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Birkhoff theorem for an $R + R^2$ theory of gravity with torsion

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We investigate the gravitational Lagrangian $L_G = (c^4/16\pi G)R - (\hbar c/16\pi\alpha_G)\, R^\alpha{}_{\beta\gamma\delta}\, R_\alpha{}^{\beta\gamma\delta}$, where the curvatures are computed from a metric-compatible connection with torsion. We prove that the Schwarzschild metric and its torsion-free connection constitute the unique O(3) spherically symmetric (including reflections) solution to the vacuum field equations of L_G.

I. INTRODUCTION

We have been investigating the theory of gravity given by the Lagrangian[1]

$$L = \frac{c^4}{16\pi G} R - \frac{\hbar c}{16\pi\alpha_G} R^\alpha{}_{\beta\gamma\delta} R^\beta{}_\alpha{}^{\gamma\delta} + L_M , \qquad (1)$$

where α_G is a new dimensionless coupling constant, all curvatures are computed from a Cartan connection (which is metric-compatible but may have nonzero torsion), and L_M is the matter Lagrangian minimally coupled to the Cartan connection. Debney, Fairchild, and Siklos[2] have proved that in vacuum, in the absence of torsion, this theory is exactly equivalent to Einstein's theory. We do not assume that the torsion is zero. Instead, we assume O(3) spherical symmetry[3] and prove the following result.

Generalized Birkhoff theorem. Let the metric and Cartan connection on a region of spacetime be an O(3) spherically symmetric vacuum solution to the field equations derived from Lagrangian (1). Then the connection has zero torsion, and the metric is the Schwarzschild metric.

This theorem is analogous to Birkhoff's theorem[4] for Einstein's theory. In the remainder of this section, we motivate the choice of Lagrangian and discuss the implications of our theorem.

Torsion was first introduced into a theory of gravity by Cartan.[5] He suggested that spacetime might be a Riemannian manifold with a connection $\Gamma^\alpha{}_{\beta\gamma}$ (hereafter called a Cartan connection) which is metric-compatible but may have nonzero torsion. The simplest extension of Einstein's theory to include torsion is the Einstein-Cartan-Sciama-Kibble (ECSK) theory, which has been studied extensively by Hehl, Trautman, and others.[6] Part of the motivation for the ECSK theory is that it may be considered a gauge theory for the Poincaré group, whereas Einstein's theory may be considered a gauge theory for only the translation group.[7] Both the Einstein theory and the ECSK theory use the energy-momentum tensor as the source for the metric. A differential equation must be solved for the metric or orthonormal frame. In addition, the ECSK theory uses the spin tensor as the source of the torsion. However, the torsion is algebraically coupled to the spin tensor and so is nondynamic. Hence, the ECSK theory does not treat the rotational and translational parts of the Poincaré group on an equal footing. This motivates one to examine theories of gravity such as Lagrangian (1) in which the torsion is differentially coupled to the spin tensor.

The squared Riemann curvature term in (1) would, by itself, be sufficient to differentially couple the torsion to the spin density. It is in fact the gravitational Lagrangian most analogous to the Lagrangian in a Yang-Mills theory of the Lorentz group, as pointed out by Yang.[8] However, by itself, the squared curvature term would not handle the rotational and translational parts of the Poincaré group symmetrically. What is more, Fairchild[9] has shown that such a theory does not have a satisfactory Newtonian limit. So we include in (1) the scalar curvature term, which is the Lagrangian for the ECSK theory, and may be regarded as a kinetic term for the translation group.[7] It is hoped that a Newtonian limit can be derived from Lagrangian (1). Fairchild[1] claims to have done this, but he fails to justify his identification of $T_{00} + V_{00}$ as the mass density. We feel it should be simply T_{00}. In that case it is necessary to show that there exists a consistent limit in which the propagating torsion is sufficiently small so that V_{00} is negligible. More work is needed on this problem.

In addition to having a Newtonian limit, a theory of gravity must make predictions which agree with the other standard experimental tests. In the absence of torsion, test particles and light waves in the theory of Lagrangian (1) move in the same manner as they do in Einstein's theory.[10] Therefore, if the gravitational field of the solar system were precisely spherically symmetric (including reflections), our generalized Birkhoff theorem would show that Lagrangian (1) makes the same predictions for solar system experiments as does

Einstein's theory. A stronger statement about the predictions of this theory for solar system experiments can be made only after checking the stability of the Schwarzschild solution under perturbations involving torsion.

