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Time functions in numerical relativity: Marginally bound dust collapse

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We explore the problem of the existence of global maximal ($K = 0$) and constant-mean-curvature ($K = K_0$) time functions in general relativity. We attempt a rigorous definition of numerical relativity so as to bridge the gap between the field and mathematical relativity. We point out that numerical relativity can in principle construct any globally hyperbolic solution to Einstein's equations. This involves the construction of Cauchy time functions. Therefore we review what is known about the existence and uniqueness of such functions when their mean curvature is specified to be a constant on each time slice. We note that in strong-field solutions which contain singularities the question of existence is intimately connected to the nature of the singularity. Defining the class of "crushing singularities" we prove new theorems showing that $K = 0$ or $K = K_0$ time functions uniformly avoid such singularities (which include both Cauchy horizons and some curvature singularities). We then study the inhomogeneous generalizations of the Oppenheimer-Snyder spherical-dust-collapse spacetimes. These Tolman-Bondi solutions are classified as to their causal structure and found to contain naked singularities of a new type if the collapse is sufficiently inhomogeneous. We calculate the $K = 0$ and $K = K_0$ time slices for a variety of these spacetimes. We find that since some extreme dust collapses lead to noncrushing singularities, maximal time slicing can hit the singularity before covering the domain of outer communications of the resulting black hole. Furthermore, the use of $K = K_0$ slices in the presence of a naked singularity is discussed.

I. INTRODUCTION

Much research is underway to develop computer codes capable of solving the time-dependent Einstein field equations of general relativity.[1] A prerequisite for the success of numerical relativity is the selection of a good spacetime coordinate system[2] in which to describe the dynamics. In particular, one must be able to construct smooth Cauchy time slices over that part of the maximal development which is of physical interest. For instance, during the birth of a black hole, one may want to follow numerically all the gravitational radiation outside until late times, without having the computer "crash" because the time slice has run into the spacetime singularity inside. Numerical experience to date[3] has shown that maximal time slicing is often useful. However, the rigorous results[4] available on maximal slices do not go far enough to *guarantee* its success in all the spacetimes of interest to numerical relativity; most crucially, almost nothing is known in general about whether maximal ($K=0$) slices avoid spacetime singularities. Similar remarks hold for constant-mean-curvature slicing ($K=K_0$).[5]

In this paper we want to evaluate what one actually knows about maximal and constant-mean-curvature slicing in the presence of spacetime singularities. First in Sec. II we attempt a rigorous definition of numerical relativity. We conclude that those spacetimes which can be constructed by numerical relativity are precisely the globally hyperbolic spacetimes. We then review the current status of existence and uniqueness theorems for $K=0$ and $K=K_0$ slices in cosmologies and asymptotically flat spacetimes. We conjecture about possible extensions of these results to problems of interest in numerical relativity, particularly the questions of avoidance of singularities by these slicings. Then, we turn the tables and define a restricted class of *crushing singularities* that automatically have the desired property that $K=0$ and $K=K_0$ slices avoid them. The problem then becomes to discover how general the class of crushing singularities is among all singularities that arise in gravitational collapse. We do not know how restrictive this class is; to gain insight we look to some simple examples.

In Sec. III we apply these ideas to Schwarzschild, Reissner-Nordström, and Kerr black holes. In Sec. IV we discuss the spherically symmetric dust spacetimes of Tolman[6] and Bondi[6] as models of gravitational collapse to a black hole; we point out a new kind of naked singularity, the "shell-focusing singularity," which has hitherto escaped notice in these models for 40 years. Finally, we numerically construct families of $K=0$ and $K=K_0$ slices in a wide variety of Tolman-Bondi marginally bound models. We conclude that much stronger constraints than just energy conditions are needed to avoid the formation of naked singularities. Furthermore, we demonstrate that even assuming "strong cosmic censorship" (global hyperbolicity) is not enough to guarantee that maximal slicing will necessarily avoid singularities.

II. NUMERICAL RELATIVITY: A DEFINITION[7]

A. Causal structure

Numerical relativity[1] is a method for obtaining solutions to the Einstein field equations. The procedure is as follows: (1) Pick an initial data set (γ_{ij}, K_{ij}) = (three-metric, extrinsic curvature) on an edgeless spacelike hypersurface S. (2) Using a spacetime metric form,

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dw^2 + 2\beta_i dw dx^i + \gamma_{ij} dx^i dx^j , \tag{1}$$

where $(\alpha, \beta_i, \gamma_{ij})$ are smooth functions of $(w, x^i) \in R \times S$, choose a gauge by specifying the lapse function α and shift vector β_i. (3) Evolve (γ_{ij}, K_{ij}) from S to the nearby surface S' using the mapping of points in S to S' defined by (α, β_i). (4) Iterate steps (2) and (3) thereby creating a spacetime M foliated by slices S_i determined by α and threaded by a coordinate congruence determined by β_i. In this section we study the causal properties of such spacetimes.

The events which can be predicted from S are contained in $D(S) = D^+(S) \cup D^-(S)$, the *domain of dependence*[8] of S. That is, $D^{\pm}(S) = \{x|$ every $\binom{\text{past}}{\text{future}}$ inextendible nonspacelike curve through x intersects $S\}$. The $\binom{\text{future}}{\text{past}}$ boundaries of $D^{\pm}(S)$ are the $\binom{\text{future}}{\text{past}}$ *Cauchy horizons* of S: $H^{\pm}(S) = D^{\pm}(S) - I^{\mp}[D^{\pm}(S)]$, where $I^{\pm}$ is the chronological $\binom{\text{future}}{\text{past}}$ of S. If S has no edge,[7] then $H^{\pm}(S)$ is null. Besides $H^{\pm}(S)$, $D^{\pm}(S)$ may have boundary points added as terminal indecomposable pasts (TIP's) or terminal indecomposable futures (TIF's) which represent singularities or points at infinity.[9] In general, $D^{\pm}(S)$ may be smoothly extended across $H^{\pm}(S)$ in an infinite number of different ways to form the complete manifold M; that is, that part of M outside $D(S)$ cannot be predicted from S. If spacetime is assumed to be analytic, not just smooth, then the number of possible extensions is greatly reduced, often to one (e.g., analytic extensions across Cauchy horizons in Reissner-Nordström or Kerr spacetime).[10] But this is too strong an assumption because most spacetimes are not analytic. Therefore, in general, S will be a *partial* Cauchy surface, that is, no nonspacelike curve in M intersects S more than once. In this case $D(S) \neq M$. If $D(S) = M$, which often is not the case, S is a (*global*) *Cauchy surface*.

In a general spacetime (M, ds^2), a *time function* w on an open neighborhood N of M is a smooth real-valued function on N with an everywhere timelike, future-pointing gradient vector $-\nabla^a w$. Then w is strictly increasing along any future-directed causal curve. A *time slice* $S(w_0)$ is a level surface $w = w_0$ = constant of w. If this time slice has the additional property that any inextendible causal curve in N intersects $S(w_0)$ exactly once, then $S(w_0)$ is a *Cauchy slice* for N. A *Cauchy time function* w on N is a time function on N such that each time slice is a Cauchy slice for N. Clearly, the procedure of numerical relativity constructs a Cauchy time function on $N \subset D(S)$ such that the level surfaces are the time slices $S(w) = S_i$ mentioned above.

The spacetime N is the *future development* $d_w^+(S)$ of the initial slice S, i.e., a solution of the Einstein equations which contains S as a Cauchy slice. This development depends both on the S chosen and on the time function w chosen. In any case, $d_w^+(S) \subset D^+(S)$. As a result $d_w^+(S)$ is *globally hyperbolic* and *stably causal*. The time slices $S(w)$ must therefore all have the same topology. *Only globally hyperbolic spacetimes N can be constructed by numerical relativity.* Conversely, Geroch[8] has shown that any globally hyperbolic spacetime admits a Cauchy time function, so that *any globally hyperbolic spacetime can in principle be constructed by numerical relativity.*

The choice of gauge breaks into two parts. Changing β_i has no effect on w and therefore on $d_w^+(S)$. It merely threads the same time slices with different coordinate congruences. Therefore we ignore β_i in the rest of this discussion.[11] Changing the specification of the lapse function leads to a different time function w' and an inequivalent $d_{w'}(S)$. These various developments are isometric in the region of their overlap and are all subsets of the *maximal development*[12] $d_{\max}(S)$ which can be identified with $D(S)$.

For a given w and S there will be a future boundary of $d_w^+(S)$. This boundary may be composed of various components, e.g., null ($\mathscr{I}^{\pm}$) and spatial (I^0) or timelike ($I^{\pm}$) infinity, spacetime singularities such as curvature singularities, Cauchy horizons $H^{\pm}(S)$, or w limit surfaces. A *w limit surface* is the future boundary in $D(S)$ of the neighborhood on which w is a Cauchy time function. We shall encounter all of these cases in our study of Tolman-Bondi spacetimes in Sec. IV.

One might also obtain inequivalent developments by choosing different slices S in the same spacetime. However, Budic, Isenberg, Lindblom, and Yasskin[13] have shown that if S is either compact, or a partial Cauchy slice extending to spatial infinity in an asympotically flat spacetime, then $D(S)$ cannot be extended to a larger globally hyperbolic spacetime N. In these cases, therefore, numerical relativity can start from a fixed initial slice S, and attempt to construct the largest subset $D(S)$ of an unknown spacetime by varying w. It is not necessary to vary S.

