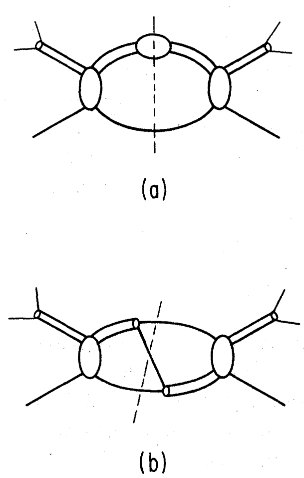

FIG. 6. (a) The unitarity contributions included in our full isobar amplitude. (b) Unitarity contributions which are missing.

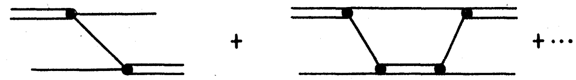

FIG. 7. Particle-resonance (π-ρ) rescattering series.

we can simulate the effect of these various smooth contributions by the appropriate use of parameters in the K matrix.

VI. CONCLUSIONS

By introducing and employing our unstable-particle Chew-Mandelstam functions $\tilde{C}(s; m^*, \mu)$, we have simplified the problem of constructing a unitary, analytic isobar model. One price is our simplification of the subenergy dependence of the amplitude, although ours is reasonable. Since most two-body amplitudes (except for S waves) are dominated by relatively narrow resonances (poles), we treat a genuine three-body problem in a way which is perhaps no more approximate than many integral-equation approaches.

While our method is appealing for treatment of the scattering of relatively narrow resonances, we do not have a prescription for broad two-body S-wave states, such as the ϵ $(\pi\pi)$ or κ $(K\pi)$, which cannot be treated as pole dominated. An idea which comes to mind is the use of Omnès functions such as Eq. (17) for the propagators. The method of Secs. III, IV, and V could be followed. Unfortunately, the "couplings" $f_\alpha(s_{12})$ of Eq. (20) do not then have a clear meaning. The problem of broad states requires more careful investigation.

The three-body problem may be considered as a coupled-channel problem involving a continuous infinity of channels. In comparisons with the three-body N/D equations of Mandelstam,[7] our method consists of truncating the set of intermediate states in the full unitarity relation, and thereby simplifying the phase-space integrations. It is possible that a scheme similar to ours could be set up with coupled N/D equations.[7,18] Although we have not done this, we believe the N/D method involves more integrals than the K-matrix approach and, therefore, would be more complicated technically for fits to data.

The principal domain of application of the present technique will be our continuing study[2] of the production and decay systematics of axial-vector me-

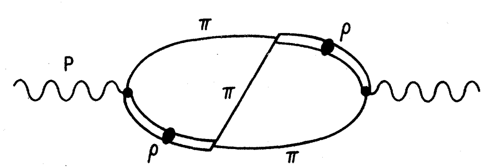

FIG. 8. Feynman graph similar to that in Fig. 2 but which contains Peierls singularities.

sons ($A_1, Q, \ldots$), and of other systems produced in "diffractive" processes. The smearing effect and correct analyticity structure of the $\tilde{C}(s; m^*, \mu)$ functions is of critical practical importance in the Q-meson system, where the ρK threshold occurs in the center of the large enhancement in the $K^*\pi$ mass distribution.

ACKNOWLEDGMENT

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APPENDIX: FINAL-STATE INTERACTIONS

Our treatment of final-state interactions is presented in detail elsewhere.[2,11] In this Appendix we elaborate on some technical aspects which are of practical interest and which will be used in future work.[19]

The matrix $D(s)$, which is the basic tool in the final-state interaction problem, is constructed as[11]

$$D(s)=(1-KC)^{-1}R\,, \tag{A1}$$

where K and C are the functions defined in Sec. II, and R is a meromorphic matrix chosen so that $\mathrm{Det}[D(s)]$ has neither poles nor zeros on the entire first sheet of the complex s plane. In dealing with unstable particles, we replace C by $\tilde{C}$, defined in Sec. III. In Ref. 11 a general method was presented for constructing R. Here we provide an explicit algebraic procedure which, although not general, turns out to be useful in practical numerical work.

We consider n coupled channels and a K matrix with n poles. We introduce a matrix g of the coupling constants g_{ij}, of rank n, and a diagonal matrix Δ for the poles

$$\Delta_{ij}=\delta_{ij}/(s_i-s)\,. \tag{A2}$$

The matrix K is therefore expressed as

$$K=g\Delta g^t\,, \tag{A3}$$

where g^t is the transpose of g. Setting

$$R=g\Delta\,, \tag{A4}$$

and defining

$$D(s)=(1-KC)^{-1}g\Delta\,, \tag{A5}$$

we may show that the determinant of $D(s)$ has no zeros at $s=s_i$. The columns of $D(s)$ behave as s^{-1} as $s\to\infty$. Therefore, if the Muskhelishvili indices are all equal to 1, Eq. (A5) is the proper D matrix. For completeness, we should search for zeros and/or poles of $\mathrm{Det}(D)$ for negative s (left-hand singularities), but even if present they will not affect results significantly if they are sufficiently distant.