It is also desirable that a theory of gravity have a locally unique vacuum solution which is spatially homogeneous, isotropic, and parity invariant. This solution would be regarded as the ground state of the gravitational field. Furthermore, it is desirable that this solution be Minkowski space so that the ground state would have no gravitational field. (For the Einstein theory with a cosmological constant, the ground state is de Sitter space, which has tidal forces.) Our theorem shows that in the theory of Lagrangian (1), as in the Einstein and ECSK theories, Minkowski space with zero torsion is the locally unique vacuum solution which is spatially homogeneous, isotropic, and parity invariant. This follows because such a solution would have to be O(3) spherically symmetric about every point. This is in contrast to the theory investigated by Horowitz and Wald,[11] who showed that their field equations have homogeneous, isotropic, vacuum solutions in addition to Minkowski.

Another reason for adding curvature squared terms to the Lagrangian [although not necessarily the term added in (1)] is the result of Stelle.[12] He shows that the addition of curvature squared terms (without torsion) to the Einstein Lagrangian yields a renormalizable, although nonunitary, theory.

The standard objection to adding curvature squared terms to the Lagrangian is that they produce field equations containing higher than second derivatives of the metric. This is true of the field equations of Lagrangian (1) when they are regarded as functions of the metric and the torsion. However, when the components of the orthonormal frame and the "mixed" components of the Cartan connection[13] are used as the independent variables, the field equations contain no higher than first derivatives of the frame (or metric) and second derivatives of the connection.[14] Without the torsion, the frames and mixed components of the connection would not be independent.

In Sec. II we give the field equations for Lagrangian (1), and in Sec. III we prove the generalized Birkhoff theorem.

II. NOTATION AND FIELD EQUATIONS

We take the signature of the metric to be $(-+++)$. The Cartan connection $\Gamma^\alpha{}_{\beta\gamma}$ may be compared with the Christoffel connection $\{{}^{\alpha}{}_{\beta\gamma}\}$ which is metric-compatible and torsion-free. Their difference is defined to be the defect tensor

$$\lambda^\alpha{}_{\beta\gamma}=\Gamma^\alpha{}_{\beta\gamma}-\{{}^{\alpha}{}_{\beta\gamma}\}.$$

The torsion tensor, Riemann tensor, Ricci tensor (asymmetric), scalar curvature, and Einstein tensor (asymmetric) are (in a coordinate basis)

$$Q^\alpha{}_{\beta\gamma}=\lambda^\alpha{}_{\gamma\beta}-\lambda^\alpha{}_{\beta\gamma}=\Gamma^\alpha{}_{\gamma\beta}-\Gamma^\alpha{}_{\beta\gamma},$$

$$R^\alpha{}_{\beta\gamma\delta}=\partial_\gamma\Gamma^\alpha{}_{\beta\delta}-\partial_\delta\Gamma^\alpha{}_{\beta\gamma}+\Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta}-\Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma},$$

$$R_{\beta\delta}=R^\alpha{}_{\beta\alpha\delta},$$

$$R=g^{\beta\delta}R_{\beta\delta},$$

$$G_{\beta\delta}=R_{\beta\delta}-\tfrac{1}{2}g_{\beta\delta}R.$$

In performing covariant derivatives, a caret over an index denotes that the correction on that index is performed using the Cartan connection, while a tilde denotes a correction with the Christoffel connection. For example,

$$\nabla_\epsilon R^{\hat\alpha}{}_{\hat\beta\tilde\gamma}{}^{\tilde\delta}=\partial_\epsilon R^\alpha{}_{\beta\gamma}{}^\delta+\Gamma^\alpha{}_{\mu\epsilon}R^\mu{}_{\beta\gamma}{}^\delta-\Gamma^\mu{}_{\beta\epsilon}R^\alpha{}_{\mu\gamma}{}^\delta-\{{}^{\mu}{}_{\gamma\epsilon}\}R^\alpha{}_{\beta\mu}{}^\delta+\{{}^{\delta}{}_{\mu\epsilon}\}R^\alpha{}_{\beta\gamma}{}^\mu.$$

The Cartan connection automatically satisfies the Bianchi identities,

$$\eta^{\mu\nu\gamma\delta}\nabla_\delta R^{\hat\alpha}{}_{\hat\beta\tilde\mu\tilde\nu}=0, \qquad (2)$$

where $\eta^{\mu\nu\gamma\delta}$ is the totally antisymmetric tensor.

