Action-principle quantization of the antisymmetric tensor field

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(Received 13 November 1978)

It is shown that the quantization of the antisymmetric tensor gauge field can readily be effected by straightforward application of the action principle. In particular, the correct number of constraint equations is found for both the massive and massless cases. When a coupling to the Maxwell field is included the two polarization states of the latter are seen to combine with the one degree of freedom of the antisymmetric tensor to yield a massive vector field.

I. INTRODUCTION

As a result of its applicability to dual-resonance models, the antisymmetric gauge field has been the object of considerable study.[1] One of the most elaborate works in this area has been that of Kaul,[2] who uses the Dirac method for its quantization. Because of the lengthy and seemingly unphysical calculations required by that treatment, it appears desirable to offer a more elementary and straightforward approach to the problem. In this work such a simplification of the quantization problem is presented using the action principle of Schwinger.[3]

In the following section the Lagrangian and field variables are introduced for the case of a massive antisymmetric tensor field. Although this is not of direct interest to dual-resonance theory, there is considerable value in a comparison between the constraint problems for the massive and massless cases. Section III presents the corresponding results for the massless field while Sec. IV goes on to consider the case in which the coupling term consists of a linear gauge-invariant interaction with the Maxwell field. The linear equations describing the latter system are soluble, and by making use of this, a direct demonstration is given of the fact that the transverse polarizations of the Maxwell field and the longitudinal polarization of the tensor field combine to yield a vector field whose mass is the coupling constant between the two fields.

II. THE MASSIVE CASE

The antisymmetric tensor field of dual-resonance models is described by $A^{\mu\nu}=-A^{\nu\mu}$ and the field $F^{\mu\nu\alpha}$ which has the property of being totally antisymmetric. Thus one makes use of the well-known result that $F^{\mu\nu\alpha}$ is equivalent to a single vector,

$$\phi^{\beta}=\tfrac{1}{6}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu\alpha}\,. \tag{2.1}$$

In consequence of this fact, the formulation to be presented here will deal exclusively with ϕ^{β}, with the reader able via (2.1) to rewrite all results in terms of $F^{\mu\nu\alpha}$. The Lagrangian of such a system is

$$\mathcal{L}=-\tfrac{1}{2}\phi^{\mu}\phi_{\mu}+\tfrac{1}{6}\epsilon^{\mu\nu\alpha\beta}\phi_{\beta}(\partial_{\mu}A_{\nu\alpha}+\partial_{\nu}A_{\alpha\mu}+\partial_{\alpha}A_{\mu\nu}) -\tfrac{1}{4}\mu^{2}A_{\mu\nu}A^{\mu\nu}+J^{\mu}\phi_{\mu}\,, \tag{2.2}$$

where J^{μ} represents a coupling term which may be either classical or quantized. The equations of motion which follow from (2.2) are

$$\phi^{\beta}=\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}A_{\mu\nu}+J^{\beta}\,, \tag{2.3}$$

$$\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}\phi_{\beta}+\mu^{2}A^{\mu\nu}=0\,, \tag{2.4}$$

or in second-order form

$$(-\partial^{2}+\mu^{2})A^{\mu\nu}-(\partial^{\mu}\partial_{\alpha}A^{\nu\alpha}-\partial^{\nu}\partial_{\alpha}A^{\mu\alpha})=-\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}J_{\beta}\,.$$

Although (2.4) immediately allows one to conclude that $\partial_{\mu}A^{\mu\nu}=0$ and consequently that

$$(-\partial^{2}+\mu^{2})A^{\mu\nu}=-\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}J_{\beta}\,,$$

the principal concern here is the demonstration of the existence of the correct number of constraint equations. To this end, one notes that (2.3) implies that ϕ^{0} is given in terms of A_{jk} as

$$\phi^{0}=-\tfrac{1}{2}\epsilon_{ijk}\partial_{i}A_{jk}+J^{0}\,,$$

or upon defining the vector

$$a_{i}\equiv\tfrac{1}{2}\epsilon_{ijk}A_{jk}$$

one has

$$\phi^{0}=-\vec{\nabla}\cdot\vec{a}+J^{0}\,. \tag{2.5}$$

Similarly from (2.4) there follows

$$A^{0k}=-\frac{1}{\mu^{2}}(\vec{\nabla}\times\vec{\phi})_{k}\,, \tag{2.6}$$

which shows that A^{0k} is a set of dependent variables. The remaining six equations in (2.3) and (2.4) contain explicit time derivatives and one consequently infers that of the ten original variables ($A^{\mu\nu}$ and ϕ^{β}) there remain six independent components (conveniently taken to be ϕ_i and a_i) appropriate to the description of a spin-one field.

