Effective Lagrangians at finite temperature

Walter Dittrich*
Institute for Advanced Study, Princeton, New Jersey 08540
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We compute the one-loop effective potential at finite temperature for scalar and spinor QED in the presence of a constant magnetic field.

I. INTRODUCTION

A few years ago Weinberg[1] and the authors of Ref. 2 computed the temperature-dependent potential of various models in field theory. It was shown how finite-temperature effects could restore a symmetry which is broken at zero temperature.[3] Several models were investigated, and the one-loop effective potential at finite temperature was then calculated using various computational methods. Weinberg used operator techniques, while Dolan and Jackiw employed functional methods.

It is the purpose of this paper to study in some detail thermal effects in QED. That is, we want to examine the Heisenberg-Euler Lagrangian at finite temperatures. To our knowledge, this problem has not been dealt with in the literature. We also offer a new approach to the computational side of the problem by employing the elegant method of proper time.[4] This method is introduced by way of illustration within a simple scalar field theory. It is assumed in this article that all chemical potentials vanish, i.e., we are interested in states of thermodynamic equilibrium in which all conserved quantum numbers have mean value zero. Pair production (or annihilation) may be considered, from the thermodynamic point of view, as the "chemical reaction" $e^+ + e^- \to \gamma$. The chemical potential of the photon gas is zero, and hence the equilibrium condition will be of the form $\mu_+ + \mu_- = \mu_\gamma = 0$. Since the number of electrons and positrons is equal, the equilibrium condition relating to pair production (annihilation) is therefore simply given by $\mu_+ = \mu_- = 0$. We will then show that the one-loop effective potential at finite temperature represents—in the language of thermodynamics—the contribution of the vacuum energy to the total free energy in presence of an external constant field. The free energy, or, equivalently, the thermodynamic potential of a relativistic Fermi (Bose) gas, will be given for massive and massless spinor and scalar particles. As a particular check of our procedure we will rederive Planck's formula by setting the mass of the particle associated with the loop equal to zero.

Our whole treatment of the problem is based on a fairly old representation of the Green's function of a charged particle in a constant external field.[5] This "momentum" representation has been rederived by Tsai[6] using Schwinger's proper-time techniques and has been applied with great success to numerous problems in QED in presence of external fields. Hence, it seems to us to be the most natural representation in the context of one-loop thermal corrections in any field theory with constant background field.

II. CALCULATION OF THE THERMAL EFFECTIVE POTENTIAL IN A SCALAR FIELD THEORY

In this section we want to consider a field theory with scalar particles, mass m, coupled to an external field $\phi(x)$:

$$[-\partial^2 + m^2 - g\phi(x)]\Delta_+(x,x'|\phi) = \delta(x-x')\,.$$

The one-loop effective potential, i.e., the physical process that describes the effect that the external field $\phi(x)$ can have on a single-particle loop, is given by the vacuum-to-vacuum amplitude

$$\langle 0_+|0_-\rangle^\phi = \exp[iW^{(1)}(\phi)]\,,$$

where

$$iW^{(1)}(\phi) = \mathrm{Tr}\ln(1-g\phi\Delta_+)^{-1} = \mathrm{Tr}\ln(\Delta_+[\phi]/\Delta_+)\,. \qquad (2.1)$$

Here the Green's function in configuration space is[7]

$$\Delta_+(x,x'|\phi) = i\int_0^\infty ds\, e^{-is\kappa^2} e^{-Y(is)} \times \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-x')} e^{-pX(is)p - f(is)p}\,, \qquad (2.2)$$

where

$$X(s) = c^{-1}\tanh(cs)\,,$$
$$f(s) = -2ic^{-2}\left(1-\frac{1}{\cosh(cs)}\right)b\,, \qquad (2.3)$$
$$Y(s) = \tfrac{1}{2}\,\mathrm{tr}\ln[\cosh(cs)] + b\cdot c^{-3}[\tanh(cs) - cs]\cdot b\,,$$

and

$$\kappa^2 = m^2 - g\phi(x)$$
$$b_\mu(x) = -g\partial_\mu \phi(x)\,, \qquad (2.4)$$
$$c^2_{\mu\nu}(x) = -2g\partial_\mu \partial_\nu \phi(x)\,.$$

