Gauge-invariant perturbations on most general spherically symmetric space-times

Ulrich H. Gerlach
Department of Mathematics, The Ohio State University, Columbus, Ohio 43210

Uday K. Sengupta
Department of Physics, The Ohio State University, Columbus, Ohio 43210

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The Einstein field equations and the conservation equations linearized around a most general (astrophysically relevant) spherically symmetric space-time are reduced to a set of equations for gauge-invariant geometrical objects defined on the two-dimensional timelike submanifold spanned by the radial and time coordinates. Odd-parity metric and matter perturbations are each expressed in terms of a vector field, matter perturbations in terms of an additional scalar field on this submanifold. Even-parity perturbations are expressed in terms of a symmetric tensor field and a scalar field for the metric and in terms of two scalars, a vector, and a symmetric tensor field for matter. The odd-parity vectorial perturbations are derivable from a single master scalar equation, and their junction conditions across the surface of a collapsing star are given.

I. INTRODUCTION

Much effort has been spent to develop gravitational detectors capable of receiving signals from various astrophysical sources, most hopefully from stellar collapse associated with supernovas or black-hole formation.[1] Currently, the theoretical justification for these undertakings is primarily based on the radiated power, spectral flux, etc. arrived at by either estimates[2] or actual computer calculations[3,4] for certain specialized and sometimes rather unrealistic models.

Considering the magnitude of the experimental effort it is interesting to note two facts: (1) No precise figures on detectable power, spectral flux, etc. are available for typical astrophysical events such as slight asymmetries in spherical collapse of the type first considered by, say, Colgate and White[5] or by May and White.[6] (2) No results are available that allow one to assert to what extent the to-be-detected gravitational radiation is diminished by the role of shear viscosity[7] of neutrinos that do seem to play an important role in such events.[8,9] This lack of theoretical knowledge is primarily due to the lack of availability of the general-relativistic equations that govern the asymmetries associated with gravitational collapse.

In this paper we purport to remedy this lack by introducing via a new formalism a set of gauge-invariant variables as well as their concomitant equations. They are so remarkably economical and flexible that they allow one to deal with perturbations away from any physically imaginable spherically symmetric space-time.

Initial investigations of perturbations of spherical space-times considered a static[10-22] or a homogeneous[23-27] background. They quite naturally single out time as a special coordinate. Such a singling out is rather infeasible for a general nonstatic background.

Treating the time and radial coordinates on an equal footing, we perform instead, what amounts to, a $2+2$ split on the background geometry and introduce the perturbations as scalar, vector, and tensor fields on the totally geodesic submanifold spanned by the time and radial coordinates. The equations for the perturbations in the metric and in the matter stress-energy tensor receive, however, their ultimate simplicity only after gauge-invariant[28] linear combinations of the fields are introduced as the new dependent variables. In terms of these gauge-independent quantities, odd-parity perturbations (metric and matter) are expressed in terms of vector fields on the totally geodesic submanifold. Similarly, even perturbations are characterized by scalar fields together with symmetric tensor fields.

II. BACKGROUND GEOMETRY

Consider any spherically symmetric space-time with a metric

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= g_{AB}dx^A dx^B + r^2(x^C)(d\theta^2+\sin^2\theta d\phi^2) \end{aligned} \tag{1a}$$

and a stress-energy tensor

$$\begin{aligned} t_{\mu\nu}dx^\mu dx^\nu &= t_{AB}dx^A dx^B \\ &\quad + \tfrac{1}{2}t^b{}_b r^2(x^C)(d\theta^2+\sin^2\theta d\phi^2)\,. \end{aligned} \tag{1b}$$

Capital latin indices A, B, C refer to some as-yet-unspecified radial and time coordinates, while lower-case latin indices $a, b, c, \ldots$ refer to θ and ϕ. The functions $r(x^C)$ and $g_{AB}(x^C)$ are scalar and tensor fields on the totally geodesic submanifold M^2 spanned by x^C $(C=0,1)$. The vector field

$$v_A = r_{,A}/r$$

is also on this submanifold. The Einstein field equations for any spherically symmetric space-time assume the form

$$8\pi G c^{-4} t_{AB} = -2(v_{A|B} + v_A v_B) + (2v_C{}^{|C} + 3v_C v^C - r^{-2}) g_{AB} \equiv \mathcal{G}_{AB}\,,$$
$$8\pi G c^{-4} \tfrac{1}{2} t^b{}_b = v_C{}^{|C} + v_C v^C - \mathcal{R} \equiv \tfrac{1}{2}\mathcal{G}^b{}_b\,. \qquad (2)$$