Under these conditions,

$$D=(\Delta^{-1}g^{-1}-g^t\tilde{C})^{-1} \tag{A6}$$

and

$$\mathrm{Im}D^{-1}=-g^t\,\mathrm{Im}\tilde{C}\,. \tag{A7}$$

If we choose to add a regular contribution to the K matrix (a constant background, for example),

$$K=g\Delta g^t+\mathcal{A}\,, \tag{A8}$$

then R retains the same form, and we obtain

$$D=[\Delta^{-1}g^{-1}-(g^t+\Delta^{-1}g^{-1}\mathcal{A})\tilde{C}]^{-1}\,, \tag{A9}$$

with

$$\mathrm{Im}D^{-1}=-(g^t+\Delta^{-1}g^{-1}\mathcal{A})\,\mathrm{Im}\tilde{C}\,. \tag{A10}$$

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*Permanent address: Laboratoire de Physique Théorique et Hautes Energies, Université Pierre et Marie Curie, Tour 16, 1 étage, 4 Place Jussieu 75230 Paris, France; Laboratoire associé au CNRS.

[1]Yu. M. Antipov *et al.*, Nucl. Phys. B63, 141 (1973); *ibid.* B63, 153 (1973); G. Ascoli *et al.*, Phys. Rev. Lett. 33, 610 (1974); G. W. Brandenburg *et al.*, *ibid.* 36, 703 (1976); further references in all of these papers.

[2]J. L. Basdevant and E. L. Berger, Phys. Rev. Lett. 37, 977 (1976); Phys. Rev. D 16, 657 (1977); Phys. Rev. Lett. 40, 994 (1978).

[3]I. J. R. Aitchison, J. Phys. G 3, 121 (1977); I. J. R. Aitchison and R. J. A. Golding, *ibid.* 4, 43 (1978); Phys. Lett. 59B, 288 (1975), and further references therein.

[4]R. S. Longacre and R. Aaron, Phys. Rev. Lett. 38, 1509 (1977), and further references therein.

[5]D. D. Brayshaw, Phys. Rev. D 18, 2638 (1978), and further references therein.

[6]V. A. Alessandrini and R. L. Omnès, Phys. Rev. 139, B167 (1965); R. Blankenbecler and R. Sugar, *ibid.* 142, 1051 (1966); D. Z. Freedman, C. Lovelace, and J. M. Namyslowski, Nuovo Cimento 43, 258 (1966).

[7]S. Mandelstam, Phys. Rev. 140, B375 (1965).

[8]G. Mennessier, J. -Y. Pasquier, and R. Pasquier, Phys. Rev. D 6, 1351 (1972).

[9]J. -L. Basdevant and R. L. Omès, Phys. Rev. Lett. 17, 775 (1966).

[10]G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960); B. W. Lee, *ibid.* 120, 325 (1960).

[11]O. Babelon *et al.*, Nucl. Phys. B113, 445 (1976); B114, 252 (1976).

[12]M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960); B. W. Lee, Nucl. Phys. B9, 649 (1969); J. -L. Basdevant and B. W. Lee, Phys. Rev. D 2, 1680 (1970);

[13]J. L. Basdevant, C. D. Froggatt, and J. L. Petersen, Nucl. Phys. B72, 413 (1974); J. L. Basdevant, P. Chapelle, C. Lopez, and M. Sigelle, *ibid.* B98, 285 (1975).

[14]R. Omès, Nuovo Cimento 8, 316 (1958); and in *High*

Energy Physics, 1965 Les Houches Lectures, edited by C. DeWitt and M. Jacob (Gordon and Breach, New York, 1965), p. 89.
[15]R. E. Peierls, Phys. Rev. Lett. 6, 641 (1961); 12, 50 (1964).
[16]R. Hwa, Phys. Rev. 130, 2580 (1963); C. Goebel, Phys. Rev. Lett. 13, 143 (1964); G. Wanders, Helv. Phys. Acta 38, 142 (1965).
[17]C. Schmid, Phys. Rev. 154, 1363 (1967).
[18]J. Bjorken, Phys. Rev. Lett. 4, 473 (1960); J. D. Bjorken and M. Nauenberg, Phys. Rev. 121, 1250 (1961).
[19]J.-L. Basdevant and E. L. Berger, following paper, Phys. Rev. D 19, 246 (1979).