Making use of the relation

$$i\frac{\partial \mathfrak{L}^{(1)}}{\partial \kappa^2} = -i\int_0^\infty ds\, \exp[-is(m^2 - \partial^2 - g\phi)] = -\Delta_+[\phi]\,, \qquad (2.5)$$

we can immediately compute the one-loop effective Lagrangian by a simple integration. However, according to Eq. (2.1) we need only the diagonal part of $\Delta_+(x, x'|\phi)$, which we take from Eq. (2.2). Together with (2.5) we then obtain

$$\mathfrak{L}^{(1)}[\phi] = -i\int_0^\infty \frac{ds}{s} e^{-is\kappa^2} e^{-Y(is)} \times \int \frac{d^4p}{(2\pi)^4} e^{-pX(is)p} e^{-f(is)p}\,. \qquad (2.6)$$

This effective Lagrangian has been worked out several times in the literature and has become known as the Coleman-Weinberg Lagrangian.[8,7] For a constant background field ϕ_0, i.e., for the special case where the external field transfers zero momenta, the result is

$$V_{\text{eff}}[\phi_0] \equiv -\mathfrak{L}^{(1)}[\phi_0] = \frac{1}{32\pi^2}\Big[(m^2 - g\phi_0)^2 \ln\Big(1 - \frac{g\phi_0}{m^2}\Big) + g\phi_0 m^2 - \tfrac{3}{2}(g\phi_0)^2\Big]\,. \qquad (2.7)$$

In the sequel we always want to assume the external field to be constant, so that its only effect is to change the mass of the scalar particle. Expression (2.6) then simplifies to

$$\mathfrak{L}^{(1)}[\phi_0] = -i\int_0^\infty \frac{ds}{s} e^{-is\kappa^2} \int \frac{d^4p}{(2\pi)^4} e^{-ip^2 s}\,, \qquad (2.8)$$

since $Y(s) \to 0$, $f \to 0$, and $X(is) \to is$ in the limit of $\phi = \phi_0 = \text{const}$.

So far, everything holds for zero temperature, $T = 0$. In order to include a finite temperature, we have to make the replacement[1,2]

$$\int \frac{d^4p}{(2\pi)^4} \to \frac{i}{\beta}\sum_{n=-\infty}^{+\infty} \int \frac{d^3p}{(2\pi)^3}\,,$$
$$p^2 = -\omega_n{}^2 + \vec{p}^2\,, \quad \beta = 1/kT\,, \qquad (2.9)$$

where $\omega_n = 2\pi i n/\beta$.

Time-translation invariance is guaranteed by observing that for constant external field

$$\Delta_+(x, x'|\phi_0) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} i\int_0^\infty ds\, e^{-is(p^2+\kappa^2)}\,. \qquad (2.10)$$

Substituting (2.9) into (2.8) yields, after performing the Gaussian integration with respect to the space components $\vec{p}$,

$$\mathfrak{L}^{(1)} = \frac{1}{\sqrt{i}}\frac{1}{8\pi^{3/2}}\int_0^\infty \frac{ds}{s^{5/2}} e^{-is\kappa^2} \frac{i}{\beta}\,\theta_3\Big(0, -\frac{4\pi s}{\beta^2}\Big)\,, \qquad (2.11)$$

where

$$\theta_3\Big(0, -\frac{4\pi s}{\beta^2}\Big) = \sum_{n=-\infty}^{+\infty} \exp\Big[-is\Big(\frac{2\pi n}{\beta}\Big)^2\Big]$$

denotes the Jacobi function. $\mathfrak{L}^{(1)}$ has to be properly normalized so as to reproduce the result (2.7) when we go to the $T = 0$ limit. This can easily be done by the following trick ($\sigma = is(4\pi^2/\beta^2)$):

$$\sum_{n=-\infty}^{+\infty} e^{-\sigma n^2} = \sum_{-\infty}^{+\infty}\Big(\frac{\pi}{\sigma}\Big)^{1/2} \exp\Big(-\frac{\pi^2}{\sigma}n^2\Big) = \sum_{-\infty}^{+\infty}\Big(\frac{1}{i\pi s}\Big)^{1/2}\frac{\beta}{2}\exp\Big(\frac{in^2\beta^2}{4s}\Big)\,,$$

which becomes by taking out $n = 0$

$$\Big(\frac{1}{i\pi s}\Big)^{1/2}\frac{\beta}{2} + {\sum_{-\infty}^{+\infty}}'\Big(\frac{1}{i\pi s}\Big)^{1/2}\frac{\beta}{2}\exp\Big(\frac{in^2\beta^2}{4s}\Big)\,. \qquad (2.12)$$