Quantization is now simply carried out by noting that the generator of variations in the field variables is

$$G=\tfrac{1}{2}\int(\vec{\phi}\cdot\delta\vec{a}-\vec{a}\cdot\delta\vec{\phi})d\sigma .$$

Upon using the fact that

$$[\chi, G]=\tfrac{1}{2}i\delta\chi$$

for all field variables χ, there thus follows the only nonvanishing equal-time commutator among the unconstrained variables $\vec{\phi}$ and $\vec{a}$:

$$[a_i(x), \phi_j(x')]=i\delta_{ij}\delta(\vec{x}-\vec{x}') .$$

Together with (2.6) and the relation

$$\phi_i=\partial_0 a_i+\epsilon_{ijk}\partial_j A^{0k}+J_i ,$$

this implies that

$$[\partial_0 a_i(x), a_j(x')]=-i\delta_{ij}\delta(\vec{x}-\vec{x}')$$

for the case that $J_i(x)$ commutes with $a_j(x)$. One thus recognizes that the present formulation is equivalent to a massive vector meson whose quantization has been carried out here as a trivial application of the action principle. In the following section the somewhat more subtle case of the massless field is examined.

III. THE MASSLESS LIMIT

The massless version of the Lagrangian equations is

$$\phi^{\beta}=\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}A_{\mu\nu}+J^{\beta} , \tag{3.1}$$

$$\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}\phi_{\beta}=0 . \tag{3.2}$$

From (3.1) there follows immediately the single constraint

$$\phi^0=-\vec{\nabla}\cdot\vec{a}+J^0 , \tag{3.3}$$

while (3.2) implies

$$\vec{\nabla}\times\vec{\phi}=0 . \tag{3.4}$$

Upon making the decomposition into transverse and longitudinal parts

$$\vec{\phi}=\vec{\phi}_T+\vec{\phi}_L ,$$

one sees that (3.4) implies

$$\vec{\phi}_T=0$$

so that of the original three components of $\vec{\phi}$ only one, namely the longitudinal part, can be nonzero.

At this point it is necessary to invoke the gauge invariance of the theory under the transformation

$$A^{\mu\nu}\to A^{\mu\nu}+\partial^{\mu}\Lambda^{\nu}-\partial^{\nu}\Lambda^{\mu} . \tag{3.5}$$

For the case in which μ and ν are both spatial indices, (3.5) implies

$$\vec{a}\to\vec{a}+\vec{\nabla}\times\vec{\Lambda} ,$$

so that with the usual splitting into transverse and longitudinal parts one can adopt the gauge

$$\vec{a}_T=0 \tag{3.6}$$

by the choice

$$\vec{\Lambda}_T=\frac{1}{\nabla^2}\vec{\nabla}\times\vec{a}_T .$$

With this step the original ten variables have been reduced to five, namely $\vec{a}_L$ (one), $\vec{\phi}_L$ (one), and A^{0k} (three). However, (3.5) for the field variables A^{0k} further allows the selection of Λ^0 such that

$$A_L^{0k}=0 . \tag{3.7}$$

Finally, one notes that (3.1) yields

$$\phi_T^i=\epsilon_{ijk}\partial_j A_T^{0k}+J_T^i$$

or

$$A_T^{0k}=\frac{1}{\nabla^2}(\vec{\nabla}\times\vec{J})_k . \tag{3.8}$$

The specification of A^{0k} in terms of $\vec{J}$ by means of Eqs. (3.7) and (3.8) completes the task of reducing the ten variables $A^{\mu\nu}$ and ϕ^{β} to the two ($\vec{\phi}_L$ and $\vec{a}_L$) necessary to describe a scalar particle. The generator G consequently becomes

$$G=\tfrac{1}{2}\int(\vec{\phi}_L\cdot\delta\vec{a}_L-\vec{a}_L\cdot\delta\vec{\phi}_L)d\sigma ,$$

from which there follows the structure of the nonvanishing commutator

$$[a_i(x), \phi_j(x')]=i\frac{\nabla_i\nabla_j}{\nabla^2}\delta(\vec{x}-\vec{x}') .$$

Thus one has precisely the reverse of the usual vector field situation in which the limit of vanishing mass leaves nontrivial transverse polarizations rather than the longitudinal mode which survives in the present case.