Varying the Lagrangian (1) with respect to the components of the orthonormal frames,[13] one obtains

$$G_{\mu\nu}-2\chi(R^\alpha{}_{\beta\gamma\mu}R^\beta{}_\alpha{}^\gamma{}_\nu-\tfrac{1}{4}g_{\mu\nu}R^\alpha{}_{\beta\gamma\delta}R^\beta{}_\alpha{}^{\gamma\delta})=8\pi Gc^{-4}t_{\mu\nu}, \qquad (3)$$

which will be called the Einstein equations. Varying the Lagrangian (1) with respect to the mixed components of the Cartan connection,[13] one obtains

$$\lambda^\gamma{}_\beta{}^\alpha-\lambda^{\gamma\alpha}{}_\beta-\delta^\gamma{}_\beta\lambda^{\alpha\delta}{}_\delta+g^{\gamma\alpha}\lambda_\beta{}^\delta{}_\delta+4\chi\nabla_\delta R^{\hat\alpha}{}_{\hat\beta}{}^{\tilde\gamma\tilde\delta}=8\pi Gc^{-4}S^\alpha{}_\beta{}^\gamma, \qquad (4)$$

which will be called the Cartan equations. Here $t_{\mu\nu}$ is the canonical energy-momentum tensor (asymmetric), $S^\alpha{}_\beta{}^\gamma$ is the canonical spin tensor, and $\chi=\hbar G/(c^3\alpha_G)$. In vacuum, $t_{\mu\nu}$ and $S^\alpha{}_\beta{}^\gamma$ are zero.

III. PROOF OF GENERALIZED BIRKHOFF THEOREM

We first give an outline of our proof. After writing out the field equations and Bianchi identities for a spherically symmetric system, we note that the Einstein equations factor, yielding three cases. In two of the cases, adding the Bianchi identities to the Cartan equations leads to a contradiction. In the third case, subtracting the Bianchi identities from the Cartan equations shows that the Einstein and torsion tensors are zero. Birkhoff's theorem for Einstein's theory then implies that the Schwarzschild metric is the unique solution. The details of the proof follow.

The most general spherically symmetric metric can be written as

$$ds^2 = -e^{2\Phi}dT^2 + e^{2\Lambda}dR^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{5}$$

where Φ, Λ, and r are arbitrary functions of R and T. We are allowing here for the possiblility that the Schwarzschild r is a bad coordinate, by making r an arbitrary function of good coordinates R and T. We write out all tensors in the orthonormal frame, with basis one-forms,

$$\vartheta^T = e^{\Phi}dT\,, \tag{6}$$

$$\vartheta^R = e^{\Lambda}dR\,, \tag{7}$$

$$\vartheta^\theta = r\,d\theta\,, \tag{8}$$

$$\vartheta^\phi = r\sin\theta\, d\phi\,. \tag{9}$$

We impose O(3) spherical symmetry on the Cartan connection by demanding that the defect tensor be invariant under rotations and reflections, and that the Christoffel symbols be computed from the spherically symmetric metric (5). The independent nonzero components of the spherically symmetric defect tensor[3] are

$$\lambda^T{}_{RT} = f(R,T)\,, \tag{10}$$

$$\lambda^T{}_{RR} = h(R,T)\,, \tag{11}$$

$$\lambda^T{}_{\theta\theta} = \lambda^T{}_{\phi\phi} = k(R,T)\,, \tag{12}$$

$$\lambda^R{}_{\theta\theta} = \lambda^R{}_{\phi\phi} = g(R,T)\,, \tag{13}$$

where, f, g, h, k are arbitrary functions of R and T. Adding the defect tensor to the Christoffel symbols, we obtain the independent nonzero components of the Cartan connection:

$$\Gamma^T{}_{RT} = e^{-\Lambda}\Phi' + f \equiv V(R,T)\,, \tag{14}$$

$$\Gamma^T{}_{RR} = e^{-\Phi}\dot\Lambda + h \equiv X(R,T)\,, \tag{15}$$

$$\Gamma^T{}_{\theta\theta} = \Gamma^T{}_{\phi\phi} = e^{-\Phi}r^{-1}\dot r + k \equiv Y(R,T)\,, \tag{16}$$

$$\Gamma^R{}_{\theta\theta} = \Gamma^R{}_{\phi\phi} = -e^{-\Lambda}r^{-1}r' + g \equiv W(R,T)\,, \tag{17}$$

$$\Gamma^\theta{}_{\phi\phi} = -r^{-1}\cot\theta\,. \tag{18}$$

Dots denote differentiation with respect to T, and primes denote differentiation with respect to R.