B. Asymptotic structure

Let us turn to the study of the asymptotic properties of spacetimes generated by numerical relativity. The simplest case is that our partial Cauchy surface is compact without boundary. There is no "asymptotic region" and so boundary conditions never arise. If the surface is compact with boundary, then suitable boundary conditions must be specified. If the surface is noncompact, but not asymptotically flat, then little is known about boundary conditions. A number of cases occur in asymptotically flat spacetimes. A spacetime might admit just one asymptotically flat infinity, as in the gravitational collapse spacetimes of Sec. IV; it might admit two such infinities, as in the complete Schwarzschild-Kruskal-Szerkeres manifold,[10] or more generally it might admit n such infinities, connected by various spacelike wormholes.[14] A completely satisfactory definition of "an asymptotically flat infinity" has not yet been agreed upon; here we shall follow the definition of Geroch and Horowitz,[15] which guarantees that each infinity consists of two null infinities $\mathscr{I}^{\pm}$, each $S^2\times R$, with the null generators complete. We shall further assume that the point I^0 at spatial infinity[16] can be attached as a boundary point to the C^1 spacetime manifold by conformal equivalence, so that all null generators of future null infinity $\mathscr{I}^+$ (respectively, past null infinity $\mathscr{I}^-$) terminate to the past (respectively, future) at I^0. We shall say that a spacetime M has *n asymptotically infinities*, or simply is *asymptotically flat*, if it has n disjoint open neighborhoods N_i, with each N_i possessing an asymptotically flat infinity $(\mathrm{Inf})_i \equiv \mathscr{I}_i^+ \cup I_i^0 \cup \mathscr{I}_i^-$ in the above sense.

For each $\mathscr{I}_i^{\pm}$ one can define the *domain of outer communications*[10] $(\mathrm{doc})_i = \{x | x \in I^-(\mathscr{I}_i^+) \cap I^+(\mathscr{I}_i^-)\}$. Points in the manifold M not in any $(\mathrm{doc})_i$ will be said to be inside *holes*. The points are in *black holes* if they are in $\mathscr{B} = M - \cup_i [I^-(\mathscr{I}_i^+)]$ or *white holes* if they are in $\mathscr{W} = M - \cup_i [I^+(\mathscr{I}_i^-)]$. Now in many maximally extended spacetimes there are many sets of infinities $(\mathrm{Inf})_i$ (e.g., Reissner-Nordström or Kerr).[10] However, numerical relativity is concerned only with the evolution of data from some partial Cauchy slice S. That surface intersects a particular set of $(\mathrm{Inf})_i$'s which must be the only set $D(S)$ intersects, that is, all other sets of infinities are hidden behind the Cauchy horizon $H(S) = H^+(S) \cup H^-(S)$.

The existence of Cauchy horizons in a complete manifold M is intimately connected to the question of whether naked singularities arise from regular initial data set on $S \subset M$. We will use the definition of *naked singularity* proposed by Penrose,[17] a singular TIP contained in the past $I^-(q)$ of some point q in M or a singular TIF contained in the future $I^+(p)$ for some point p in M. Such a singularity violates *strong cosmic censorship*,[17] i.e., the hypothesis that M be globally hyperbolic. For this reason, such a singularity must by hypothesis occur behind a Cauchy horizon of S, since as remarked before, $D(S)$ is globally hyperbolic. Now a naked singularity may be *local* [see Fig. 1(b)], i.e., $H^+(S) \cap \mathscr{I}^+ = \phi$ or *global* [Fig. 1(c)] in which case $H^+(S) \cap \mathscr{I}^+ \neq \phi$ and not all of $\mathscr{I}^+$ can be predicted from S. In the latter case we can only assume M is *partially asymptotic predictable*.[18]

By definition, strong cosmic censorship (global hyperbolicity) obtains for every spacetime N generated by numerical relativity. However, that does not mean that one is unable to investigate the formation of naked singularities in numerical relativity. As seen in the examples in Sec. IV, one can use asymptotically null slices (defined below) to approach the null surface $H^+(S)$. If the naked singularity is global, one will find that regardless of how nearly null the slices are, $d_w^+(S)$ always becomes singular before all of $\mathscr{I}^+$ can be covered. This implies that $H^+(S) \cap \mathscr{I}^+$ is not empty and therefore that a global naked singularity exists to the future of S.

There is some evidence that in any generic (no Killing vector) solution to Einstein's equations, Cauchy horizons will not appear.[19] Calculations on some solutions which contain Cauchy horizons[20] (e.g., Reissner-Nordström, Kerr, some "whimper" cosmologies) indicate that these horizons will change drastically if perturbed. The general idea is that perturbations will "blue-shift[21]" near a Cauchy horizon and grow into curvature singularities. A strong version of this notion is that all spacetimes of physical interest will be globally hyperbolic, i.e., that the maximally analytically extended spacetime will obey strong cosmic censorship. If that turns out to be true then numerical relativity would be able, in principle, to provide all physically realistic solutions to the Einstein field equations.

Even if the singularity is achronal and inside a black hole (i.e., disjoint from the doc), one still has the problem that various time functions w lead to different future boundaries on $d_w^+(S)$. The time slices of w can hit the singularity at a point or small neighborhood in $S(w)$, they can uniformly wrap up around the singularity, or they can simply not probe an open neighborhood of the singularity ("avoidance"). By *singularity* we mean "future (or past) boundary of $D(S)$, not counting infinity" so that we may consider part of $H^{\pm}(S)$ as part of the singularity. This differs from the usual definition of geodetic incompleteness, but is the appropriate definition for numerical relativity, because γ_{ij} becomes singular or degenerate at all of the boundary of $D(S)$, including $H^{\pm}(S)$. The ideal time function would be one which covers both the doc and the inside of the black hole, i.e., one which spanned $d^+_{\max}(S) \equiv D(S)$. No satisfactory prescription for choosing such a time function has yet been made. One of the major purposes of this paper is to discuss the behavior of Cauchy time functions near such singularities (see Sec. II D below).

In the compact case, the required time function should define slices which are compact partial Cauchy slices. In the case where $D(S)$ is asymptotically flat, the partial Cauchy slices S are of two types. If for any set of neighborhoods N_i of $(\mathrm{Inf})_i$, $S - \cup N_i$ is compact, S is said to be an *asymptotically flat* partial Cauchy slice. This implies that S intersects any neighborhood of the asymptotically flat *spatial* infinities I^0_i of $D(S)$ but has no other infinities.

Alternatively, S is a future *asymptotically null partial Cauchy* slice of M if S lies in a globally hyperbolic, asymptotically flat neighborhood $N \subseteq M$; if for each $\mathscr{I}^+_i$ of $\bar{N}$ (the conformal completion of N) $\bar{S} \cap \mathscr{I}^+_i$ is a smooth cut (two-dimensional spacelike cross section), if $S \cup \{\cup_i [\mathscr{I}^+_i \cap \bar{I}^-(S, \bar{N})]\}$ is a Cauchy slice of N; and finally if for any set N_i of neighborhoods of $\mathscr{I}^+_i$, $S - \cup N_i$ is compact. Here $I^{\pm}(S,N)$ is the chronological $\binom{\text{future}}{\text{past}}$ of S in N. Therefore S itself is not a Cauchy slice of N, but S in union with some pieces of the $\mathscr{I}^+_i$ is. The pieces of $\mathscr{I}^+_i$ are necessary for the complete prediction (or actually retrodiction) of all of N because some gravitational radiation may have already crossed $\mathscr{I}^+_i$ to the past of S, and it is necessary to save data describing this radiation. The slice S does carry complete data for its own future development $D^+(S) = I^+(S, N)$.

Henceforth, we shall restrict attention to those asymptotically flat spacetimes which admit an asymptotically flat slice S near every $(\mathrm{Inf})_i$. This is so spacetime itself will have no other infinities.

Past asymptotically null slices could similarly be defined and used. In fact, up to now, numerical relativists have employed none of these slices and boundary conditions. Instead, the spatial mesh has been truncated at some large but finite radius (e.g., $r = 50M$), and an approximate outgoing radiation condition imposed there. The boundary in spacetime is therefore a timelike cylinder.[1] In principle this is an awkward boundary condition for Einstein's equations, but in practice is has seemed to work. One area of future work should be to justify this "engineering" approach in terms of the rigorous definitions given above.

C. Maximal and constant-mean-curvature slices

One class of time functions has received considerable attention, the maximal and constant-mean-curvature slicings. The investigation of their properties in various model spacetimes will occupy the remainder of this paper. The *convergence* K of a time function is

$$K = \nabla_\mu [\nabla^\mu w / (-g_{\lambda\nu} \nabla^\lambda w \nabla^\nu w)^{1/2}] , \tag{3}$$

which is equivalent to the mean extrinsic curvature of the level surfaces of w. For a given initial surface S, K has some dependence on spatial position in S, that is, some initial value $K(x^i)$. A useful way to consider the generation of a time function[2] is to give a prescription for $\partial_w K(x^i)$. Many aspects of the problem simplify if one requires K to be independent of spatial position on the slice, that is $K = K(w)$ only. If $K = 0$, and the slice is compact or asymptotically flat, then the slice is said to be a *maximal slice*. If $K(w) = K_0$ on $S(w)$, and the slice is compact or asymptotically null, then it is called a *slice of constant mean curvature*. A *maximal time function* w has level surfaces with $K(w) = 0$, whereas a *constant-mean-curvature time* function has level surfaces $K(w) = K_0(w)$ where K_0 is a constant on each level surface.

If one wishes to study a spacetime numerically using these time functions, a series of questions must be asked. First, can one find a *single* spacelike hypersurface S in the spacetime for which $K(S) = K_0$? This is the slice on which one wishes to pose initial data. Second, if so, does there exist a *family* of such slices and is the future boundary of $d^+_w(S)$ by a maximal or constant-curvature time function w nonsingular? This is the evolution of the initial data. Third, does $d^+_w(S) = d^+_{\max}(S)$? Fourth, if not, can $d^+_w(S)$ be extended by selecting a different w? That is, how much of the domain of dependence of S can be reached by w?

Some significant progress has been made on these problems in the last five years.[4,5] The kinds of questions which can now be answered rigorously are of the first type and partially of the second. That is, for spacetimes "sufficiently close" to a spacetime known to contain a $K = K_0$ or $K = 0$ slice, one can prove the existence of such a slice in the nearby spacetime. If the original spacetime has a family of such slices, then so does the nearby spacetime. However, these theorems have been proved only for the simplest cases, i.e., spacetimes with compact slices or those asymptotically flat spacetimes of the topology $R^3 \times R$. We review these theorems below.

Of principal interest to numerical relativity are those spacetimes which are "far away" from any known spacetimes. Usually such strong-field spacetimes develop singularities. Thus the theorems we need are *global* answers to the second question, i.e., can one construct a maximal or constant-mean-curvature time function which covers the *entire* manifold in a nonsingular fashion?

