New null experiment to test the inverse square law of gravitation

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A "source-independent" null experiment for the Newtonian law of gravity is proposed. The new scheme involves detection of the Laplacian of the gravitational potential $\nabla^2\phi(\vec{r})$ in free space, i.e., where $\rho(\vec{r}) = 0$. This quantity should vanish identically regardless of the mass distribution in the rest of the universe if the inverse square law is exact. A departure from null in $\nabla^2\phi(\vec{r})$ could be measured by monitoring the breathing mode of a sphere or by summing three gravity gradients at a single point along any three orthogonal directions. The "source-independent" behavior of the proposed technique will allow a test of the inverse square law in the intermediate range between 1 m and 10 km in which large geological objects might be used as sources.

I. INTRODUCTION

The Newtonian inverse square law of gravitation agrees with astronomical data to a very high accuracy and has been widely accepted as the correct law in the static weak-field limit at all distances. Contributing to this belief is the fact that most viable metric theories of gravitation converge to Newton's theory in the nonrelativistic limit. However, it has been pointed out recently by various authors[1-3] that existing experimental data cannot exclude a possibility that Newton's law may be violated at distances less than 10^3 km. From a theoretical point of view, an additional short-range field could be added to the Newtonian component without being detected at large distances. The potential due to a point mass M could have the form

$$\phi(r) = -\frac{GM}{r}\left(1 + \alpha e^{-\beta r}\right), \tag{1}$$

where r is the separation between the source M and the test mass. Such a departure from the inverse square law could be conceived of in a generalized scalar-tensor theory of gravitation.[4-6] A value of α as large as $\frac{1}{3}$ has been predicted.[1,5] A most likely range of force in which a significant departure could have gone undetected would be $10\text{ m} \lesssim \beta^{-1} \lesssim 1\text{ km}$ or $\beta^{-1} \ll 1$ cm where experimental data are poorest.[1,3]

After performing a modified Cavendish experiment for gravitation, Long[7] has published a result asserting that the inverse square law is violated by over 0.3% between 4.5 and 30 cm. If $\alpha = \frac{1}{3}$ is assumed, the data could be interpreted to yield $\beta^{-1} = 0.6$ cm or 2 m. The implications of Long's result, if verified by others, are significant. First, one will have to alter the mass scale of all astronomical objects which in turn may affect our understanding of the universe. Moreover, the new force may play an important role in connecting gravity to other branches of physics.[1,6] Long's experiment should be repeated with greater precision in order to obtain a unique set of values for α and β. A null experiment will be desirable in which deviation from the Newtonian force law is detected directly.

In this paper we will first review the traditional source-dependent null experiments. We will find that the uncertainties in the source geometry and density severely limit the precision of measurements. We will then propose a new source-independent null experiment. In this experiment, the measured quantity will be identically zero for an arbitrary source if the inverse square law is exact.

II. TRADITIONAL SOURCE-DEPENDENT NULL EXPERIMENTS

On contemplating a null experiment, one might first consider a gravitational analog of the electrostatic experiment by Cavendish and Maxwell.[8] In this classic experiment, they tested for electric field inside a charged spherical shell, obtaining an upper limit of 5×10^{-5} in $|\epsilon|$ where the force law is written as $r^{-2+\epsilon}$. A series of improvements have been made on this experiment[9,10] and now the experimental upper limit in $|\epsilon|$ stands at 3×10^{-16}. One could attempt a similar experiment in which gravitational force inside a spherical shell of uniform mass density is measured as a test of the inverse square law of gravitation. For the general potential of Eq. (1), one can show that the specific force (force per unit test mass) inside a thin shell is

$$-\vec{\nabla}\phi(r) = -\frac{\alpha}{2\beta R}e^{-\beta R}\frac{GM\vec{r}}{r^3}\Big[(1+\beta r)e^{-\beta r} - (1-\beta r)e^{\beta r}\Big], \quad r<R \tag{2}$$

where R is the radius of the shell, r is the test mass position measured from the origin, and M is the mass of the shell. For the inverse square law ($\alpha=0$), Eq. (2) becomes $\nabla\phi(\vec{r})=0$ as in the case of electrostatics.