In the light of this simple formula we can rewrite (2.11) according to

$$\mathfrak{L}^{(1)}[\phi_0, T] = \frac{1}{16\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-is\kappa^2} + \text{c.t.} + \frac{1}{\sqrt{i}}\frac{1}{8\pi^{3/2}}\int_0^\infty \frac{ds}{s^{5/2}} e^{-is\kappa^2}\frac{i}{\beta}\Big[\theta_3\Big(0, -\frac{4\pi s}{\beta^2}\Big) - \Big(\frac{1}{i\pi s}\Big)^{1/2}\frac{\beta}{2}\Big]\,. \qquad (2.13)$$

The first term, including the contact terms, is identical with expression (2.7), i.e., it represents the $T = 0$ result, whereas the second term contains the combined effect of external field and temperature. We can easily isolate the pure temperature contribution by putting $\phi_0 = 0$ in Eq. (2.13). We then obtain

$$\begin{aligned}\mathfrak{L}^{(1)}[0, T] &= \frac{1}{8\pi^{3/2}}\frac{\sqrt{i}}{\beta}\int_0^\infty \frac{ds}{s^{5/2}} e^{-ism^2}\Big[\sum_{-\infty}^{+\infty}\Big(\frac{1}{i\pi s}\Big)^{1/2}\exp\Big(i\frac{n^2\beta^2}{4s}\Big) - \Big(\frac{1}{i\pi s}\Big)^{1/2}\Big]\frac{\beta}{2}\\ &= \frac{1}{16\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-ism^2}\Big[\sum_{-\infty}^{+\infty}\exp\Big(i\frac{\beta^2 n^2}{4s}\Big) - 1\Big]\\ &= \frac{1}{16\pi^2}{\sum_{-\infty}^{+\infty}}'\int_0^\infty \frac{ds}{s^3} e^{-ism^2}\exp\Big(i\frac{\beta^2 n^2}{4s}\Big)\,.\end{aligned} \qquad (2.14)$$

The integral occurring in (2.14) can be found in Ref. 9 $[s \to (1/i)s]$,

$$\int_0^\infty x^{\nu-1} \exp\left(-\alpha \frac{1}{x} - \gamma x\right) = 2\left(\frac{\alpha}{\gamma}\right)^{\nu/2} K_\nu(2\sqrt{\alpha\gamma}\,)\,, \tag{2.15}$$

so that we end up with

$$\mathcal{L}^{(1)}[0, T] = -\frac{1}{8\pi^2} \sum_{n=1}^\infty 2\frac{4m^2}{\beta^2} K_2(\beta m n)$$
$$= -\frac{m^2}{\beta^2\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} K_2(\beta m n)\,. \tag{2.16}$$

$K_2(\beta m n)$ denotes the modified Bessel function.[9]

At this point it is instructive to investigate further the limiting situation of massless scalar particles. In this case, $m=0$, Eq. (2.14) leads to

$$-V_{\text{eff}}[T] \equiv \mathcal{L}^{(1)}[0, T]$$
$$\underset{m=0}{=} -\frac{1}{8\pi^2} \sum_{n=1}^\infty \left(\frac{2}{n\beta}\right)^4$$
$$= -\frac{1}{3}\frac{\pi^2 k^4}{15} T^4\,. \tag{2.17}$$

Finally, we turn to the computation of the second term of Eq. (2.13) for $\phi_0 \neq 0$, which we call $\Delta\mathcal{L}^{(1)}[\phi_0, T]$:

$$\Delta\mathcal{L}^{(1)}[\phi_0, T] = \frac{2}{16\pi^2} \sum_{n=1}^\infty \int_0^\infty \frac{ds}{s^3} e^{is\kappa^2} \exp\left(i\frac{\beta^2 n^2}{4s}\right)\,. \tag{2.18}$$

Obviously, the evaluation of (2.18) only changes the mass term in (2.16) by $m^2 \to \kappa^2 = m^2 - g\phi_0$. Here then is our result of the Coleman-Weinberg effective potential at finite temperature:

$$\mathcal{L}^{(1)}[\phi_0, T] = [\text{Eq. (2.7)}] - \frac{\kappa^2}{\beta^2\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} K_2(\beta\kappa n)$$
$$= [\text{Eq. (2.7)}] - \frac{\pi^2}{45\beta^4} + \frac{m^2 - g\phi_0}{12\beta^2} + \cdots\,, \tag{2.19}$$

where the last line shows the first two terms of a high-temperature expansion.