It would seem that this question is intimately related to the character of the singularities which form. This in turn is related to the matter content of the spacetime. We will make some conjectures below that if the spacetimes are *vacuum*, the slicings tend to be well behaved. This is because, as discussed in Sec. II D, we believe that vacuum globally hyperbolic spacetimes will have singularities of a type (we call crushing) which have nice properties with respect to $K=0$ or $K=K_0$ slicings.

The restriction to vacuum is important here; presumably it can be weakened to allow matter fields which obey well-behaved hyperbolic equations. We certainly need a stronger assumption about the matter fields than one of the usual energy conditions on the stress-energy tensor, because, just as for the cosmic censorship problem, there exist counterexamples involving pressureless perfect fluid matter ("dust"), which we shall discuss in Sec. IV.

We now review what is currently known about local existence of $K=0$ or $K=K_0$ slices in various classes of spacetimes. We organize these theorems and conjectures by the topology and asymptotic conditions satisfied by these spacetimes.

First, we summarize the results for the compact case. Simple examples are the fluid filled Robertson-Walker $k=+1$ cosmologies, which admit one maximal slice or $K=K_0$ slice for each value of K_0. The maximal slice corresponds to the moment of maximal expansion of the universe. On the other hand, there exist spatially compact cosmologies, for instance, the T^3-identified vacuum Kasner models, which have no maximal slice because they expand forever.[23]

The general uniqueness theorem is

Theorem 2.1 (Ref. 24): Suppose a spacetime (M,g) contains a compact C^∞ spacelike Cauchy hypersurface S. Let the timelike convergence condition $R_{ab}V^aV^b \geqslant 0$ for all timelike vectors V^a hold in (M,g). If S is a $K=K_0$ surface, then there is no other smooth compact spacelike hypersurface in M with $K=K_0$.

The above also holds for $K=0$ if (1) (M,g) is not identically flat and if (2) S is not time symmetric with $R_{ab}Z^aZ^b=0$, where Z^a is the unit normal to S. In case (2), the theorem holds if either $R_{ab}=0$ at each point of S or at some point of S, $Z^cZ^dZ_{[a}R_{b]cd[e}Z_{f]} \neq 0$. For cases not covered by (1) or (2) see Marsden and Tipler.[11]

The best existence theorem to date is

Theorem 2.2 (Ref. 25): If $\tilde{M}$ is a spacetime sufficiently "close" to a spacetime M which contains a compact $K=0$ or $K=K_0$ slice S, then unless S is totally geodesic, there exists a $K=0$ or $K=K_0$ slice, respectively, in $\tilde{M}$. In the special case S is totally geodesic (time symmetric and therefore with $K=0$), there exists a $K=K_0'$ surface with K_0' possibly different from 0.

Theorem 2.2 covers a limited number of spacetimes, i.e., those which are slightly deformed from some known analytic spacetime (e.g., Robertson-Walker $k=+1$ or T^3-identified Kasner) which contains a $K=0$ or $K=K_0$ slice. One would like to prove existence for a much more general intrinsically defined class of spacetimes, e.g., globally hyperbolic spacetimes obeying some energy condition. However, such a theorem is not true because we can find simple counterexamples in the spherically dust-filled Tolman-Bondi spacetimes (to be discussed in a later paper). Therefore, we limit our conjecture about the type of theorems which might be proved to include only vacuum spacetimes. This conjecture is essentially due to York.[26]

Conjecture 2.3: A vacuum spacetime M which is the maximal Cauchy development of a compact slice S admits a unique Cauchy slice of constant mean curvature K_0 for each value of K_0 in some range $K_{\min}<K_0<K_{\max}$; "usually" $K_{\min}=-\infty$ and $K_{\max}=+\infty$. These slices cover M. In particular there exists a Cauchy time function w, unique to trivial reparametrization, whose level surfaces $w=\text{const}$ are all these slices.

Turning from the compact to the noncompact case, we consider spacetimes of topology $R^3\times R$. These are of two classes. First, such a spacetime may not be asymptotically flat, e.g., $k=0$ Robertson-Walker. It is difficult to prove theorems here because the three-space is infinite with no well-posed asymptotic conditions.[28] In particular, there are no results on general existence of $K=0$ or $K=K_0$ slices for such nonasymptotically flat $R^3\times R$ spacetimes known to us.

We now consider the asymptotically flat case, which is much more difficult and diverse than the compact case because of the existence of boundary conditions. The fundamental difference between $K=0$ and $K=K_0\neq 0$ slices is their asymptotic properties: Maximal slices end on spatial infinity I^0 while $K=K_0>0$ end on past null infinity $\mathscr{I}^-$ and $K=K_0<0$ end on future null infinity $\mathscr{I}^+$. These latter asymptotically null slices are a special case of the $S(\tau)$ used by Hawking and Ellis[7] (p. 313) in their discussion of black holes.

For maximal slices one has a complete knowledge of existence and uniqueness in Minkowski spacetime:

Theorem 2.4 (Ref. 29): In Minkowski spacetime

M the spacelike, asymptotically flat, $K=0$ hypersurfaces are the 4-parameter family of time planes. Three of the parameters are the components of the boost velocity and the other is the time translation parameter. For a given choice of boost, the one parameter family of maximal slices covers all of M.

This theorem can be extended to discuss the existence of maximal slices in asymptotically flat spacetimes of topology $R^3\times R$, i.e., spacetime deformable to a portion of Minkowski spacetime.

Theorem 2.5 (Ref. 30): For spacetime sufficiently "close[23]" to Minkowski spacetime, there exists a four-parameter family of maximal slices which cover the spacetime.

Note that vacuum spacetimes of topology $R^3\times R$ can be very far from Minkowski spacetime, e.g., strong gravitational waves[31] which focus themselves into a black hole and singularity. The above theorem does not state that such a spacetime can be completely covered by regular maximal or $K=K_0$ slices.

Even more difficult is the case where wormholes are present. The topology of the time slices is then no longer R^3 and there are several infinities present. For the static, spherically symmetric and analytic black hole spacetimes one can calculate a two-parameter family of spherically symmetric maximal slices.[32] In these spacetimes, unlike the R^4 case above, singularities to the future of a Cauchy slice are inevitable.[33] Thus black holes or naked singularities must occur. This means that the existence of *global* maximal time functions, which cover the doc, is the crucial question. For the vacuum case of several throats in the time slice, we conjecture the following:

Conjecture 2.7: A vacuum globally hyperbolic spacetime M with n different asymptotically flat infinities, which is a maximal Cauchy development, admits exactly a $4n$-parameter family of maximal Cauchy slices. Specifically, at each spatial infinity I^0_i, four parameters analogous to the one time displacement and three boosts in Theorem 2.4 can be freely given as boundary conditions for the slice. In particular there exist (many) maximal time functions which are Cauchy time functions for the domain of outer communications.

The existence of $K=K_0\neq 0$ slices in asymptotically flat spacetimes is currently in doubt.[22] Goddard[5] conjectures that the four-parameter family of hyperboloids of radius $-3/K_0$ are the only $K=K_0$ slices of Minkowski spacetimes. From this he conjectures that in a general spacetime a $K=K_0$ slice must interest $\mathscr{I}^\pm$ in some special kind of cut, e.g., in a good cut (a *good cut*[34] is a cross section of $\mathscr{I}^\pm$, such that the shear of the ingoing normal null geodesics vanishes through order r^{-2}).[27]

Conjecture 2.8: In a vacuum globally hyperbolic spacetime with n different asymptotically flat infinities, which is a maximal Cauchy development, a sufficient condition for the existence of a $K=K_0$ asymptotically null slice (for given K_0) is the existence of good cuts. In particular, for $K_0<0$ (respectively, >0) any good cut on each $\mathscr{I}^+_i$ (respectively, $\mathscr{I}^-_i$) can be freely given as the boundary of the slice.

Since a two-parameter family of good cuts exists[34] for axially and reflection symmetric spacetime (e.g., collapse of an axially symmetric nonrotating star, head on collision of two nonrotating black holes), $K=K_0$ slicing probably can be used in such a spacetime.

This finishes our review of known theorems and possible extensions to vacuum spacetimes. One would like to develop more powerful techniques which allow one to prove existence for even wider classes of spacetimes.

There are so far two main approaches to solving these global problems. One stated by Choquet-Bruhat, Fischer, and Marsden is to try to extend the notion of spacetimes being "near" a fidicual spacetime to the notion of spacetimes being "connected by a curve of spacetimes" to a fidicual spacetime. However, we have examples of spacetimes which physically seen infinitesimally near each other, in which one spacetime has a $K=K_0$ slice and the other does not. Therefore, even if such a globalization could be proved, the "curve of spacetime" might be very restricted in its physical content.

A different approach is to characterize intrinsically a class of spacetimes, say by giving their allowed singularity structure. Then one attempts to prove existence of $K=0$ or $K=K_0$ global Cauchy time functions in such spacetimes. If such a theorem can be proved, then the physical content is answered by studying how broad is this class of singularities. Recently, Tipler and Marsden[4] have made significant progress on this question by showing that under certain restrictions maximal slices avoid a class of singularities called "strong curvature singularities," so that existence of a maximal slice can still be proven provided all singularities are of this class. For the necessary restrictions see Theorem 3C of Marsden and Tipler.[4]

We have adopted a different approach to characterizing singularities, which is the subject of the next section.

D. Crushing singularities and avoidance theorems

To discuss the behavior of $K=0$ or $K=K_0$ time functions in general globally hyperbolic manifolds M, one must devise a characterization of the future boundary[35] of M which is adapted to questions of time slicing. The definition should include both Cauchy horizons and nontimelike singularities. In model spacetimes we noticed that such boundaries are uniformly approached by the "natural time functions" in those spacetimes, e.g., τ = constant in Friedmann (singularity) and r = constant in Schwarzschild (singularity) or Reissner-Nordström (Cauchy horizon). On such slices, $K \to \infty$ uniformly as the boundary is approached (see Sec. III below). Therefore, we have selected out this feature as essential and defined its generalization: the *crushing singularity*.