The independent nonzero components of the Riemann tensor are

$$R^T{}_{RTR} = [(Xe^{\Lambda})^{\cdot} - (Ve^{\Phi})']e^{-\Phi-\Lambda} \equiv -A\,, \tag{19}$$

$$R^T{}_{\theta T\theta} = R^T{}_{\phi T\phi} = e^{-\Phi}r^{-1}(Yr)^{\cdot} + VW \equiv -C\,, \tag{20}$$

$$R^T{}_{\theta R\theta} = R^T{}_{\phi R\phi} = e^{-\Lambda}r^{-1}(Yr)' + XW \equiv D\,, \tag{21}$$

$$R^R{}_{\theta T\theta} = R^R{}_{\phi T\phi} = e^{-\Phi}r^{-1}(Wr)^{\cdot} + YV \equiv -G\,, \tag{22}$$

$$R^R{}_{\theta R\theta} = R^R{}_{\phi R\phi} = e^{-\Lambda}r^{-1}(Wr)' + YX \equiv H\,, \tag{23}$$

$$R^\theta{}_{\phi\theta\phi} = r^{-2} + Y^2 - W^2 \equiv L\,, \tag{24}$$

and the nonzero components of the Einstein tensor are

$$G_{TT} = 2H + L\,, \tag{25}$$

$$G_{TR} = -2D\,, \tag{26}$$

$$G_{RT} = -2G\,, \tag{27}$$

$$G_{RR} = 2C - L\,, \tag{28}$$

$$G_{\theta\theta} = G_{\phi\phi} = C - H + A\,. \tag{29}$$

The independent Einstein equations (3) in vacuum are

$$2H + L - 4\chi(D^2 + C^2 - H^2 - G^2 - \tfrac12 L^2 + \tfrac12 A^2) = 0\,, \tag{30}$$

$$-2D + 8\chi(CD - HG) = 0\,, \tag{31}$$

$$-2G + 8\chi(CD - HG) = 0\,, \tag{32}$$

$$2C - L - 4\chi(D^2 + C^2 - H^2 - G^2 + \tfrac12 L^2 - \tfrac12 A^2) = 0\,, \tag{33}$$

$$C - H + A + 2\chi(L^2 - A^2) = 0\,. \tag{34}$$

The independent Cartan equations (4) in vacuum are

$$-2(W + e^{-\Lambda}r^{-1}r') + 4\chi[e^{-\Lambda}r^{-2}(r^2A)' - 2YG + 2WC] = 0\,, \tag{35}$$

$$-2(Y - e^{-\Phi}r^{-1}\dot r) - 4\chi[e^{-\Phi}r^{-2}(r^2A)^{\cdot} + 2YH - 2WD] = 0\,, \tag{36}$$

$$-[X + Y - e^{-\Phi-\Lambda}r^{-1}(re^{\Lambda})^{\cdot}] - 4\chi\{e^{-\Phi-\Lambda}r^{-1}[(Dre^{\Phi})' + (Cre^{\Lambda})^{\cdot}] + VG + XH + YL\} = 0\,, \tag{37}$$

$$-[W - V + e^{-\Phi-\Lambda}r^{-1}(re^{\Phi})'] - 4\chi\{e^{-\Phi-\Lambda}r^{-1}[(Hre^{\Phi})' + (Gre^{\Lambda})^{\cdot}] + VC + XD + WL\} = 0\,. \tag{38}$$

The independent Bianchi identities (2) are

$$-e^{-\Lambda}r^{-2}(r^2L)' + 2YD - 2WH = 0\,, \tag{39}$$

$$e^{-\Phi}r^{-2}(r^2L)^{\cdot} + 2YC - 2WG = 0\,, \tag{40}$$

$$e^{-\Phi-\Lambda}r^{-1}[(Gre^{\Phi})' + (Hre^{\Lambda})^{\cdot}] + VD + XC + YA = 0\,, \tag{41}$$

$$e^{-\Phi-\Lambda}r^{-1}[(Cre^{\Phi})' + (Dre^{\Lambda})^{\cdot}] + VH + XG + WA = 0\,. \tag{42}$$

Based upon equations (30)–(42), we now prove that the defect and Einstein tensors are zero. First, the Einstein equations (30)–(34) can be manipulated into the equivalent form

$$G = D\,, \tag{43}$$

$$G[1 - 4\chi(C - H)] = 0\,, \tag{44}$$

$$(H+C)[1-4\chi(C-H)]=0\,, \tag{45}$$

$$(A-L)+2(C-H)=0\,, \tag{46}$$

$$(A+L)[1+8\chi(C-H)]=0\,. \tag{47}$$

These equations split into three cases.

Case I. $C-H=(4\chi)^{-1}$, so that $G=D$ and $L=-A=(4\chi)^{-1}$.

Case II. $C-H=-(8\chi)^{-1}$, so that $G=D=0$, $H=-C=(16\chi)^{-1}$, and $A-L=(4\chi)^{-1}$.