III. HEISENBERG-EULER LAGRANGIAN AT FINITE TEMPERATURE

We now examine the effect that an external constant electromagnetic field A_μ can have on a single fermion loop. The starting point is the vacuum persistence amplitude

$$\langle 0_+ | 0_- \rangle^A = \exp\{iW^{(1)}[A]\}\,,$$

where

$$iW^{(1)}[A] = -\operatorname{Tr}\ln(1 - e\gamma A G_+)^{-1}$$
$$= -\operatorname{Tr}\ln(G_+[A]/G_+)\,. \tag{3.1}$$

The calculation is performed along the same lines as in Sec. II. Hence, we need first of all the Green's function $G_+[A]$ in a constant electromagnetic field. If we choose the particular case of parallel fields, e.g., in the z direction, then $F_{12} = -F_{21} = H$ and $F_{30} = -F_{03} = E$. Under these circumstances the Green's function is given by[5,6]

$$G_+(x, x'|A) = \exp\left[ie\int_{x'}^x A_\mu(\xi)d\xi^\mu\right] \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} G_+(p|A)\,, \tag{3.2}$$

with

$$G_+(p|A) = i\int_0^\infty ds \exp\left[-is\left(m^2 + \frac{\tan z}{z}\vec{p}_\perp^{\,2} + \frac{\tanh z'}{z'}\vec{p}_\parallel^{\,2} - i\epsilon\right)\right] \frac{e^{i\sigma_3 z}}{\cos z} \frac{e^{\alpha_3 z'}}{\cosh z'} \left(m - \frac{e^{-\alpha_3 z'}}{\cosh z'}(\vec{\gamma}\cdot\vec{p})_\parallel - \frac{e^{-i\sigma_3 z}}{\cos z}(\vec{\gamma}\cdot\vec{p})_\perp\right)\,. \tag{3.3}$$

Here we used the notation $(\vec{a}\cdot\vec{b})_\parallel = -a^0 b^0 + a_3 b_3$, $(\vec{a}\cdot\vec{b})_\perp = a_1 b_1 + a_2 b_2$, $z = esH$, $z' = esE$, $\sigma_3 = i\gamma_1\gamma_2$, and $\alpha_3 = -\gamma_0\gamma_3$. Note that in absence of an external electric field, the longitudinal momentum components remain unchanged. In particular, there is no modification of p_0^2 (the magnetic field does not transfer energy). In the sequel we want to focus on the *magnetic* part of (3.3) only. Since the trace in Eq. (3.1) operates in spinor as well as coordinate space, we only need to consider

$$\operatorname{tr} G_+(x, x) = im\int \frac{d^4p}{(2\pi)^4} \int_0^\infty ds \frac{1}{\cos z} \exp\left[-is\left(m^2 + \vec{p}_\parallel^{\,2} + \frac{\tan z}{z}\vec{p}_\perp^{\,2} - i\epsilon\right)\right] \operatorname{tr}(e^{i\sigma_3 z})\,, \tag{3.4}$$

and use of $i\partial\mathcal{L}^{(1)}/\partial m = \mathrm{tr}G_+(x,x)$ then yields immediately

$$\mathcal{L}^{(1)}[H] = \frac{i}{2}\int^{\infty}\frac{ds}{s}e^{-ism^2}\frac{1}{\cos z}\int\frac{d^4p}{(2\pi)^4}\exp\left[-is\left(\vec{p}_{\parallel}^{\,2}+\frac{\tan z}{z}\,\vec{p}_{\perp}^{\,2}-i\epsilon\right)\right]4\cos z\,. \tag{3.5}$$