IV. COUPLING TO THE MAXWELL FIELD

An interesting example of a possible coupling of the antisymmetric tensor gauge field is obtained by the inclusion of a linear interaction with the electromagnetic field.[4] As the coupling to the latter must be by means of a conserved current, one finds that the appropriate Lagrangian is

$$\mathcal{L} = -\tfrac{1}{2}\phi^\mu\phi_\mu + \tfrac{1}{6}\epsilon^{\mu\nu\alpha\beta}\phi_\beta(\partial_\mu A_{\nu\alpha} + \partial_\nu A_{\alpha\mu} + \partial_\alpha A_{\mu\nu})$$
$$+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)$$
$$+\tfrac{1}{6}gA_\beta\epsilon^{\mu\nu\alpha\beta}(\partial_\mu A_{\nu\alpha} + \partial_\nu A_{\alpha\mu} + \partial_\alpha A_{\mu\nu}) . \tag{4.1}$$

From (4.1) there follow the equations

$$\phi^\beta = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha A_{\mu\nu} , \tag{4.2}$$
$$\partial_\mu(\phi_\nu + gA_\nu) - \partial_\nu(\phi_\mu + gA_\mu) = 0 , \tag{4.3}$$
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu , \tag{4.4}$$
$$\partial_\nu F^{\mu\nu} = g\phi^\mu , \tag{4.5}$$

which will be used to reduce the 20 components A^μ, ϕ^μ, $F^{\mu\nu}$, and $A^{\mu\nu}$ to the six appropriate to a massive vector field.

As before one infers from (4.2) that

$$\phi^0 = -\vec{\nabla}\cdot\vec{a} \tag{4.6}$$

while (4.3) yields

$$\vec{\nabla}\times(\vec{\phi} + g\vec{A}) = 0$$

or

$$\vec{\phi}_T = -g\vec{A}_T , \tag{4.7}$$

where again use has been made of a decomposition into (three-dimensional) transverse and longitudinal parts. The three constraints (4.6) and (4.7) are immediately increased to seven by inclusion of the Maxwell field constraints

$$\tfrac{1}{2}\epsilon_{ijk}F_{jk} = (\vec{\nabla}\times\vec{A})_i$$

and

$$\partial_k F_L^{0k} = g\phi^0 ,$$

which eliminate F_{ij} and F_L^{0k} as independent variables.

Since the Lagrangian (4.1) is invariant under both

$$A^\mu \to A^\mu + \partial^\mu\Lambda$$

and

$$A^{\mu\nu} \to A^{\mu\nu} + \partial^\mu\Lambda^\nu - \partial^\nu\Lambda^\mu ,$$

one is able to select Λ and $\vec{\Lambda}_T$ such that

$$\vec{A}_L = 0 \tag{4.8}$$

and

$$\vec{a}_T = 0 .$$

Using (4.8) there follows immediately the eleventh constraint

$$F_L^{0k} = -\partial_k A^0 ,$$

which expresses A^0 nonlocally in terms of F_L^{0k}. The gauge function Λ^0 again allows the choice

$$A_L^{0k} = 0$$

while (4.2) implies

$$\phi_T^i = \epsilon_{ijk}\partial_j A_T^{0k}$$

or

$$A_T^{0k} = \frac{1}{\nabla^2}g(\vec{\nabla}\times\vec{A}_T)_k .$$

One thus has a complete set of 14 constraints which reduce the original twenty field components to the six $\vec{a}_L$, $\vec{\phi}_L$, $\vec{A}_T$, and F_T^{0k}. The generator of variations in the field variables is seen by inspection to be

$$G = \tfrac{1}{2}\int (A_T^k\,\delta F_T^{0k} - F_T^{0k}\delta A_T^k + \vec{\phi}_L\cdot\delta\vec{a}_L - \vec{a}_L\cdot\delta\vec{\phi}_L)d\sigma ,$$

so that the only nonvanishing commutators among the six remaining dynamically independent variables are

$$[F_T^{0k}(x), A_T^l(x')] = i\left(\delta_{kl} - \frac{\nabla_k\nabla_l}{\nabla^2}\right)\delta(\vec{x} - \vec{x}') ,$$
$$[a_i(x), \phi_j(x')] = i\frac{\nabla_i\nabla_j}{\nabla^2}\delta(\vec{x} - \vec{x}') .$$

Finally one notes that Eqs. (4.2)–(4.6) can be manipulated by elementary means to give the equations of motion

$$(-\partial^2 + g^2)\vec{A}_T = 0 ,$$
$$(-\partial^2 + g^2)\vec{\nabla}\cdot\vec{a}_L = 0 ,$$

thereby completing the task of displaying the physical content of the theory in terms of its equivalence to a vector field of mass g. There is, of course, the interesting parallel here to the familiar example in gauge theories in which a massless vector field mixes with a scalar field to give a massive vector meson. Precisely the same phenomenon has occurred in the case of the antisymmetric tensor gauge field, which is merely a more complex formulation of a spin-zero field which happens to possess the possibly useful feature of gauge invariance.

This research was supported in part by the U. S. Department of Energy.

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