Case III. $C-H\neq-(8\chi)^{-1}$ and $C-H\neq(4\chi)^{-1}$, so that $G=D=0$ and $A=-L=-2C=2H$.

Next, we compare the Cartan equations (35)–(38) with the Bianchi identities (39)–(42) in each of the three cases.

Case I. We add 4χ times (41) to (37), and 4χ times (42) to (38), and use the conditions of case I on the resulting two equations to show that $Y=W=0$. From the definitions (20) and (23) of C and H, this implies $C=H=0$, which contradicts $C-H=(4\chi)^{-1}$, ruling out case I.

Case II. We add 4χ times (39) to (35), and 4χ times (40) to (36), and use the conditions of case II to show that $Y=W=0$. As before, this implies $C=H=0$, which contradicts $C-H=-(8\chi)^{-1}$, ruling out case II.

Case III. We subtract 4χ times (39) from (35), 4χ times (40) from (36), 4χ times (41) from (37), and 4χ times (42) from (38), and use the conditions of case III as well as Eqs. (14)–(17) to show that $f=g=h=k=0$. This implies that the defect and hence the torsion tensors are zero. The conditions of case III directly imply that $G=D=2H+L=2C-L=C-H+A=0$. This implies that the Einstein tensor, computed with torsion, is zero, but since the torsion is zero, the Einstein tensor computed from the Christoffel connection is also zero. Birkhoff's theorem for Einstein's theory then says that the metric is the Schwarzschild metric. This is, in fact, a solution since with zero torsion, any solution to the vacuum field equations of Einstein's theory is a solution to Eqs. (3) and (4) in vacuum.[15] Hence the Schwarzschild metric, with zero torsion, is the unique solution.

This completes the proof of our generalized Birkhoff theorem.

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[1]The spinor version of this theory has been investigated by E. E. Fairchild, Jr., Phys. Rev. D 16, 2438 (1977). The vector version, with and without torsion and metric-compatibility, has been proposed by F. Mansouri and L. N. Chang, Phys. Rev. D 13, 3192 (1976).

[2]G. Debney, E. E. Fairchild, Jr., and S. T. C. Siklos, Gen. Rel. Grav. 9, 879 (1978). Note that they take "vacuum" to mean torsion-free as well as no matter. We take it to mean only that there are no matter fields present.

[3]A spacetime is O(3) spherically symmetric if there is an action of the group O(3) on the spacetime manifold which leaves the metric and torsion invariant and whose orbits are generically two-spheres. The inclusion of reflections as well as rotations reduces the number of torsion functions from eight to four.

[4]G. D. Birkhoff, *Relativity and Modern Physics* (Harvard University Press, Cambridge, Mass., 1927), p. 253.

[5]E. Cartan, C. R. Acad. Sci. (Paris) 174, 593 (1922); Ann. Ec. Norm. Sup. 40, 325 (1923); 41, 1 (1924); 42, 17 (1925).

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[7]Y. M. Cho, Phys. Rev. D 14, 2521 (1976).

[8]C. N. Yang, Phys. Rev. Lett. 33, 445 (1974). This theory was previously investigated by G. Stephenson, Nuovo Cimento 9, 263 (1958). Neither paper allows for torsion.

[9]E. E. Fairchild, Jr., Phys. Rev. D 14, 384 (1976).

[10]In any theory of gravity in which the matter is coupled to the gravitational field by minimal coupling to the connection within the matter Lagrangian, it is possible to compute the motion of test particles and light waves from the conservation laws. These are derived by Noether's theorem from the matter Lagrangian and do not depend on the choice of gravitational field equations. See P. B. Yasskin, Ph.D. thesis, Univ. of Maryland, 1979 (unpublished).

[11]G. T. Horowitz and R. M. Wald, Phys. Rev. D 17, 414 (1978). See their Ref. 12.

[12]K. S. Stelle, Phys. Rev. D 16, 953 (1977).

[13]The orthonormal vector frame e_α, and its dual one-form frame θ^α may be expanded in the coordinate bases as $e_\alpha=e_\alpha{}^a\,\partial_a$, and $\theta^\alpha=\theta^\alpha{}_a\,dx^a$. The mixed components of the connection $\Gamma^\alpha{}_{\beta c}$ are defined by $\nabla_{\partial_c}e_\beta=\Gamma^\alpha{}_{\beta c}\,e_\alpha$. We take $\theta^\alpha{}_a$ and $\Gamma^\alpha{}_{\beta c}$ as the independent variables.

[14]This observation appears in P. B. Yasskin, Ref. 10.

[15]See E. E. Fairchild, Jr., Ref. 1.