In order to incorporate temperature in (3.5), we simply make the replacement

$$\int\frac{d^4p}{(2\pi)^4}\to\frac{i}{\beta}\sum_{-\infty}^{+\infty}\int\frac{d^3\vec{p}}{(2\pi)^3}\,,\quad \vec{p}_{\parallel}^{\,2}=-\omega_n^{\,2}+p_3^{\,2}\,,\quad \beta=\frac{1}{kT}\,, \tag{3.6}$$

where

$$\omega_n=\begin{cases}(i\pi/\beta)(2n+1), & \text{Fermi-Dirac}\\ (i\pi/\beta)2n, & \text{Bose-Einstein}.\end{cases}$$

Time-translation invariance is guaranteed by observing that we can write the gauge-sensitive phase factor in (3.2) as $\exp[\frac{1}{2}ieH(x_1+x_1')(x_2-x_2')]$.

When we make the substitution (3.6) in (3.5), we obtain after performing the trivial Gaussian integrations

$$\mathcal{L}^{(1)}[H,T]=\left(\frac{\pi}{i}\right)^{1/2}\frac{1}{4\pi^2}\int_0^{\infty}\frac{ds}{s^{5/2}}\,e^{-ism^2}(esH)\cot(esH)\frac{i}{\beta}\sum_{n=-\infty}^{+\infty}e^{is\omega_n^{\,2}}+\text{c.t.}\,,$$

whereby

$$\sum_{n=-\infty}^{\infty}e^{is\omega_n^{\,2}}=\begin{cases}\displaystyle\sum_{-\infty}^{+\infty}\exp\left[-is\left(\frac{2\pi n}{\beta}\right)^2\right]=\theta_3\left(0-\frac{4\pi s}{\beta^2}\right), & \text{for bosons}\\ \displaystyle\sum_{-\infty}^{+\infty}\exp\left(-is\frac{4\pi^2}{\beta^2}(n+\tfrac{1}{2})^2\right)=\theta_2\left(0,-\frac{4\pi s}{\beta^2}\right), & \text{for fermions}\,.\end{cases}$$

So far we found for the spinor case

$$\mathcal{L}^{(1)}[H,T]=\left(\frac{\pi}{i}\right)^{1/2}\frac{1}{4\pi^2}\int^{\infty}\frac{ds}{s^{5/2}}\,e^{-ism^2}(esH)\cot(esH)\frac{i}{\beta}\,\theta_2\left(0,-\frac{4\pi s}{\beta^2}\right).$$

This expression has to be properly normalized, i.e., we want to reproduce the free theory if we switch off the external environment (H,T). This can be achieved by extracting the $n=0$ term in the above series. Hence, we employ formula (2.12) in a slightly modified version:

$$\begin{aligned}\sum_{n=-\infty}^{+\infty}e^{-\sigma(n-z)^2}&=\sum_{n=-\infty}^{+\infty}\left(\frac{\pi}{\sigma}\right)^{1/2}\exp\left(-\frac{\pi^2}{\sigma}n^2-2\pi izn\right)\\ &=\left(\frac{1}{i\pi s}\right)^{1/2}\frac{\beta}{2}\left[1+\sum_{n=-\infty}^{+\infty}{}'\exp\left(i\frac{n^2\beta^2}{4s}\right)e^{-i\pi n}\right],\end{aligned}$$

where we have used $z=-\frac{1}{2}$, $\sigma=is4\pi^2/\beta^2$. Thus our result so far can be expressed as

$$\mathcal{L}^{(1)}[H,T]=\mathcal{L}^{(1)}[H]+\Delta\mathcal{L}^{(1)}[H,T]\,,$$

whereby $\mathcal{L}^{(1)}[H]$ denotes the $T=0$ Heisenberg-Euler Lagrangian[10]:

$$\begin{aligned}\mathcal{L}^{(1)}[H]\equiv\mathcal{L}^{(1)}[H,T=0]&=\frac{1}{8\pi^2}\int^{\infty}\frac{ds}{s^3}\,e^{-ism^2}[(esH)\cot(esH)-1-\tfrac{1}{3}(esH)^2]\\ &=\frac{m^4}{32\pi^2}\frac{1}{h^2}[4\zeta'(-1,h)+h^2-\tfrac{1}{3}-(2h^2-2h+\tfrac{1}{3})\ln h]\,,\quad h=\frac{m^2}{2eH}\,,\end{aligned} \tag{3.7}$$

and the temperature effect is contained in

$$\Delta\mathcal{L}^{(1)}[H,T]=\left(\frac{\pi}{i}\right)^{1/2}\frac{1}{4\pi^2}\int^{\infty}\frac{ds}{s^{5/2}}\,e^{-ism^2}(esH)\cot(esH)\frac{i}{\beta}\left[\theta_2\left(0,-\frac{4\pi s}{2}\right)-\left(\frac{1}{i\pi s}\right)^{1/2}\frac{\beta}{2}\right]. \tag{3.8}$$