Definition 2.9 A *future crushing function* f on a globally hyperbolic neighborhood N is a Cauchy time function on N with some range $c<f<0$ ($c<0$ is a constant), such that the convergence K of f obeys $\lim K = \infty$ as $f \to 0^-$, uniformly. Similarly a *past crushing function* h on N is a Cauchy time function on N with some range $0<h<d$ ($d>0$ is a constant), such that the convergence K of h obeys $\lim K = -\infty$ as $h \to 0^+$, uniformly.

From now on we shall generally assume that a neighborhood N, as well as a spacetime M, is globally hyperbolic. Furthermore, we shall just treat the future case, and leave the past case implicit as an obvious dual.

Definition 2.10: We shall say that a *spatially compact* spacetime M has a *future crushing singularity* if there is a neighborhood N in M, such that N contains a Cauchy slice of M, and such that N admits a future crushing function.

Definition 2.11: We shall say that an *asymptotically flat* spacetime M has a *future crushing singularity* if the interior int$\mathcal{B}$ of all black holes, contains a neighborhood N such that N contains a Cauchy slice of int$\mathcal{B}$, and such that N admits a future crushing function.

That there are incomplete timelike curves in M at a crushing singularity follows immediately from

Theorem 2.12 (Hawking and Ellis,[7] p. 274): If a spacetime M

(1) obeys the strong energy condition;

(2) has a Cauchy slice S;

(3) has $K>k>0$ everywhere on S, where K is the convergence of S and k is a constant,

then no future-directed timelike curve from S has proper length greater than $3/k$.

Proposition 2.13: If a globally hyperbolic and either spatially compact or asymptotically flat spacetime M obeys the strong energy condition and has a future crushing singularity defined on a neighborhood N, then no point $p \in M - N$ lies in the chronological future $I^+(N)$ of N.

Proof, Compact case: Assume p exists; there is a timelike curve C leaving N and passing through p. However, by Definition 2.9, C will cross Cauchy slices in N with $K>k$ for arbitrarily large constant k, so the proper length of C to the future of N must be less any constant $3/k$ greater than 0 by Theorem 2.12, which is a contradiction. Asymptotically flat case; replace M by int$\mathcal{B}$ and proceed the same way. Q.E.D.

Therefore, without loss of generality, we can take $N=M$ in Definition 2.10 and N = int$\mathcal{B}$ in Definition 2.11. For any smaller N, we can extend f to the past easily, and we need not extend it to the future at all. Any neighborhood N appearing in these definitions could be considered a neighborhood of the singularity. Crushing functions are not unique; if a spacetime M admits one crushing function it will admit many. This makes it easier to guess a crushing function for any given M, and therefore to establish that M has a crushing singularity; see examples below. We hope that a crushing function is in some sense "asymptotically unique" so that if we were to build a singularity structure on the future causal boundary of M, this structure would be independent of choice of crushing function.

There is an important issue that we will not address in this paper. One would like to view a "singularity" as a point set which is constructed as a topological boundary on spacetime. Can a crushing singularity be viewed in this way? And if so are its properties independent of the choice of neighborhood, Cauchy slice and crushing function? The answers seem to be "yes" to these questions; we shall return to this issue in a future paper.

Proposition 2.13 can be reworded in a more interesting form:

Corollary 2.14: A spacetime M that obeys the strong-energy condition and has a future crushing singularity is a maximal Cauchy development into the future of any one of its Cauchy slices S, i.e., there is no larger spacetime $M' \supset M$ with $I^+(M,M')$ nonempty, and with S a Cauchy slice of M'.

The usual proof[12] of the existence of a maximal Cauchy development assumes Zorn's Lemma, equivalent to the axiom of choice, to which some pragmatic relativists object.[36] Corollary 2.14 provides a constructive sufficient criterion for, at least, deciding if a given spacetime is a maximal Cauchy development.

Conjecture 2.15: A globally hyperbolic, spatially compact or asymptotically flat, vacuum spacetime that is a maximal Cauchy development and has a singularity has a crushing singularity.

Corollary 2.14 is likely to be useful in the practice of numerical relativity. If one succeeds in constructing a spacetime with a demonstrable future crushing singularity, for example by the use of space slices of constant mean curvature K and by the demonstration that the computer can evolve the spacetime to as large a coordinate time K_0 as one likes, so that one's time coordinate is itself a crushing function, then one has achieved the goal of constructing the maximal Cauchy development of the initial data.

A globally hyperbolic spacetime M with a future crushing singularity has future-incomplete timelike curves which terminate at the future causal boundary. However, M still might be smoothly extendible across this boundary into a larger spacetime M', such that the initial Cauchy slice S of M fails to be a Cauchy slice of M'. So a future crushing singularity implies *either* a true spacetime singularity, *or* a Cauchy horizon; both cases will be illustrated below. In particular, physical objects may or may not be actually crushed at the boundary; it is our coordinate system based on a Cauchy time function that is hypothesized to be crushed. In this sense our definitions are a moderate abuse of the usual terminology. For instance, if one were to try to extend the definition to the spacetime $D(S)$ where S is the unit hyperboloid contained in the future light cone of the origin of Minkowski spacetime (which is neither spatially compact nor asymptotically flat) one would conclude that this subset of Minkowski spacetime has a past crushing singularity, even though the spacetime is perfectly well behaved. However, a crushing singularity is always a singularity in numerical relativity, because the three metric of the time slice is always singular as the crushing function approaches 0.

Now we shall show that the avoidance of crushing singularities by maximal slices, and slices of constant mean curvature, follows trivially from the definitions. The particular form of Definitions 2.10 and 2.11 that we need is:

Corollary 2.16: Given N and f as in Definition 2.10 or 2.11 (future case); given any constant $K_0>0$. There exists a unique neighborhood $N(K_0) \subset N$, of the form $\{p \in N \mid f(p)>b\}$ for some constant b, $c<b<0$, such that $K>K_0$ on $N(K_0)$, and finally such that $N(K_0)$ is the largest neighborhood with these properties.

Since f restricted to $N(K_0)$ is still a crushing function, $N(K_0)$ itself suffices to define a future crushing singularity for M.

Theorem 2.17 (Choquet-Bruhat[37]): Let a spatially compact spacetime M have a future crushing singularity. Then no Cauchy slice of constant-mean-curvature K_0 on M intersects the neighborhood $N(K_0)$ of Corollary 2.16.

Theorem 2.18: Let an asymptotically flat spacetime M have a future crushing singularity. Then no partial Cauchy maximal $(K=0)$ or constant-mean-curvature $(K=K_0)$ slice in M intersects the neighborhood $N(0)$ or $N(K_0)$, respectively, of Corollary 2.16.

Proof: Let S be a Cauchy slice of M. Then $S \cap B$ is contained in a compact set in S, with boundary. The boundary of $S \cap B$ lies to the past of any Cauchy slice of int$B$, hence it can be neglected in the argument of Choquet-Bruhat. Q.E.D.

Therefore, a future crushing singularity has a neighborhood $N(K_0)$ which all Cauchy slices of constant-mean-curvature K_0 uniformly avoid. We expect that most of all of the present existence theorems which depend on the absence of singularities, or on the hypothesis that singularities are of Robertson-Walker type[5] or are strong-curvature singularities,[4,5] can also be pushed through under the hypothesis that spacetime has a crushing singularity.

For instance,

Conjecture 2.19 (J. A. Wheeler[38]): Let M be a spatially compact, globally hyperbolic, spacetime having both a future and past crushing singularity. Then there exists a Cauchy constant-mean-curvature time function w on M such that its time slices cover M and wrap up around the singularities uniformly.

This is a different version of Conjecture 2.3. Here we allow any matter or vacuum content. In Conjecture 2.3 we expect vacuum to lead to spacetimes with crushing functions. Here we restrict ourselves to cosmologies that have them. J. E. Marsden and F. J. Tipler (private communication) have proved a theorem similar to Conjecture 2.19 using their methods (cf. Ref. 4).

III. STATIONARY BLACK HOLES

The avoidance behavior referred to in Theorem 2.18 was first noted for spherical maximal slices of the Schwarzschild black hole by Estabrook *et al.*[3] It has since been observed in a variety of other contexts. In this section, we review these particular results and use our avoidance theorem to extend these to general results.

Example 3.1: Complete Schwarzschild metric. The whole two-parameter family of *spherically symmetric* maximal Cauchy slices has been constructed and studied by[4] Estabrook *et al.*, Reinhart, and Brill. The main result is that no such slice intersects the region $r>3M/2$ inside the black hole $\mathcal{B}$ (here r is the Schwarzschild radial coordinate).

Proposition 3.2: The complete Schwarzschild metric has a future crushing singularity.

Proof: The radial coordinate r is a Cauchy time function for the region $(r<2M)$, i.e., for the interior of the black hole int$\mathcal{B}$. The convergence of this time function is

$$K = (3M - 2r)/[r^2(2M/r - 1)^{1/2}] . \qquad (4)$$

Therefore r is a future crushing function on int$\mathcal{B}$. Q.E.D.

Corollary 3.3: No maximal Cauchy slice of the Schwarzschild metric intersects the region $r < 3M/2$.

Proof: $K>0$ on $N(0) = (r<3M/2)$; appeal to Theorem 2.18. Q.E.D.

Corollary 3.3 was already known to hold true, by explicit constructions, for the special case of spherically symmetric slices.[39] This extends the result to the whole eight-parameter family of slices postulated in Conjecture 2.7; most of these slices are "boosted" and hence not spherically symmetric. It is not known if these slices actually exist.[40] Thus for maximal slicing there will exist a w limit surface which forms all of the future boundary of $d_w^+(S)$ for any S in the slicing. This result also holds for constant mean curvature slicings.

Corollary 3.4: No $K=K_0$ partial Cauchy slice of the Schwarzschild metric intersects the region $r< r_{\text{lim}}$ where $0< r_{\text{lim}}<2M$ is the solution of equation (4) with $K=K_0$.

Proof: Same as for Corollary 3.3.

Notice that the limit surface is closer to (further from) the singularity than the limit surface for maximal slices if $K_0>0$ ($K_0<0$). The other main point to remember is that the $K=K_0$ slices end up on $\mathcal{I}^+$ ($\mathcal{I}^-$) depending on whether $K_0<0$ ($K_0>0$).

