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Parallel transport of a vector along a circular orbit in Schwarzschild space-time

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The parallel propagator on a circular orbit in Schwarzschild space-time is given. By use of it, we find that the geodesic precession of a gyroscope in free fall which moves along a timelike circular orbit of radius r_0 is $\|\vec{\Omega}\| = (1/r_0)(m/r_0)^{1/2}[(1-3m/r_0)^{-1/2}-1)]$.

As is well known, the parallel transport of a vector v^i along a curve c: $x^i = x^i(u)$ in a Riemannian manifold of n dimensions is given by

$$\frac{\delta v^i}{\delta u} = \frac{dv^i}{du} + \Gamma^i_{jk}\frac{dx^j}{du}V^k = 0\,. \tag{1}$$

When we now write v^{i_0} for the vector at a point p_0, say at $u=u_0$, the solution v^i of (1) is expressed in terms of the parallel propagator[1,2] $\bar{g}^i{}_{j_0}$ as

$$v^i = \bar{g}^i{}_{j_0} v^{j_0}\,. \tag{2}$$

Since the vector v^{i_0} may be chosen arbitrarily at p_0, substitution from (2) in (1) gives

$$\frac{\delta \bar{g}^i{}_{j_0}}{\delta u} = \frac{d\bar{g}^i{}_{j_0}}{du} + \Gamma^i_{lk}\frac{dx^l}{du}\bar{g}^k{}_{j_0} = 0\,. \tag{3}$$

This is the equation of a parallel propagator generating parallel vector fields along the curve c.

In the following we shall solve (3) on a circular orbit in Schwarzschild space-time. In Schwarzschild space-time the metric form is

$$\Phi = \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1-\frac{2m}{r}\right)dt^2\,. \tag{4}$$

The circular orbit lying in the equatorial plane with an affine parameter u is given by

$$r = r_0\,,\quad \theta = \frac{\pi}{2}\,,\quad \phi = \frac{u}{\alpha r_0^2}\,,\quad t = \frac{\beta u}{1-2m/r_0}\,,$$
$$\alpha^2\beta^2 = \frac{1}{mr_0}\left(1-\frac{2m}{r_0}\right)^2, \tag{5}$$

where α and β are constants depending on the initial conditions. This orbit is spacelike, null-like, or timelike for $2m < r_0 < 3m$, $r_0 = 3m$, or $3m < r_0$, respectively. When we take account of (4) and (5), the equation of the parallel propagator (3) reduces to

$$\frac{d\bar{g}^1{}_{j_0}}{du} + \frac{1}{\alpha r_0^2}\Gamma^1_{33}\bar{g}^3{}_{j_0} + \frac{\beta}{1-2m/r_0}\Gamma^1_{44}\bar{g}^4{}_{j_0} = 0\,, \tag{6}$$

$$\frac{d\bar{g}^2{}_{j_0}}{du} = 0\,, \tag{7}$$

$$\frac{d\bar{g}^3{}_{j_0}}{du} + \frac{1}{\alpha r_0^2}\Gamma^3_{31}\bar{g}^1{}_{j_0} = 0\,, \tag{8}$$

$$\frac{d\bar{g}^4{}_{j_0}}{du} + \frac{\beta}{1-2m/r_0}\Gamma^4_{41}\bar{g}^1{}_{j_0} = 0\,, \tag{9}$$

where $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, and $x^4 = t$, and $j = 1,2,3,4$. In order to solve Eqs. (6)–(9), differentiating (6) and substituting (8) and (9) in the result, we get

$$\frac{d^2\bar{g}^1{}_{j_0}}{du^2} + \gamma^2\bar{g}^1{}_{j_0} = 0\,, \tag{10}$$

where $\gamma^2 = (1-3m/r_0)/\alpha^2 r_0^4$.