In the case of scalar QED, Eqs. (3.7) and (3.8) have to be multiplied by $-\frac{1}{2}$, $\cot(esH) \to (\sin esH)^{-1}$, $\theta_2(0, -4\pi s/\beta^2) \to \theta_3(0, -4\pi s/\beta^2)$, and $-\frac{1}{3}(esH)^2$ has to be replaced by $\frac{1}{6}(esH)^2$.

If we for the moment disregard the magnetic field, $H = 0$, we are left with

$$\mathcal{L}^{(1)}[0, T] = \frac{1}{8\pi^2} \sum_{n=-\infty}^{\infty}{}' (-1)^n \int_0^\infty \frac{ds}{s^3} e^{-ism^2} \exp\left(i\frac{\beta^2 n^2}{4s}\right),$$

which according to (2.15) is given by

$$\mathcal{L}^{(1)}_{1/2}[0, T] = \frac{2m^2}{\beta^2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} K_2(\beta m n) . \tag{3.9}$$

Likewise, for the scalar case we obtain

$$\mathcal{L}^{(1)}_{0}[0, T] = \frac{m^2}{\beta^2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(\beta m n) . \tag{3.10}$$

As a useful check of our procedure we now can relate the results (3.9) and (3.10) to well-known findings of statistical thermodynamics.[11] For this reason we investigate the limiting situation of massless fermions (bosons) somewhat closer. This case, $m = 0$, is not an unreasonable assumption at high temperatures. If we therefore set $m = 0$ in $\Delta\mathcal{L}^{(1)}$ of (3.8), we end up with

$$\frac{W^{(1)}}{\text{Vol}} = -\frac{F}{\text{Vol}} = \begin{cases} \mathcal{L}^{(1)}_{1/2}[0, T] = \frac{2}{3}\frac{7}{120}\pi^2 k^4 T^4, \text{ Fermi-Dirac} & (3.11) \\ \mathcal{L}^{(1)}_{0}[0, T] = \frac{2}{6}\frac{1}{15}\pi^2 k^4 T^4, \text{ Bose-Einstein}. & (3.12) \end{cases}$$

We must emphasize that these expressions are derived for charged pairs, hence the factor of 2. Equation (3.12), in particular, is identical with the Stefan-Boltzmann Law for the free energy per unit volume of a gas of scalar massless particles in thermal equilibrium. It is also the free energy per unit volume of a photon gas if we think of the equal particle-antiparticle contribution in (3.12) represented by the two polarization states of the photon.[11] Hence, (3.12) leads to the well-known expression for the total energy of blackbody radiation:

$$E = -3F = V\frac{\pi^2 k^4}{15(\hbar c)^3} T^4 .$$

With the aid of the definition of the Bernoulli numbers

$$\int_0^\infty \frac{x^{2n-1}}{e^x + 1} dx = (1 - 2^{1-2n})(2\pi)^{2n} \frac{|B_{2n}|}{4n}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \zeta(4) = \frac{\pi^4}{90} = \frac{1}{6} \int_0^\infty \frac{x^3}{e^x - 1} dx,$$

we can rewrite expressions (3.11) and (3.12) in the form

$$\mathcal{L}^{(1)}_{1/2}[0, T] = \frac{2}{3}\frac{1}{\pi^2} \int_0^\infty \frac{\epsilon^3}{e^{\epsilon/kT} + 1} d\epsilon , \tag{3.13}$$

$$\mathcal{L}^{(1)}_{0}[0, T] = \frac{1}{3}\frac{1}{\pi^2} \int_0^\infty \frac{\epsilon^3}{e^{\epsilon/kT} - 1} d\epsilon , \tag{3.14}$$