Example 3.5: Maximal asymptotically flat Cauchy development in the Reissner-Nordström metric, for $e<M$. This is the region outside of the inner or Cauchy horizon, $r_- < r < \infty$, where $r_\pm = M \pm (M^2 - e^2)^{1/2}$, in the usual coordinates. The spherically symmetric maximal slices here have been given by Duncan.[4] We find that $f=(r-r_-)$ is a crushing function for the whole region $r_- < r < r_+$ inside the black hole int$\mathcal{B}$.

Proposition 3.6: The Reissner-Nordström metric has a future crushing singularity, for $e<M$.

This is a case where the "singularity" is actually a Cauchy horizon in our definition.

Corollary 3.7: No maximal Cauchy slice of the Reissner-Nordström metric intersects the region

$$r< r_{\text{lim}} = 3M/4 + (9M^2/16 - e^2/2)^{1/2} . \qquad (5)$$

The spherically symmetric special case of this was shown by Duncan.[4] The external case $e=M$ seems special here.

Corollary 3.8: No $K=K_0$ partial Cauchy slice of the Reissner-Nordström metric intersects the region $r< r_{\text{lim}}$ where $r_- < r_{\text{lim}} < r_+$ is given by the solution to the equation

$$K_0 = (3M - 2r - e^2/r)/[r^2(2M/r - 1 - e^2/r^2)^{1/2}] . \qquad (6)$$

Example 3.9: Maximal asymptotically flat Cauchy development in the Kerr metric, for $a<M$. This is the region outside of the inner or Cauchy horizon, $r_- < r < \infty$, where $r_\pm = M \pm (M^2 - a^2)^{1/2}$, in the usual coordinates. We find that $f=(r-r_-)$ is a future crushing function for the whole region $r_- < r < r_+$ inside the black hole int$\mathcal{B}$, so that:

Proposition 3.10: The Kerr metric has a future crushing singularity for $a<M$.

Here again the "singularity" is actually a Cauchy horizon.

Corollary 3.11: No maximal Cauchy slice of the Kerr metric intersects the region

$$r< r_{\text{lim}} = \tfrac{1}{2}\left(M + \{M^3 + [M^6 + (\tfrac{4}{3}a^2 - M^2)^3]^{1/2}\}^{1/3} + \{M^3 - [M^6 + (\tfrac{4}{3}a^2 - M^2)^3]^{1/2}\}^{1/3}\right); \qquad (7)$$

this follows by a straightforward calculation of K for the Cauchy time function f.

The situation is different here in that no slice $r=\text{const}$ inside a Kerr black hole is a maximal slice,[41] in particular the limit surface $r=r_{\text{lim}}$ of Corollary 3.11 is not itself a maximal slice. Corollary 3.11 is a conservative estimate, and the true limit surface for maximal Cauchy slices must lie somewhat to the past of the slice $r=r_{\text{lim}}$. We conjecture that this limit slice is a maximal Cauchy slice for int $\mathcal{B}$.

Corollary 3.12: No $K=K_0$ partial Cauchy slice of the Kerr metric intersects the region $r< r_{\text{lim}}$, where $r_- < r_{\text{lim}} < r_+$ is the solution of the equation

$$K_0 = \min_\theta K(r_{\text{lim}}), \qquad (8)$$

where the minimum over θ is taken since $K=K(r,\theta)$ in Kerr.

We see from the above examples that the interior of a black hole, which is noncompact, is in many ways like a spatially compact solution. Rather than families of maximal slices there seems to be only one. In fact, the interior $\mathcal{B}$ of Schwarzschild is isometric to a vacuum Kantowski-Sachs-Thorne cylindrical homogeneous cosmology.[42] It thus seems reasonable that an analog to Conjecture 2.3 also holds for black hole interiors, stationary or nonstationary:

Conjecture 3.13: Any vacuum black hole $\mathcal{B}$ contains a unique maximal Cauchy slice for int$\mathcal{B}$. More generally, it contains one Cauchy slice for int$\mathcal{B}$ of constant curvature K_0 for each value of K_0.

We have been able to prove this only for the case of the Schwarzschild black hole.[43] Additional support for the maximal slice case comes from numerical evolutions of dynamic nonspherical black holes, such as the collisions of black holes[1] or nonspherical star collapse.[44] In these cases, the maximal slices seem to "wrap up" around a unique limit maximal slice inside the final black hole.

IV. TOLMAN-BONDI SPACETIMES

A. Review of the Tolman-Bondi (TB) metric

To study the problems discussed in Sec. II on selecting Cauchy time functions in numerical relativity, we turn now to the main examples of this paper, the Tolman-Bondi[6] (TB) spacetimes. Our emphasis in this paper will be on the marginally bound dust cloud universes. In later papers we will treat bound collapse and closed universes. The great advantage of discussing the TB spacetimes is that one knows the explicit form of the entire spacetime metric. Thus one can use it as a laboratory to test maximal and constant-mean-curvature time functions in the presence of spacelike singularities, Cauchy horizons, and naked singularities. From our analysis in Sec. II, it seems likely that the spherical restriction will not unduly restrict our results. The experience of numerical relativity in the nonspherical spacetimes containing colliding black holes and stellar collapse bears out this judgement.

Many well-known solutions of the Einstein field equations are spherically symmetric with either dust or vacuum matter content, e.g., the Schwarzschild black hole, the homogeneous Friedmann universes, the Oppenheimer-Snyder star collapse, inhomogeneous universes with delayed cores or black holes, etc. These can all be grouped under one family of exact solutions, the Tolman-Bondi spacetimes, with metric

$$ds^2 = -dt^2 + X^2(r,t)\,dr^2 + Y^2(r,t)\,d\Omega^2, \qquad (9)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the two-sphere metric. The Einstein equations become[6]

$$X(r,t) = Y'(r,t)/W(r), \qquad (10)$$

$$\dot{Y}^2(r,t) = W^2(r) - 1 + 2\left[\int_0^r W(\tilde{r})M'(\tilde{r})\,d\tilde{r}\right]\Big/ Y(r,t), \qquad (11)$$

$$M'(r) = 4\pi\rho(r,t)X(r,t)Y^2(r,t), \qquad (12)$$

where the dot signifies $\partial/\partial t$ and the prime signifies $\partial/\partial r$. Let us examine the invariant meaning of this coordinate representation.

The time slicing (t = constant) has been chosen by the demand that the normals to the slices lie along the trajectories of freely falling test particles. That is, r = constant labels a matter shell with t measuring the proper time elapsed along its geodesic path. The total proper mass of the matter within this shell $M(r)$ is independent of time since the r coordinate is comoving with the matter. However, the proper area of the shell, $4\pi Y^2(r,t)$, is time dependent. Thus the mass shells r = constant are moving with respect to the trajectories of constant areal radius $Y(r,t)$ = constant. At any time, the three velocity v^r relating these two trajectories is just $\dot{Y}$. The relative binding energy of the shell r, in the field of the mass $M(r)$ within it, is given by the function $W(r)$. If $W(r_0)=1$ the shell $r=r_0$ is marginally bound; if $W(r_0)$ is greater (less) than one, the shell is unbound (bound). As Bondi[6] describes, Eq. (11) is thus the general relativistic generalization of the Newtonian energy equation. Finally, to avoid having to treat "shell-crossing singularities," in which matter shells cross each other, we will always require

$$X(r,t) > 0. \qquad (13)$$

This together with an assumption of positive mass density, $\rho(x,t)>0$, implies that mass increases with coordinate r:

$$M'(r) \geq 0. \qquad (14)$$

For this paper, we shall restrict ourselves to the marginally bound case

$$W(r) = 1, \quad 0 \leq r < \infty. \qquad (15)$$

Equations (10) and (11) then immediately integrate[6] to give

$$Y(r,t) = \left\{\frac{9M(r)}{2}[t_0(r) - t]^2\right\}^{1/3}, \qquad (16)$$

$$X(r,t) = \{M'(r)[t_0(r) - t] + 2M(r)\,t_0'(r)\} \times \{6M^2(r)[t_0(r) - t]\}^{-1/3}, \qquad (17)$$

where $t_0(r)$ is an integration function. Since the area of the matter shell at r = constant goes to zero when $Y(r,t)=0$, we see that $t=t_0(r)$ is the proper time when the matter shell hits the physical singularity. The range of coordinates is thus

$$0 \leq r < \infty, \quad -\infty < t < t_0(r). \qquad (18)$$

In order to prevent shell crossing, Eq. (5) must hold, restricting $t_0(r)$ by

$$t_0'(r) \geq 0. \qquad (19)$$

Having specified $W(r)$ by Eq. (15), it appears that we still have two freely specified functions $t_0(r)$ and $M(r)$ of r. However, since r only serves to label spherical shells, we still have the coordinate freedom to relabel by any function of r, so that this freedom can be used to fix either t_0 or M. Finally, we demand $M(r) \to$ constant as $r\to\infty$ so that spacetime is asymptotically flat.

The two simplest choices lead to well-known spacetimes. Suppose we choose $t_0'(r) = t_0(r) = 0$. Then we use our coordinate gauge to set $M(r) = r^3$. Evaluating (16) and (17) yields the line element

$$ds^2 = -dt^2 + (9t^2/2)^{2/3}(dr^2 + r^2 d\Omega^2), \qquad (20)$$

which is just the marginally bound ($k=0$) Friedmann solution. On the other hand, setting $M'(r)=0$ and choosing $t_0(r)=r$,

$$ds^2=-dt^2+\left[\frac{4M}{3}(r-t)^{-1}\right]^{2/3}dr^2+\left[\frac{9M}{2}(r-t)^2\right]^{2/3}d\Omega^2, \tag{21}$$

which is just the Eddington-Finkelstein patch of extended Schwarzschild spacetime, written in Lemaitre coordinates.[45] If we sew these two spacetimes together along $r=1$, we obtain the Oppenheimer-Snyder[46] solution of a homogeneous collapsing marginally bound dust cloud.