The general solution of (10) is expressed as

$$\bar{g}^1{}_{j_0} = A_{j_0}e^{i\gamma u} + B_{j_0}e^{-i\gamma u}\,, \tag{11}$$

with the vectors A_{j_0} and B_{j_0} determined by coincidence limits

$$\lim_{u\to 0}\bar{g}^1{}_{j_0} = \delta^1{}_j\,, \tag{12}$$

$$\lim_{u\to 0}\frac{d\bar{g}^1{}_{j_0}}{du} = -\left(\frac{1}{\alpha r_0^2}\Gamma^1_{33}\bar{g}^3{}_{j_0} + \frac{\beta}{1-2m/r_0}\Gamma^1_{44}\bar{g}^4{}_{j_0}\right)\Bigg|_{u=0}\,. \tag{13}$$

Similarly, $\bar{g}^3{}_{j_0}$ and $\bar{g}^4{}_{j_0}$ are obtained by substituting this $\bar{g}^1{}_{j_0}$ in (8) and (9). Thus we get the parallel propagator on the timelike circular orbit $(r_0 > 3m)$ as follows:

$$(\bar{g}^{i}{}_{j_0}) = \begin{bmatrix} \cos\gamma u & 0 & \frac{r_0(1-2a)}{(1-3a)^{1/2}}\sin\gamma u & -\frac{(1-2a)\sqrt{a}}{(1-3a)^{1/2}}\sin\gamma u \\ 0 & 1 & 0 & 0 \\ -\frac{\sin\gamma u}{r_0(1-3a)^{1/2}} & 0 & 1-\frac{2(1-2a)}{1-3a}\sin^2\frac{\gamma u}{2} & \frac{2(1-2a)\sqrt{a}}{r_0(1-3a)}\sin^2\frac{\gamma u}{2} \\ -\frac{\sqrt{a}\sin\gamma u}{(1-2a)(1-3a)^{1/2}} & 0 & -\frac{2r_0\sqrt{a}}{1-3a}\sin^2\frac{\gamma u}{2} & 1+\frac{2a}{1-3a}\sin^2\frac{\gamma u}{2} \end{bmatrix}, \quad (14)$$

where $a = m/r_0$. Formulas (14) may be applied to the null-like and spacelike orbits by taking the limits $r_0 \to 3m$ and $2m < r_0 < 3m$, respectively.

We now consider, with use of this parallel propagator, the geodesic precession of a gyroscope in free fall which moves along a timelike circular orbit. Taking the affine parameter u for the proper time, we have the constants α and β, from (4) and (5), such that

$$\alpha = \frac{1}{r_0}\left(\frac{r_0}{m}\right)^{1/2}\left(1-\frac{3m}{r_0}\right)^{1/2}; \quad \beta = \frac{1-2m/r_0}{(1-3m/r_0)^{1/2}}. \quad (15)$$

We introduce the co-moving frame $\{\vec{e}_{\hat{i}}\}$ and its dual basis $\{\vec{\omega}^{\hat{i}}\}$ attached to the gyroscope as follows:

$$\vec{e}_{\hat{i}} = s_i{}^k\vec{e}_k, \quad \vec{\omega}^{\hat{i}} = t^i{}_k\vec{\omega}^k \quad (16)$$

and these transformations matrices are given by

$$(s_i{}^k) = \begin{bmatrix} (1-2a)^{1/2} & 0 & 0 & 0 \\ 0 & \frac{1}{r_0} & 0 & 0 \\ 0 & 0 & \frac{(1-2a)^{1/2}}{r_0(1-3a)^{1/2}} & \frac{\sqrt{a}}{[(1-2a)(1-3a)]^{1/2}} \\ 0 & 0 & \frac{\sqrt{a}}{r_0(1-3a)^{1/2}} & \frac{1}{(1-3a)^{1/2}} \end{bmatrix} \quad (17)$$

and

$$(t^i{}_k) = \begin{bmatrix} \frac{1}{(1-2a)^{1/2}} & 0 & 0 & 0 \\ 0 & r_0 & 0 & 0 \\ 0 & 0 & \frac{r_0(1-2a)^{1/2}}{(1-3a)^{1/2}} & -\frac{\sqrt{a}(1-2a)^{1/2}}{(1-3a)^{1/2}} \\ 0 & 0 & -\frac{r_0\sqrt{a}}{(1-3a)^{1/2}} & \frac{1-2a}{(1-3a)^{1/2}} \end{bmatrix}, \quad (18)$$

where $a = m/r_0$, $\{\vec{e}_i\}$ is the coordinate basis $\{\partial/\partial x^i\}$, and $\{\vec{\omega}^i\}$ its dual basis $\{dx^i\}$.