with the correct Planck and Fermi distribution, respectively, in contrast to the Lagrangians of Ref. 12 which are not related to temperature Green's functions. Finally, in order to evaluate $\Delta\mathcal{L}^{(1)}[H, T]$ for nonvanishing magnetic field H, we first observe that it can be rewritten as

$$\Delta\mathcal{L}^{(1)}_{1/2}[H, T] = \left(\frac{eH}{2\pi}\right)^2 \sum_{n=1}^{\infty} (-1)^{n+1} \int_0^\infty \frac{dz}{z^3} e^{-\rho z - \gamma/z} z \coth z ,$$

$$\rho = \frac{m^2}{eH}, \quad \gamma = (\tfrac{1}{2}n\beta)^2 eH . \tag{3.15}$$

There is little known about the sum and integral of the combined field-temperature expression (3.15). Notice, however, that in the range $z \lesssim 1$ and $z \gtrsim 10$, $\coth z$ can be replaced to a very high degree of accuracy by $1 + 1/z$. In the high-temperature limit this is certainly an excellent approximation since the dominant contributions to the integral stem from small z values. In the weak-field limit we can further improve our result, although we need not incorporate this assumption in performing the integration. Here then is our final result for the temperature- and field-dependent effective Lagrangian in spinor QED (for pairs) in the high-temperature limit:

$$\mathcal{L}^{(1)}_{1/2}[H, T] = [\text{Eq. (3.7)}] + \frac{eHm}{\pi^2\beta} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} K_1(\beta m n) + \frac{2m^2}{\pi^2\beta^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} K_2(\beta m n),$$

whereby the first important terms are contained in

$$\mathcal{L}^{(1)}_{1/2}[H,T] \cong [\text{Eq. } (3.7)] + 2\left[\frac{eH}{24}\left(1-\frac{m^2}{eH}\right)k^2T^2 + \frac{1}{3}\frac{7}{120}\pi^2k^4T^4\right]. \tag{3.16}$$

Finally, in the weak-field limit this equation becomes

$$\mathcal{L}^{(1)}_{1/2}[H,T] \cong \frac{1}{360\pi^2}\left(\frac{eH}{m}\right)^4 + \frac{eH}{12}\left(1-\frac{m^2}{eH}\right)k^2T^2 + \frac{2}{3}\frac{7}{120}\pi^2k^4T^4 . \tag{3.17}$$

Similarly, for scalar QED we find

$$\mathcal{L}^{(1)}_0[H,T] = \mathcal{L}^{(1)}_0[H] + \Delta\mathcal{L}^{(1)}_0[H,T],$$

where

$$\mathcal{L}^{(1)}_0[H] = \frac{1}{64\pi^2}\left[\,[2m^4 - \tfrac{2}{3}(eH)^2]\left(1+\ln\frac{m^2}{2eH}\right) - 3m^4 - (4eH)^2\zeta'\left(-1;\frac{m^2+eH}{2eH}\right)\right] \tag{3.18}$$

and

$$\Delta\mathcal{L}^{(1)}_0[H,T] = \frac{1}{2}\left(\frac{eH}{2\pi}\right)^2\sum_{n=1}^{\infty}\int_0^{\infty}\frac{dz}{z^3}\,e^{-\rho z-\gamma/z} \times \frac{z}{\sinh z}, \tag{3.19}$$

with ρ and γ as stated in (3.15). In the high-temperature weak-field limit where we approximate $1/\sinh z \approx 1/z$, we find at last

$$\mathcal{L}^{(1)}_0[H,T] \cong \frac{7}{16}\frac{1}{360\pi^2}\left(\frac{eH}{m}\right)^4 + 2\left(\frac{\pi^2}{90}k^4T^4 - \frac{m^2}{24}k^2T^2\right). \tag{3.20}$$

In the framework of statistical thermodynamics expression (3.16) is just the contribution of the zero-point energy to the free-energy density of a Fermi gas of electron-positron pairs in thermal equilibrium in a prescribed constant magnetic field. Formula (3.16) shows distinctly how the equilibrium system of charged fermion pairs is modified by the presence of magnetic fields.

In the light of various recent attempts to formulate a theory of astrophysical objects which are surrounded by magnetic fields, it becomes more and more important to know the thermodynamic properties of a high-temperature electron-positron plasma. Our work presented in this paper is an attempt to give a first understanding of the total energy and equilibrium radiation in a plasma of electron-positron pairs.

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