In performing the test, however, there is an important difference between the two physical phenomena. In electrostatics, any imperfection in the sphericity of the shell is exactly compensated for by a redistribution of electric charges, ensuring perfect cancellation of the field inside. This is not true in a gravitational experiment. Gravitational "charges" in a rigid body do not have a mobility to make the shell surface a gravitational equipotential. Therefore, unlike in its electrostatic counterpart, dimensional uncertainties and density inhomogeneity of the source (shell) will put severe limitations on the precision to which Newton's law can be tested.

There are two other source geometries that provide a null environment for the inverse square law: two-dimensional (infinitely long cylindrical shell) and one-dimensional "spherical" shell (infinite plane). Inside a cylindrical shell, the gravitational force on a test mass due to the shell is zero. For the infinite plane, it is the force gradient that vanishes on either side of the plane. In practice, however, one has to use a truncated cylindrical shell (long cylinder) or plane (disk). In this instance one no longer performs a null experiment, but instead looks for departure from the Newtonian field due to the missing mass. In spite of this drawback, experimenters have compromised by choosing a long cylinder[11,12] or a disk[13] as the gravitating body because they are inherently easier to fabricate than a sphere. It seems that uncertainties associated with the source dimensions and density will limit the resolution of the force law to one part in 10^4 in these experiments even with such simplified source geometries. This resolution is at least eleven orders of magnitude inferior to the value already achieved in electrostatics.[10] In a direct force measurement, a most careful Cavendish-type experiment[14] has not yielded resolution in G better than one part in 10^3 after two hundred years of elaboration on the experimental technique and apparatus. Although this poor resolution is partly due to the weakness of gravitational interaction, it is important to realize that little improvement can be expected even when more sensitive detectors are developed.

In view of this unpleasant situation, one is forced to seek a new experimental scheme in which the measured quantity does not significantly depend on the source geometry. This is a particularly important requirement for tests of the force law in the range between 100 m and 10 km, where a natural object such as a rock or a mountain may have to be used as a gravitating body. In the following section, we discuss a new approach to the problem and propose a different null experiment which does not require knowledge of exact mass distribution of the source. Because of its essentially source-independent characteristic, the new experiment will allow a test of Newton's law on the geological scale as well as laboratory dimensions. As the detector technology improves, the experimental uncertainty in the force law, which is better than the present source-limited value of one part in 10^4, may be obtained.

III. NEW SOURCE-INDEPENDENT NULL EXPERIMENT

In the gravitational null or near-null experiments discussed in Sec. II, the experimenter tries to create artificially a region over which either the gravitational potential ϕ or the force $-\vec{\nabla}\phi$ is a constant. Such quantities depend on the *global* mass distribution as can be seen from the integrals (for r^{-2} force in an inertial frame)

$$\phi(\vec{r})=-G\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3x' , \tag{3}$$

$$\vec{\nabla}\phi(\vec{r})=G\int_V \rho(\vec{r}\,')\,\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d^3x' , \tag{4}$$

where V is the volume of the source and $\rho(\vec{r}\,')$ is the mass density. Notice, however, that one more differential operation on the potential yields a quantity which is specified solely by *local* mass density, i.e.,

$$\nabla^2\phi(\vec{r})=-4\pi G\rho(\vec{r}) . \tag{5}$$

This well-known Poisson's equation has significant implication, as we will see, for a new possibility of testing the force law. Unlike Eqs. (3) and (4), Eq. (5) holds *locally independent of mass distribution* in the rest of the universe. Outside the source where $\rho(\vec{r})=0$, $\nabla^2\phi(\vec{r})$ vanishes identically regardless of the source geometry and density variation. In a noninertial frame rotating at an angular velocity $\vec{\Omega}$, the centrifugal acceleration will contribute an additional term $2\Omega^2$ on the right-hand side of Eq. (5). The local characteristic of $\nabla^2\phi(\vec{r})$ is a consequence of a unique mathematical property of the Newtonian potential function $1/|\vec{r}-\vec{r}\,'|$ under the Laplacian operation[15]

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)=-4\pi\delta(\vec{r}-\vec{r}\,') . \tag{6}$$

Therefore, any modification of the force law will in general produce a nonlocal term on the right-hand side of Eq. (5). For the potential given by

Eq. (1), it is easy to show that

$$\nabla^2\phi(\vec{r}) = -G\alpha\beta^2\int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\exp(-\beta|\vec{r}-\vec{r}'|)d^3x'$$

$$\equiv \beta^2\phi_s(\vec{r}), \tag{7}$$

where $\rho(\vec{r})=0$ is assumed and $\phi_s(\vec{r})$ denotes the short-range component of the potential.