The vertical bars refer to covariant derivatives and $\mathcal{R}$ to the Gaussian curvature on the submanifold. The partial trace $t^a{}_a = t^2{}_2 + t^3{}_3$ is a scalar on this submanifold. Equations (2) cast into a geometric form the well-known coordinate-dependent equations for a spherically symmetric space-time.[29]

III. GEOMETRICAL PERTURBATION OBJECTS

Consider general perturbation fields away from Eqs. (1a) and (1b). It has "odd" harmonic components, whose parity is $(-1)^{l+1}$ and whose form (suppress the angular integers l and m) is

$$h_{\mu\nu} dx^\mu dx^\nu = h_A(x^C) S_a(\theta,\phi)(dx^A dx^a + dx^a dx^A) + h(x^C)(S_{a:b} + S_{b:a}) dx^a dx^b, \qquad (3a)$$

$$\Delta t_{\mu\nu} dx^\mu dx^\nu = \Delta t_A(x^C) S_a(\theta,\phi)(dx^A dx^a + dx^a dx^A) + \Delta t (S_{a:b} + S_{b:a}) dx^a dx^b\,. \qquad (3b)$$

The covariant derivative of the transverse $(S_a{}^{:a}=0)$ vector harmonic S_a on the unit two-sphere is indicated by colon. It is clear that the three metric expansion coefficients h_0, h_1, and h, and the three perturbed stress-energy expansion coefficients Δt_0, Δt_1, and Δt can be assembled into two covectors and two scalar fields

$h_A dx^A$ and h,

$\Delta t_A dx^A$ and Δt

on the submanifold M^2.

Similarly, an "even"-parity $[(-1)^l]$ perturbation mode

$$h_{\mu\nu} dx^\mu dx^\nu = h_{AB}(x^C) Y(\theta,\phi) dx^A dx^B + h_A Y_{,a}(dx^A dx^a + dx^a dx^A) + r^2[K Y(\theta,\phi)\gamma_{ab} + G Y_{,a:b}] dx^a dx^b, \qquad (4a)$$

$$\Delta t_{\mu\nu} dx^\mu dx^\nu = \Delta t_{AB}(x^C) Y(\theta,\phi) dx^A dx^B + \Delta t_A Y_{,a}(dx^A dx^a + dx^a dx^A) + r^2[\Delta t^1 Y(\theta,\phi)\gamma_{ab} + \Delta t^2 Y_{,a:b}] dx^a dx^b \qquad (4b)$$

is characterized by two symmetric tensors, two covectors, and four scalar fields

$h_{AB} dx^A dx^B$, $h_A dx^A$, K, and G,

$\Delta t_{AB} dx^A dx^B$, $\Delta t_A dx^A$, Δt^1, and Δt^2,

on the submanifold M^2.

It is a straightforward process to insert the odd-parity perturbation fields, Eqs. (3a) and (3b), into the linearized field equations. They reduce to a vectorial together with a scalar equation defined on a two-dimensional submanifold M^2. Similarly, the reduced field equations for the even-parity expressions Eqs. (4a) and (4b) become a symmetric tensorial together with a vectorial and two scalar equations. All these equations, which contain no gauge assumptions, are gauge dependent.

IV. GAUGE-INVARIANT GEOMETRICAL OBJECTS

In this paper we instead reformulate them in terms of gauge-invariant[28] geometrical objects. These gauge-invariant objects are found as follows:

First write down the gauge change induced by the infinitesimal vector fields:

$$\xi_\mu^{\rm odd} dx^\mu = M(x^C)\left[-(\sin\theta)^{-1}\left(\frac{\partial Y}{\partial\phi}\right) d\theta + \sin\theta\left(\frac{\partial Y}{\partial\theta}\right) d\phi\right],$$

"odd parity",

$$\xi_\mu^{\rm even} dx^\mu = \xi_A(x^C) Y(\theta,\phi) dx^A + \xi(x^C)\left[\left(\frac{\partial Y}{\partial\theta}\right) d\theta + \left(\frac{\partial Y}{\partial\phi}\right) d\phi\right],$$

"even parity".