B. Causal structure

Two of the most important hypersurfaces in collapse spacetimes are the apparent horizon and the event horizon.[47] The apparent horizon (ah) was defined by Hawking to be the outer boundary of the region of trapped surfaces in a given slice. For our purposes in this paper, we shall generalize the definition of the apparent horizon to the boundary of the region of trapped two spheres in spacetime. To find the boundary of the region of trapped two spheres, we search for two spheres Y = constant whose outward normals are null,

$$\nabla Y\cdot\nabla Y=-\dot{Y}^2+X^{-2}Y'^2=0. \tag{22}$$

Inserting Eqs. (10) and (11) for the case $W(r)=1$, leads to the condition

$$Y(r,t)=2M(r), \tag{23}$$

a simple generalization of our experience in Schwarzschild spacetime. Using Eq. (16) this can be written as

$$t_{\rm ah}=t_0(r)-\tfrac{4}{3}M(r). \tag{24}$$

So the apparent horizon precedes the singularity $t=t_0(r)$ by an amount of comoving time $4M(r)/3$. The induced metric on the apparent horizon is [using Eqs. (16), (17), and (24)]

$$ds^2=4M'(r)\,[t_0'(r)-\tfrac{1}{3}M'(r)]dr^2+4M^2(r)d\Omega^2. \tag{25}$$

In view of the inequalities (14) and (19), we see that portions of the apparent horizon can be past-pointing timelike, past-pointing null, or spacelike if

$$M'(r)>0,\quad \begin{cases} 0\le t_0'<\tfrac{1}{3}M' & \text{(past timelike)}, \quad (26a)\\ 0<t_0'=\tfrac{1}{3}M' & \text{(past null)}, \quad (26b)\\ 0<\tfrac{1}{3}M'<t_0' & \text{(spacelike)}. \quad (26c)\end{cases}$$

In the case of vacuum, $M'(r)=0$, the apparent horizon is future-pointing null and coincides with the null event horizon of the Schwarzschild spacetime. In our Friedmann example above (26a) obtains and the apparent horizon is timelike for all r. In the Oppenheimer-Snyder spacetime the abrupt change from past-directed timelike to future-directed null occurs because the density ρ does not fall off smoothly to zero as the surface of the star is approached.

The event horizon, in contrast, is always future-pointing null. A radial future-pointing null geodesic in the general line element (9) satisfies

$$\frac{dt}{dr}=X(r,t). \tag{27}$$

Even for $W(r)=1$ this equation cannot be solved in closed form except for very special cases such as our three examples. It can be numerically integrated in general as described below. Before giving details we shall summarize the causal structure of the marginally bound, asymptotically flat TB spacetimes in the form of Penrose-Carter[10] diagrams. There exist past and future null infinities of the usual sort described in Sec. II. The curvature singularity at $t=t_0(r)$ is spacelike everywhere except at the origin. A delicate situation occurs in general at the origin on the singularity, $r=0$. If $\lim t_0/M=0$ as $r\to 0^+$, then the singularity is spacelike at $r=0$, and the causal structure is of the familiar form, Fig. 1(a). The Oppenheimer-Snyder model belongs to this case. Here the "singularity grows faster than the speed of light" after the initial collapse of the dust at the origin, and no light rays can escape the growing singularity.

On the other hand, if $\lim t_0/M=\infty$ as $r\to 0^+$, then, surprisingly, there is a piece of past-null singularity at the origin, and the causal structure will be as shown in either Fig. 1(b) or 1(c). Note that in either of these cases a Cauchy horizon exists for any regular partial Cauchy surface. The choice between Figs. 1(b) and 1(c) depends on the details of t_0 and M for r well away from 0 in a complicated way; specific examples will be given below. In this case the "singularity grows slower than the speed of light" after the initial collapse, and light rays can escape its immediate vicinity. For Fig. 1(b), all these escaping rays are eventually trapped within the final black hole, so that there is no naked singularity visible from infinity, but we do have a local breakdown of predictability inside the black hole, i.e., a locally naked singularity as discussed in Sec. II. For Fig. 1(c), the black hole grows more slowly, and some of the escaping rays reach future null infinity, so that a global naked singularity occurs. We shall call either of these a *shell-focusing singularity*. In the critical case $\lim t_0/M$ = finite const, all these three cases occur, as will be discussed below. We can find C^∞

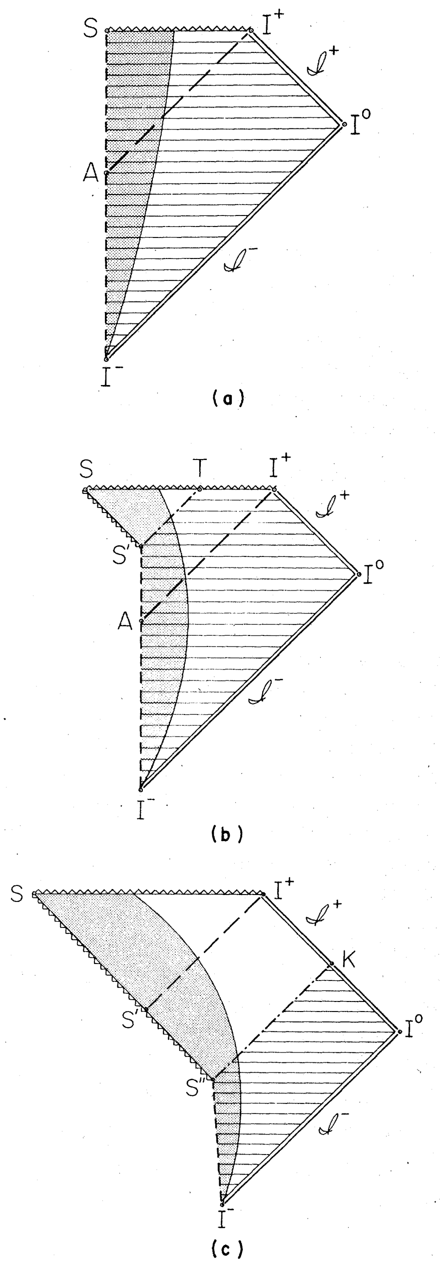

FIG. 1. Penrose-Carter diagrams of the three possible causal structures for inhomogeneous marginally bound dust-sphere collapse. The line with short dashes to the left is the center of the star. The shaded region is where $T_{\mu\nu} \neq 0$. The infinities are labeled by I^+ (future timelike), $\mathscr{I}^+$ (future null), I^0 (spacelike), $\mathscr{I}^-$ (past null), and I^- (past timelike). The jagged line is the singularity. Note that points along lines $SS'S''$ all map into the single coordinate point $(t, M) = (0, 0)$. The horizontally striped region is the globally hyperbolic region accessible to numerical relativity. The three types of causal structure are (a) generalization of Oppenheimer-Snyder collapse, (b) locally naked singularity, (c) globally naked singularity. The line with long dashes is the absolute event horizon. For causal structures (b) and (c), a Cauchy horizon exists (dot-dash line).

spacetimes which exhibit all three kinds of causal structure.

It is not very surprising that there exist inhomogeneous gravitational collapse spacetimes with a breakdown of global hyperbolicity near the singularity, or even with a global naked singularity. It is a bit more surprising that these phenomena occur in the family of Tolman-Bondi spacetimes, which are thought to be well understood. Two general ideas have been discussed to avoid naked singularities: (1) to forbid matter fields, such as dust, that can form density singularities even in a flat background spacetime[48]; (2) to allow such density singularities and describe them as distributions ("δ-function singularities") in the spacetime curvature.[49] Our Tolman-Bondi examples with "shell-focusing singularities" demonstrate that idea (2) does not work in general; for a Tolman-Bondi spacetime with the causal structure of Fig. 1(c), there is a density singularity along the past-null singularity which is partially visible from future null infinity, but it seems that the spacetime cannot be extended to a larger distributional spacetime, or any other spacetime that is stably causal, in which the past-incomplete null rays from $\mathscr{I}^+$ end up at $\mathscr{I}^-$. The familiar "shell-crossing" singularities[50] can be handled by either idea (1) or (2). Of course, idea (1) gets rid of our "shell-focusing singularities" as well.

To be precise, the nonextendibility property of our spacetime is as follows: No spherically symmetric spacetime M with the causal structure of Fig. 1(c) can be extended to a larger spacetime M' such that (1) $M = M'$ in some neighborhood of infinity $\mathscr{I}^+ \cup I^0 \cup \mathscr{I}^-$; (2) in M' each past-directed null geodesic from $\mathscr{I}^+$ has a unique extension until it reaches $\mathscr{I}^-$; (3) M' is stably causal; (4) M' is spherically symmetric. For assume M' exists. The family F^+ of past-incomplete radial null geodesics g^+ from $\mathscr{I}^+$ in M is a three-parameter family (indexed by a retarded time and two angles; we identify geodesics which differ only in parameter along the curve). Each g^+ in F^+ must extend to a complete radial null geodesic in M', and therefore must eventually link up with a member of the family F^- of future-incomplete radial null geodesics g^- from $\mathscr{I}^-$ in M. But no such g^- can be incomplete at the future singularity [line SI^+ in Fig. 1(c)] because the extension of g^+ to such a g^- would destroy stable causality (as can be seen by introducing a Hawking time function). Then the only available g^- are the two-parameter subfamily of F^-, which are incomplete at S''; but this subfamily can never cover the three-parameter family F^+. This contradiction means M' with properties (1)–(4) cannot exist. We believe that hypothesis (4) of spherical symmetry could be eliminated by an argument which uses detailed information about the null geodesics of Tolman-Bondi spacetimes, but we have not completed such an argument.

Our examples show only that shell-focusing singularities occur in spherical dust collapse. We believe it possible that they also occur for perfect fluids with pressure, e.g., equation of state $p=\rho/3$, in the case of imploding spherical shocks, but this remains uninvestigated. Deviation from spherical symmetry, or deviation from perfect-fluid behavior, seem likely to eliminate these singularities. Therefore we do not believe they will occur in nature. Their main utility is as counterexamples to definition or possible theorems about cosmic censorship.