Transforming from the basis $\{\vec{e}_i\}$ to $\{\vec{e}_{\hat{i}}\}$, we can calculate the parallel propagator $\bar{g}^{\hat{i}}{}_{\hat{j}_0}$ in the co-moving frame by the formula $\bar{g}^{\hat{i}}{}_{\hat{j}_0} = t^i{}_k s_{j_0}{}^{l_0}\bar{g}^k{}_{l_0}$, with the result

$$(\bar{g}^{\hat{i}}{}_{\hat{j}_0}) = \begin{bmatrix} \cos\gamma u & 0 & \sin\gamma u & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\gamma u & 0 & \cos\gamma u & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (19)$$

Let S^i be the spin vector of the gyroscope. Since the spin vector S^i is defined to remain orthogonal to the velocity dx^i/du, i.e., $S_i dx^i/du = 0$, the following relation holds:

$$S^4 = \frac{r_0(m/r_0)^{1/2}}{1-2m/r_0}S^3. \quad (20)$$

The spin vector $S^{\hat{i}}$ in the co-moving frame can similarly be obtained by $S^{\hat{i}} = t^i{}_k S^k$, and then the component $S^{\hat{4}}$ reduces to zero by use of (20). Let $S^{\hat{\alpha}_0}$ $(\alpha = 1,2,3)$ be the spin vector at the point P_0 $(u=0)$ on the circular orbit, and let $S^{\hat{\alpha}}$ be the parallel transport of $S^{\hat{\alpha}_0}$ at the point P $[u = u_p = 2\pi r_0(r_0/m)^{1/2}(1-3m/r_0)^{1/2}]$ where the gyroscope returns to the initial spatial point along the orbit. We can then define the scalar product of $S^{\hat{\alpha}_0}$ and $S^{\hat{\alpha}}$ by parallel transporting $S^{\hat{\alpha}}$ to the point P_0 along the time axis x^4.

As one can see from (19), the spin vector $S^{\hat{\alpha}}$ is turning around the basis vector $\vec{e}_{\hat{2}}$. We may therefore set $S^{\hat{2}_0} = 0$ for seeking the geodesic precession frequency $\vec{\Omega}$ of the gyroscope. Then we have the angle between $S^{\hat{\alpha}_0}$ and $S^{\hat{\alpha}}$,

$$\cos\delta = \frac{S^{\hat{\alpha}_0}S_{\hat{\alpha}}}{\|S^{\hat{\alpha}_0}\|\,\|S^{\hat{\alpha}}\|} = \cos\gamma u_p, \quad (21)$$

where $S^{\hat{\alpha}} = \bar{g}^{\hat{\alpha}}{}_{\hat{\beta}_0}S^{\hat{\beta}_0}$, and $g_{\hat{i}\hat{j}} = \eta_{\hat{i}\hat{j}}$ in our co-moving frame. Since $\gamma u_p = 2\pi(1-3m/r_0)^{1/2}$, we get

$$\delta = \pm 2\pi\left(1-\frac{3m}{r_0}\right)^{1/2} + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots. \quad (22)$$

Further, we have $\delta \to 0$ according to $3m/r_0 \to 0$ and have $\delta > 0$, so that

$$\delta = 2\pi\left[1 - \left(1 - \frac{3m}{r_0}\right)^{1/2}\right]. \tag{23}$$

Hence we obtain the geodesic precession frequency as follows:

$$\|\vec{\Omega}\| = \frac{\delta}{u_p} = \frac{1}{r_0}\left(\frac{m}{r_0}\right)^{1/2}\left[\left(1 - \frac{3m}{r_0}\right)^{-1/2} - 1\right] \tag{24}$$

and its direction is in the $\vec{e}_2$ direction.

If we use the coordinate time $t = u/(1 - 3m/r_0)^{1/2}$, the approximation of (24) to lowest order becomes $\|\vec{\Omega}\| = \frac{3}{2}(m/r_0^2)(m/r_0)^{1/2}$, which coincides with Schiff's formula.[3]

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