The result, the Lie derivative with respect to ξ_μ, is

$$\left.\begin{aligned}\bar h_A - h_A &= -r^2(M/r^2)_{,A},\\ \bar h - h &= -M,\end{aligned}\right\}\quad \text{(metric)} \qquad (5a)$$

$$\left.\begin{aligned}\Delta\bar t_A - \Delta t_A &= -\tfrac{1}{2} t^a{}_a r^2 (M/r^2)_{,A},\\ \Delta\bar t - \Delta t &= -\tfrac{1}{2} t^a{}_a M,\end{aligned}\right\}\quad \text{(matter)} \qquad (5b)$$

and

$$\left.\begin{aligned}\bar h_{AB} - h_{AB} &= -(\xi_{A|B} + \xi_{B|A}),\\ \bar h_A - h &= -\xi_A - r^2(\xi/r^2)_{,A},\\ \bar K - K &= -2V^A\xi_A,\\ \bar G - G &= -2\xi/r^2,\end{aligned}\right\}\quad \text{(metric)} \qquad (6a)$$

$$\left.\begin{aligned} &\Delta\bar{t}_{AB}-\Delta t_{AB}=-t_{AB|C}\xi^C-t_{CB}\xi^C{}_{|A}-t_{CA}\xi^C{}_{|B},\\ &\Delta\bar{t}_A-\Delta t_A=-t_{AB}\xi^C-\tfrac{1}{2}r^2t^a{}_a(\xi/r^2)_{,A},\\ &\Delta\bar{t}^1-\Delta t^1=-\tfrac{1}{2}r^{-2}(r^2t^a{}_a)_{,A},\xi^A,\\ &\Delta\bar{t}^2-\Delta t^2=-t^a{}_a\xi, \end{aligned}\right\}\text{(matter)}\quad(6b)$$

for odd- and even-parity metric and stress-energy perturbations.

Second, construct the gauge-invariant geometrical objects by taking those linear combinations of the gauge-transformed (barred) quantities in Eqs. (5) [or Eqs. (6)] which are independent of the odd-parity (or even-parity) gauge generator $M(x^C)$ (or ξ_A and ξ). The set of gauge-invariant odd-parity geometrical objects is

$$k_A=h_A-r^2(h/r^2)_{,A},\quad\text{(matter)}\quad(7a)$$

$$\left.\begin{aligned} &L_A=\Delta t_A-(t^a{}_a/2)h_A,\\ &L=\Delta t-(t^a{}_a/2)h. \end{aligned}\right\}\text{(matter)}\quad(7b)$$

For even parity there is an analogous set of gauge-invariant geometrical objects which is

$$\left.\begin{aligned} &k_{AB}=h_{AB}-(p_{A|B}+p_{B|A}),\\ &k=K-2v^Ap_A, \end{aligned}\right\}\text{(matter)}\quad(8a)$$

$$\left.\begin{aligned} &T_{AB}=\Delta t_{AB}-t_{AB}{}^{|C}p_C\\ &\qquad-2(t_{CA}p^C{}_{|B}+t_{CB}p^C{}_{|A}),\\ &T_A=\Delta t_A-t_A{}^Cp_C-r^2(t^a{}_a/4)G_{,A},\\ &T^1=\Delta t^1-(p^c/r^2)(r^2t^a{}_a/2)_{,c},\\ &T^2=\Delta t^2-(r^2t^a{}_a/2)G \end{aligned}\right\}\text{(matter)}\quad(8b)$$

where

$$p_A\equiv h_A-\tfrac{1}{2}r^2G_{,A}.$$

V. FIELD EQUATIONS IN TERMS OF GEOMETRICAL GAUGE INVARIANTS

Finally, introduce these gauge-invariant geometrical quantities into the afore-mentioned gauge-dependent reduced linearized Einstein field equations. The resultant simplification is considerable.

The odd-parity equations become

$$k^A{}_{|A}=\kappa L\qquad(2\le l),\quad(9a)$$

$$-[r^4(r^{-2}k^A)^{|C}-r^4(r^{-2}k^C)^{|A}]_{|C}+(l-1)(l+2)k^A=\kappa r^2L^A\qquad(1\le l).\quad(9b)$$

The even-parity equations are

$$\begin{aligned} &r^{-2}[r^2(k_{AB|C}-k_{AC|B}-k_{BC|A})]^{|C}-[l(l+1)/r^2+\mathcal{G}^C{}_C+\mathcal{G}^a{}_a]k_{AB}+g_{AB}[r^{-2}(r^2k_{CD})^{|C|D}-\mathcal{G}^{CD}k_{CD}]+k^C{}_{C|A|B}\\ &\quad-g_{AB}\left[r^{-2}(r^2k^C{}_{C|D})^{|D}-\frac{l(l+1)}{r^2}k^C{}_C-\tfrac{1}{2}(\mathcal{G}^D{}_D+\mathcal{G}^a{}_a)k_C{}^C\right]\\ &\quad+2(v_Ak_{,B}+v_Bk_{,A}+k_{,A|B})-g_{AB}\left[2k_{,C}{}^{|C}+6v^Ck_{,C}-\frac{(l-1)(l+2)}{r^2}k\right]=-\kappa T_{AB}\qquad(0\le l),\quad(10a) \end{aligned}$$