C. Maximal and constant-mean-curvature time functions

We have described the Tolman-Bondi spacetimes in geodesic, or Gaussian-normal, coordinates. In general such a coordinate system breaks down because of focusing of the time lines,[7] according to the Landau-Raychaudhuri equation. In our case we have thrown away all those solutions, "shell-crossing singularities,[50]" in which breakdown of the coordinates occurs before the spacetime singularity. Even so, t is not a Cauchy time function because the slices $t=\text{const}$ intersect the singularity for $t\geq t_0(0)$. We seek to change w and S so that $d_w(S)$ will remain nonsingular while including the domain of outer communications of any black hole formed. In particular we have devised a method to generate maximal and constant mean curvature time functions for any TB spacetime.

For any spherically symmetric metric,

$$ds^2=g_{AB}(X^C)dX^A dX^B+Y^2(X^C)d\Omega^2 \quad (A,B,C=1,2)\,, \tag{28}$$

the spherically symmetric maximal slices can be obtained from the variational principle[51]

$$0=\delta\int\left(Y^4\frac{dX^A}{d\lambda}\frac{dX^B}{d\lambda}g_{AB}\right)^{1/2}d\lambda\,. \tag{29}$$

In fact this is identical to the variational principle for timelike geodesics of the unphysical two-dimensional Lorentzian metric

$$dv^2=-Y^4 g_{AB}\,dX^A dX^B\,. \tag{30}$$

Now specialize to the marginally bound, asymptotically flat Tolman-Bondi spacetime. A spherically symmetric slice can be described by a function $t(r)$, so that the slice is the level surface $w=0$ of the function $w=t-t(r)$ in spacetime, restricted of course to be a time function near $w=0$. Then from Eq. (3) of Sec. II, or from the variational principle for the special case $K=0$, the convergence of a slice is

$$K=X(X^2-t'^2)^{-3/2}\left[-t''+\frac{2Y'}{X^2Y}t'^3+\left(2\frac{\dot X}{X}+2\frac{\dot Y}{Y}\right)t'^2+\left(\frac{X'}{X}-2\frac{Y'}{Y}\right)t'-X^2\left(\frac{\dot X}{X}+2\frac{\dot Y}{Y}\right)\right]. \tag{31}$$

Upon setting $K=0$ this becomes a second-order ordinary differential equation in $t(r)$ whose solutions are the maximal slices; or, setting $K=K_0=\text{const}$ gives slices of constant-mean-curvature K_0. It is convenient to introduce a new variable T in addition to t, and rewrite this equation as two first-order equations:

$$\frac{dT}{dr}=(T^2-1)\left[\frac{2Y'}{Y}T+X\left(\frac{\dot X}{X}+2\frac{\dot Y}{Y}\right)\right]-(1-T^2)^{3/2}XK_0\,, \tag{32a}$$

$$\frac{dt}{dr}=XT\,. \tag{32b}$$

The boundary conditions for this system are the following: At $r=0$, exactly one free parameter may be given, $t(0)$; the value $T(0)$ must be 0 (otherwise the slice will have a cone singularity; the point $r=0$ is a singular point of the system). The boundary condition as $r\to\infty$ is free; the slice must cross spatial infinity I^0 for $K=0$, or the slice must cross null infinity $\mathscr{I}^{\pm}$ for $K_0\lessgtr 0$. Therefore, for each K_0, the solutions form a one-parameter family. Some solutions of these equations will run into the singularity instead of getting out to infinity; these solutions should be discarded. Equations (32a) and (32b) are readily integrated numerically by a standard fourth-order, adaptive-step, Runge-Kutta algorithm.

There is an energylike quantity E which is useful for locating the limit slices discussed in the last section,

$$E=-Y^2(T+\dot Y)(1-T^2)^{-1/2}-\tfrac{1}{3}K_0Y^3\,. \tag{33}$$

It obeys the equation

$$E'=YM'T(1-T^2)^{-1/2}\,. \tag{34}$$

Therefore, when the slice emerges from the dust cloud into vacuum ($M'=0$), E becomes conserved along the remainder of the slice. Using the fact that, along a limit slice $r_{\text{sch}}=r_{\text{lim}}$ of Corollary 3.4, $Y'=0$, one can show that

$$E_{\text{lim}}=r_{\text{lim}}^2\,(2M/r_{\text{lim}}-1)^{1/2}-\tfrac{1}{3}K_0 r_{\text{lim}}^3\,. \tag{35}$$

Therefore for each $K=K_0$ slice one can compute E, and the limit slice is the one that has $E=E_{\text{lim}}$ in the Schwarzschild exterior.

The main question we shall address is under what conditions on $t_0(r)$ and $M(r)$ do nonsingular maximal slices completely cover the region of spacetime outside the black hole, i.e., the domain of outer communications? Just as for causal structure, the key issue is the behavior of t_0/M as $r\to 0^+$.

Case 1: $\lim t_0/M=0$ as $r\to 0^+$. In this case we have a proof that maximal slicing always works. (See Theorem 4.1 below.) The essential point is that

$$f=t-t_0(r) \qquad (36)$$

can be shown to be a crushing function on some neighborhood N of the singularity. The computation of the convergence K [Eq. (3)] of f to prove this fact is straightforward and we omit it.

Case 2: $\lim t_0/M=\infty$ as $r\to 0^+$. In this case there is always a "shell-focusing singularity" at the origin, which may [Fig. 1(c)] or may not [Fig. 1(b)] be visible from infinity. The singularity is never a crushing singularity. If the singularity is globally naked [Fig. 1(c)] then maximal slicing always fails to cover the doc because the region outside the absolute event horizon is not even globally hyperbolic. If [Fig. 1(b)] the singularity is locally naked then maximal slicing sometimes works and sometimes fails, as we have found by numerical examples, to be discussed below.

Case 3: $\lim t_0/M=\zeta=\text{const}>0$ as $r\to 0^+$. In this critical intermiate case, behavior depends on the value of ζ.

Case 3a: $0<\zeta<\zeta_1=[(1921+533\sqrt{13})/72]^{1/2}\cong 7.3056$. By tedious and boring calculations it can be shown that the singularity is a crushing singularity, the causal structure is Fig. 1(a), and therefore maximal slicing always works following Theorem 4.1 (see below).

Case 3b: $\zeta_1<\zeta<\zeta_2=\frac{26}{3}+5\sqrt{3}\cong 17.3269$. Here the singularity is never a crushing singularity, but the causal structure is still Fig. 1(a). Maximal slicing sometimes works and sometimes breaks down. These examples are important because they show a "nice" causal structure, Fig. 1(a), is not sufficient for maximal slicing to work.

Case 3c: $\zeta_2<\zeta<\infty$. The singularity is never a crushing singularity; it may or may not be naked. Maximal slicing does not work. The case for mean-constant-curvature slices can be similarly analyzed. We reserve this for the section on numerical results. Let us now turn to the global existence theorem for maximal slices for the TB spacetimes of Case 1.

Theorem 4.1: In a marginally bound Tolman-Bondi spacetime M (as described above) with $t_0(r)$ and $M(r)$ continuous, nondecreasing, and piecewise smooth on $0\le r<\infty$; if

(1) $M(r)=\text{const}=M_0$ outside of some radius, $r>R$ (hence spacetime is the vacuum Schwarzschild solution for $r>R$);

(2) $\lim t_0/M=0$ as $r\to 0^+$; then any point p outside the absolute event horizon lies on a unique spherically symmetric maximal Cauchy slice.

Proof: We use standard methods which are reviewed in Sec. 6.7 of Hawking and Ellis,[7] and will just sketch the proof. Define a two-dimensional manifold-with-boundary $\bar{M}_2$ with coordinates (t,r), $-\infty<t\le t_0(r)$, $0\le r<\infty$, and define the spacetime $(\bar{M}_2,dv^2)$ from Eq. (30). The set of all compact spherically symmetric C^1 slices S in (M,ds^2) that span the two-sphere on which p lies, becomes the set $C'(p)$ of all C^1 timelike curves from $(t(p),r(p))$ to $r=0$ in $(\bar{M}_2,dv^2)$. Under the C^0 topology, $C'(p)$ forms a closure $C(p)$, the space of all C^0 causal curves from $(t(p),r(p))$ to $r=0$. The length functional $L=\int dv$ is upper semicontinuous on $C'(p)$ and can be extended by continuity to $C(p)$. A segment of a curve running along the singularity then has zero length. Now $C(p)$ is compact in the C^0 topology, and therefore the upper semicontinuous functional L achieves its maximum value. This maximum must be achieved by a curve which is C^1 and which is a geodesic of dv^2. As discussed above [Eq. (30)] this curve then corresponds to a C^1 maximal slice S in (M,ds^2). If S intersects the axis $r=0$ in the regular part of spacetime, it must be smooth there or it will not be maximal. Also, S can be smoothly extended through and radially outward from p. During this extension, S cannot intersect the singularity, because p is outside the absolute event horizon, S is spacelike, and there are no naked singularities from hypothesis (2). Therefore S can be extended outward until $r>R$, at which point it must match onto one of the explicitly known maximal slices of the Schwarzschild spacetime. All these latter slices are known to reach I^0, so S can be extended to I^0. At this point we are nearly done; the only thing which could go wrong is that S might hit the singularity, and perhaps run along it for a way, inside p, and therefore fail to be a Cauchy slice in (M,ds^2); the crushing function f helps us to exclude this possibility. Given a slice S which corresponds to a maximum of L. Define a one-parameter C^0 deformation $S(\epsilon)$, with $S(0)=S$, by

$$t(r,\epsilon)=\max\{t(r),\,t_0(r)-\epsilon\}\,, \qquad (37)$$

where $t(r)$ describes the slice S and $t(r,\epsilon)$ describes the slice $S(\epsilon)$. If ϵ is small enough that the two-sphere labeled by $(t_0(r)-\epsilon,r)$ lies for all r in the neighborhood $N(0)$ of the crushing function f, then it follows by a short computation that $L(\epsilon)$ is strictly increasing in ϵ. Therefore any slice S which intersects $N(0)$ cannot achieve the global maximum of L, and the true global maximum avoids the singularity and thus is a Cauchy slice of (M, ds^2). Finally, uniqueness follows from the mixed energy condition, e.g., as in Brill and Flaherty.[4] Q.E.D.