In general relativity, Eq. (5) is replaced by Einstein's field equation

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu},$$

where units are chosen such that $G=c=1$. Therefore, all components of the Ricci tensor $R_{\mu\nu}$ vanish in free space. In particular, $R_{00}\equiv R^{\alpha}{}_{0\alpha0}=4\pi\rho$ depends on the local mass density alone. Since the tidal force sensors discussed in this paper actually measure $R^{\alpha}{}_{0\alpha0}$, the output of the devices will vanish even in the limit of strong gravitational fields.[16] In the absence of a Yukawa-type force and in the presence of a cosmological term, the experiment could in principle measure the cosmological constant Λ since

$$R^{\alpha}{}_{0\alpha0} = 4\pi\rho - \tfrac{1}{2}\Lambda.$$

Detection of Λ in such a laboratory experiment, however, is beyond reasonable hope with our present technology.

The new null test involves measuring the quantity $\nabla^2\phi(\vec{r})$ outside a gravitational source and comparing it with zero. In order to discriminate the signal against a direct current (dc) level arising from the constant mass density of the detector itself and from a rotational motion of the platform due to the earth's spin, etc., and avoid the $1/f$ noise spectrum of the measuring instrument, one will have to perform an alternating current (ac) experiment by modulating $|\vec{r}-\vec{r}'|$ periodically and detecting the signal at appropriate frequencies. The question that needs to be answered is whether one can design a detector that measures $\nabla^2\phi(\vec{r})$ directly. We give a few alternatives for such a detector below.

A. Spherical shell or solid sphere as detector

In Sec. II we discussed an experiment which uses a spherical shell as a source of null gravitational force environment. It is interesting to note that the same object can be used as a null detector of $\nabla^2\phi$. In the latter experiment, one utilizes the fact that the purely radial "breathing" mode of a sphere does not couple to an external source of arbitrary shape if the inverse square law is correct. When $\nabla^2\phi=0$, any source outside a sphere (or spherical shell) tends to produce a tidal distortion with zero divergence for displacement (no breathing). Hence the breathing mode of a sphere can be excited by gravitational coupling to a source only when the inverse square law is violated. This is shown mathematically below.

The spatial eigenfunctions of radial modes of a thin spherical shell can be written as

$$\vec{\psi}_{lm}(\theta,\phi) = (a_l\vec{n} + b_l R\vec{\nabla})Y_{lm}(\theta,\phi), \tag{8}$$

where a_l and b_l are constants, $\vec{n}=\vec{r}/r$, and R is the radius of the shell. The driving force on each mode due to a gravitational potential $\phi(\vec{r})$ is

$$f_{lm} = -\int_S d^2x\,\vec{\psi}_{lm}(\theta,\phi)\cdot\vec{\nabla}\phi(\vec{r}). \tag{9}$$

Here we used a normalization condition

$$\int d^2x\,\vec{\psi}_{lm}\cdot\vec{\psi}^{*}_{l'm'} = \delta_{ll'}\delta_{mm'}.$$

For the monopole (breathing) mode, $Y_{00}=(4\pi)^{-1/2}$ so that the driving force becomes

$$f_{00} = -\frac{a_0}{(4\pi)^{1/2}}\int_S d^2x\,\vec{n}\cdot\vec{\nabla}\phi(\vec{r})$$

$$= -\frac{a_0}{(4\pi)^{1/2}}\int_V d^3x\,\nabla^2\phi(\vec{r}), \tag{10}$$

where the last step was obtained by applying Gauss's divergence theorem. It is clear from Eq. (10) that the monopole mode will remain unexcited if the force law is Newtonian and the source is located outside the spherical volume V. The same argument can be readily extended to a solid sphere using the superposition principle.[17]