$$k_{,A}-k_{AC}{}^{|C}+k^C{}_{C,A}-v_Ak^C{}_C=-\kappa T_A\qquad(0\le l),\quad(10b)$$

$$\begin{aligned} &-(k_{,C}{}^{|C}+2v^Ck_{,C}+\mathcal{G}^a{}_ak)+[k_{CD}{}^{|C|D}+2v^Ck_{CD}{}^{|D}+2(v^{C|D}+v^Cv^D)k_{CD}]\\ &\qquad-\left[k^C{}_{C|D}{}^{|D}+v^Ck^D{}_{D|C}+\mathcal{R}k^C{}_C-\frac{l(l+1)}{r^2}k^C{}_C\right]=-\kappa T^1\qquad(0\le l),\quad(10c) \end{aligned}$$

$$k^C{}_C=-\kappa T^2\quad(0\le l),\quad(10d)$$

where $\kappa=16\pi Gc^{-4}$. Thus both the odd- as well as the even-parity linearized field equations are totally independent of gauges. In vacuum these equations reduce to the form given by Zerilli.[30]

The even-parity tensorial gauge-invariant differential equation (10a) can be simplified with the following identity applicable to an arbitrary symmetric tensor on a two-dimensional manifold with Gaussian curvature,

$$(k_{AB|C}-k_{AC|B}-k_{BC|A})^{|C}+k^C{}_{C|A|B}+g_{AB}(k_{CD}{}^{|C|D}-k^C{}_{C|D}{}^{|D})=\mathcal{R}(k^C{}_Cg_{AB}-2k_{AB}).\quad(11)$$

It eliminates all second derivatives of k_{AB} in Eq. (10a), and consequently, with the help of the background Eq. (2), one obtains

$$2v^C(k_{AB|C}-k_{AC|B}-k_{BC|A})-[(l-1)(l+2)/r^2+\mathcal{G}^C{}_C+\mathcal{G}^a{}_a]k_{AB}$$
$$-2g_{AB}v^C(k_{DE|C}-k_{DC|E}-k_{EC|D})g^{ED}+g_{AB}(2v^{C|D}+4v^Cv^D-\mathcal{G}^{CD})k_{CD}$$
$$+g_{AB}\left[\frac{l(l+1)}{r^2}+\tfrac{1}{2}(\mathcal{G}^C{}_C+\mathcal{G}^a{}_a)\right]k^D{}_D+\mathcal{R}(k^C{}_Cg_{AB}-2k_{AB})+2(v_Ak_{,B}+v_Bk_{,A}+k_{,A|B})$$
$$-g_{AB}\left(2k_{,C}{}^{|C}+6v^Ck_{,C}-\frac{(l-1)(l+2)}{r^2}k\right)=-\kappa T_{AB}\,. \qquad (12)$$

A linear combination of tensorial equation (12) and the vectorial equation (10b) results in further simplication and one obtains a differential equation involving the gauge-invariant tensor k_{AB} only. Thus, define

$$U_{AB}=-\frac{1}{r^2}[(r^2T_A)_{|B}+(r^2T_B)_{|A}]\,.$$

Then the traceless equation

$$-\kappa[(T_{AB}-\tfrac{1}{2}g_{AB}T^C{}_C)-(U_{AB}-\tfrac{1}{2}g_{AB}U^C{}_C)]=-k_{AB|C}{}^{|C}+k^C{}_{C|A|B}+2v^C(k_{AB|C}-k_{AC|B}-k_{BC|A})$$
$$-v_A(k_{CD|B}-k_{BC|D}-k_{DB|C})g^{CD}-v_B(k_{CD|A}-k_{AC|D}-k_{DA|C})g^{CD}$$
$$-\left[\frac{l(l+1)}{r^2}+\mathcal{G}^C{}_C+\mathcal{G}^a{}_a\right](k_{AB}-\tfrac{1}{2}k^C{}_Cg_{AB})$$
$$-[2(v_{A|B}+2v_Av_B)-(v_D{}^{|D}+2v_Dv^D)g_{AB}]k^C{}_C \qquad (13)$$

is a differential equation in the tensor k_{AB} only.