A null test would involve observation of the monopole mode of a sphere. A source can be moved sinusoidally at the eigenfrequency of the mode in order to build up the amplitude of the sphere. The problem is similar to the detection of scalar gravitational waves by a sphere.[18] While the new null experiment does not necessitate accurate determination of geometry and density of the source, essentially the same burden is now on the detector. In the new scheme one could employ a solid sphere which is easier to fabricate than a spherical shell. However, its high-resonance frequency may present difficulty in modulating the source-detector separation for an ac experiment.

B. Three-dimensional gravity gradiometer

In Cartesian coordinates, $\nabla^2\phi$ is simply

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}. \tag{11}$$

The right-hand side of this equation is a sum of gravity gradients in three orthogonal axes. Therefore, one could measure $\nabla^2\phi$ directly by using three gravity gradiometers oriented orthogonal to one another.[19] The same job might be done in principle with a single one-dimensional gradiometer by pointing its sensitive axis along the three orthogonal axes and summing the outputs for each position of the source. In practice three orthogonal gradiometers mounted rigidly on a common platform may give superior performance in axes alignment and data rate and will be easier to handle. A satisfactory method of accurately matching the three gradiometers should be found before one can use a three-dimensional system reliably.

It is interesting to note that all sensitive experiments on static gravity have been done with torsion pendula to date.[20-24] Since the extremely low-resonance frequency of torsional motion allows a large deflection of the test mass under gravitational force, very sensitive gravity gradiometers could be constructed using a torsion balance.[22,23] However, such a device would necessarily confine the experimenter to the two-dimensional space of the pendulum motion. The superconducting gradiometer developed by Worden and Everitt[25] utilizes a low-frequency magnetic suspension of test masses and a low-noise Josephson-junction magnetometer (SQUID) to obtain an improved sensitivity. This gradiometer is inherently one-dimensional and would not be applicable for our three-dimensional experiment. The best candidate for a $\nabla^2\phi$ experiment seems to be the superconducting displacement differencing gravity gradiometer developed by Paik, Mapoles, and Wang.[26] In this device two cylindrical superconducting test masses are suspended by diaphragms of the same material from a rigid coaxial rim. Typical resonant frequencies of the diaphragm suspension are between 50 and 100 Hz. The sensitive axis of this device can be oriented in any direction without altering its basic characteristics.

A three-axis gravity gradiometer could be constructed extending the principles employed in the device of Paik *et al.*, and the suspension frequency lowered below 10 Hz to improve the sensitivity. A detailed analysis of a practical apparatus and experimental sensitivity will be published elsewhere.

IV. CONCLUSIONS

The Newtonian law of gravity should be tested in a new range between 1 m and 10 km. The proposed experiment in which $\nabla^2\phi$ is measured will allow a resolution of the force law in this range as well as a true null experiment in the laboratory for the first time. The new method does not require precise knowledge of mass distribution for the source. As a result, experimental uncertainty in the force law can continue to decrease as detector technology improves.

In a sensitive gravity experiment, vibration isolation is always a challenge because of the difficulty of distinguishing gravity from platform accelerations. In this regard, a gradient measurement is a superior experimental technique to a force measurement because the unique tensor characteristic of gravity gradient cannot be simulated by accelerations. The equivalence principle does not prohibit distinction of gravity from an acceleration when the force is measured at more than one point.[27] When three gradients are summed, the output becomes proportional to $\nabla^2\phi$, which is a more fundamental quantity than ϕ or $\nabla\phi$ from a field-theoretic point of view because of its intrinsic local nature. Furthermore, $\nabla^2\phi$ is a null quantity for the inverse square law in free space allowing an ideal null experiment for one of the most basic laws of nature.

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[16]Edison Liang has pointed this out to the author.

[17]There may be an induced variation of internal mass density $\rho(\vec{r},t)$ of the sphere which could give rise to a nonvanishing value of $\nabla^2\phi(\vec{r},t)$ in the integrand of Eq. (10). However, such self-gravity of the detector will be a second-order effect at best.

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