The linearized conservation equations $\Delta(t_{\mu\nu}{}^{;\nu})=0$ are also gauge invariant. For odd-parity perturbations they reduce to the single scalar equation

$$(r^2L^A)_{|A}=(l-1)(l+2)L \quad (1\le l)\,. \qquad (14)$$

For even parity there is a scalar and a vector equation,

$$r^{-2}(r^2T^A)_{|A}+T^1+[1-l(l+1)]T^2/r^2=\tfrac{1}{2}t^a{}_a(k-\tfrac{1}{2}k^C{}_C)+\tfrac{1}{2}t^{AB}k_{AB} \quad (0\le l)\,, \qquad (15a)$$

$$r^{-2}(r^2T_{AB})^{|B}-T_Al(l+1)/r^2-2v_AT^1+v_AT^2l(l+1)/r^2=\tfrac{1}{2}k_{BC|A}t^{BC}+k_{CB}{}^{|B}t^C{}_A-\tfrac{1}{2}k^C{}_{C|B}t^B{}_A-k_{,C}t^C{}_A$$
$$+(\tfrac{1}{2}k_{,A}-kv_A)t^a{}_a+2v^Bk_{BC}t^C{}_A+k^B{}_Ct^C{}_{A|B} \quad (0\le l)\,. \qquad (15b)$$

These conservation equations are directly implied by the corresponding linearized field equations provided one again uses identity Eq. (11).

VI. ODD-PARITY MASTER EQUATION

The solutions to the odd-parity equations (9) and (14) are most easily obtained from a scalar master equation. This is accomplished particularly rapidly in terms of Cartan's calculus of differential forms. Rewrite the equations in the form

$$*d(r^2*L_Cdx^C)=(l-1)(l+2)L \quad (1\le l)\,, \qquad (16a)$$

$$*d(*k_Cdx^C)=\kappa L \quad (2\le l)\,, \qquad (16b)$$

$$-*dr^4*d(r^{-2}k_Cdx^C)+(l-1)(l+2)k_Cdx^C$$
$$=\kappa r^2L_Cdx^C \quad (1\le l)\,, \qquad (16c)$$

where $*k_Cdx^C=k^B\epsilon_{B\,C}dx^C$ is the Hodge dual of k_Cdx^C with respect to the metric $g_{AB}dx^Adx^B$ and an orientation of the submanifold M^2, and d is the exterior derivative. The odd-parity equations (16) are solved as follows. Let Π be the scalar function,

$$\Pi = *d(r^{-2}k_C dx^C)$$

Take the exterior derivative of Eq. (16c) and obtain the odd-parity master equation

$$-*dr^{-2}*dr^4\Pi + (l-1)(l+2)\Pi = \kappa *d(L_C dx^C)\,. \quad (17)$$

Solve for Π. For a homogeneous background this equation reduced to Eq. (II-12b) in Ref. 3. There (with $r^2\Pi = \pi_1$) it was found with the help of an algebraic computer code.

For $l \neq 1$ the solution $k_C dx^C$ is then obtained from Eq. (16c):

$$k_C dx^C = [\kappa r^2 L_C dx^C + *d(r^4\Pi)][(l-1)(l+2)]^{-1}, \quad l \geq 2\,. \quad (18)$$

For the nonradiative case $l=1$ Eq. (16a) implies that $r^2 * L_C dx^C$ is closed. Hence there exists locally a scalar T such that

$$r^2 * L_C dx^C = dT\,.$$

After integration Eq. (16c) therefore becomes

$$-r^4 *d(r^{-2}k_c\, dx^C) = \kappa T\,. \quad (19)$$

Express $r^{-2}k_C dx^C$ in terms of two scalars Φ and ψ,

$$r^{-2}k_C dx^C = *d\Phi + d\psi\,.$$

Thus Eq. (19) becomes

$$*d*d\Phi = -\kappa r^{-4}T \quad (l=1)\,.$$

One sees therefore that $l=1$ odd-parity perturbations are composed of a particular solution, $*d\Phi$, which can be associated with rotating matter and a complementary solution $d\Psi$ which may be nonzero even in the absence ($T=0$) of ostensible spinning matter.

The junction conditions[31] across a timelike hypersurface Σ of discontinuity (e.g., the history of the surface of a collapsing star) are the continuity of Π and of the projections of k_A onto Σ and onto the unit normal to Σ.

VII. CONCLUSION

The Einstein field equations linearized around the most general spherically symmetric space-time assume their final most striking simplicity in terms of gauge-invariant metric and matter geometrical objects. The generality of these simple equations allows one to consider perturbations not only on any spherically symmetric (in general matter occupied) space-time but also, at a glance, with respect to any coordinate system. With these equations at hand the way stands open to considering astrophysically interesting perturbed spherical-collapse problems.

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