Hypothesis (1) is a restrictive form of asymptotic flatness, and presumably could be weakened. Hypothesis (2) guarantees that "shell-focusing singularities" are absent; as seen above many counterexamples exist if this is dropped. Theorem 4.1 shows in particular that the Oppenheimer-Snyder model admits a unique spherically symmetrical maximal Cauchy slicing, which covers all of spacetime outside the black hole.

D. Numerical results

We constructed a large store of numerical solutions to the slicing equations, Eqs. (32a) and (32b), in the following representative family of dust collapses:

$$M(r)=r^3, \quad t_0(r)=\zeta r^p, \quad \text{for } 0\leq r\leq 1 \tag{38a}$$

$$M(r)=1, \quad t_0(r)=r^2-1+\zeta, \quad \text{for } 1<r<\infty \tag{38b}$$

where $\zeta\geq 0$ is a constant and $p\geq 1$ is an integer. The region $0\leq r\leq 1$ is a dust collapsing core, inhomogeneous unless $\zeta=0$ (when we have Oppenheimer-Snyder); the region $1<r<\infty$ is the Schwarzschild exterior solution. For $p\geq 4$, Theorem 4.1 applies, and maximal slicing is guaranteed to work. For $1\leq p\leq 3$ we determined by numerical search when maximal slicing works and when it does not. The results are summarized in Table I. Note that the degree of differentiability of the metric is high, even C^∞. [The metric is actually one lower degree than given when expressed in comoving coordinates (t,r); as usual one must go to nongeodestic coordinates, e.g., minimal-shear coordinates,[2] to see the full differentiability.] We showed above that if a TB solution has a crushing singularity, then maximal slicing works. However, the converse is not true. Even in the case when the singularity fails to be crushing, maximal slicing can sometimes work. It would seem that, as argued by Smarr and York,[2] it is the inhomogeneity of the density distribution which determines if maximal slices converge to a limit slice before hitting the singularity. If $p\geq 4$ the singularity is "broad" and maximal slices, generated in numerical relativity by elliptic equations[2] on each slice, "feel" the curvature mounting soon enough to halt evolution in the strong field region. If $p<4$, the singularity is too "pointy" and the elliptic equations do not "feel" the curvature until it is too late.

We finish this section by exhibiting several examples generated by numerically integrating Eqs. (32a) and (32b). First, we examine maximal slicing. In Fig. 2 we slice the $M=1$ Oppenheimer-Snyder solution. We show [Fig. 2(a)] the interior of the star $(0\leq r\leq 1)$ in the TB coordinates. The choices of t_{in}, the value of the TB time coordinate

TABLE I. This is a classification of the causal structure of a two-parameter family (ζ,p) of Tolman-Bondi dust-collapse spacetimes $(W=1)$. The parameter p determines the differentiability class of the metric. For $p<4$, the causal structure [Penrose-Carter diagrams in Fig. 1(a)–1(c)] is determined by ζ which characterizes the degree of inhomogeneity of the collapse. The singularity is described by the properties discussed in the text. Maximal slicing works if a family of Cauchy maximal slices exists which covers all of the spacetime outside of the black hole, i.e., the domain of outer communications.

				Singularity			
p	Differentiability	ζ	Penrose-Carter diagram	Crushing ?	Totally spacelike ?	Globally naked ?	Maximal slicing work ?
1	C^1	$0<\zeta<2.4532$	(b)	No	No	No	Yes
		$2.4532<\zeta<6.3084$	(b)	No	No	No	No
		$6.3084<\zeta<\infty$	(c)	No	No	Yes	No
2	C^∞	$0<\zeta<3.6453$	(b)	No	No	No	Yes
		$3.6453<\zeta<9.0307$	(b)	No	No	No	No
		$9.0307<\zeta<\infty$	(c)	No	No	Yes	No
3 (Self-similar case)	C^3	$0<\zeta<7.3056$	(a)	Yes	Yes	No	Yes
		$7.3056<\zeta<\sim 7.4$	(a)	?	Yes	No	Yes
		$\sim 7.4<\zeta<17.3269$	(a)	No	Yes	No	No
		$17.3269<\zeta<\infty$	(c)	No	No	Yes	No
≥ 4	C^∞ (p even) C^p (p odd)	$0<\zeta<\infty$	(a)	Yes	Yes	No	Yes
(Oppenheimer-Snyder)	C^∞	$\zeta\equiv 0$	(a)	Yes	Yes	No	Yes

Interior: $0\leq r\leq 1$, $t_0(r)=\zeta r^p$, $M(r)=r^3$

Exterior: $1\leq r<\infty$, $t_0(r)=r^2-1+\zeta$, $M(r)=1$

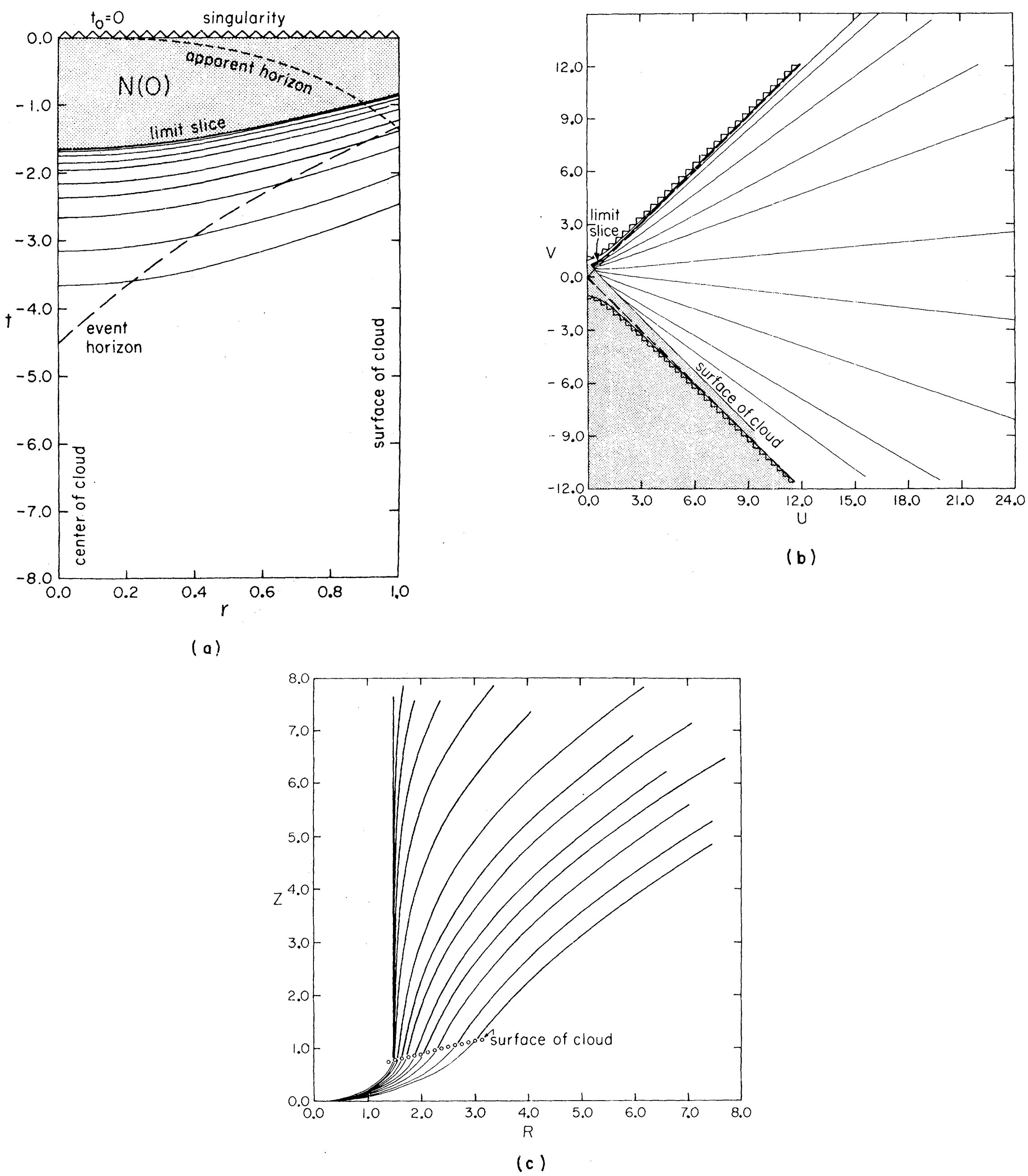

FIG. 2. Maximal slicing of Oppenheimer-Snyder collapse. (a) The dust cloud interior in TB coordinates (t,r). The left side is the center of the cloud $r=0$, the right side is the surface of the cloud $r=1$. The singularity is at $t=t_0(r)=0$. The apparent horizon (see text for our definition) and event horizon are labeled. In the special case of $t_0=0$, the location of the event horizon can be analytically determined (Ref. 53). The maximal slices have been chosen so that they are evenly spaced in $t_{\rm Sch}$ as $r\to\infty$. Note that these slices approach a limit slice. Furthermore, in this homogeneous case, the slices are self-similar inside the cloud. The region between the limit slice and the singularity is the neighborhood $N(0)$ of theorem 2.18. (b) The maximal slices in the vacuum exterior to the cloud shown in (a). We plot these slices in the Kruskal diagram. The surface of the cloud is shown and the shaded region below this curve is to be thrown away. The spacetime inside and outside the surface are matched along the surface. Note the limit slice ($r_{\rm Sch}=\frac{3}{2}M$) avoids the singularity but allows all events in the domain of outer communications to be covered by the maximal slices. (c) Isometric embedding diagrams of the maximal slices. The dotted line labeled surface of cloud separates the dust from the vacuum. Note that the limit slice approaches the $r_{\rm Sch}=\frac{1}{2}$ $(3M)$ cylinder.

at $r=0$, were made so that the slices would be equally spaced in time ($\Delta t_\infty = 2.1M$) at spatial infinity. This is unnatural in our scheme, but natural in numerical relativity. As the slices leave the dust filled interior, they emerge into an Eddington-Finkelstein patch of $M=1$ Schwarzschild spacetime. We represent this in Kruskal coordinates [Fig. 2(b)]. Note that the maximal slicing fills the spacetime exterior to the black hole (the doc). The slices wrap up around a "limit slice" which

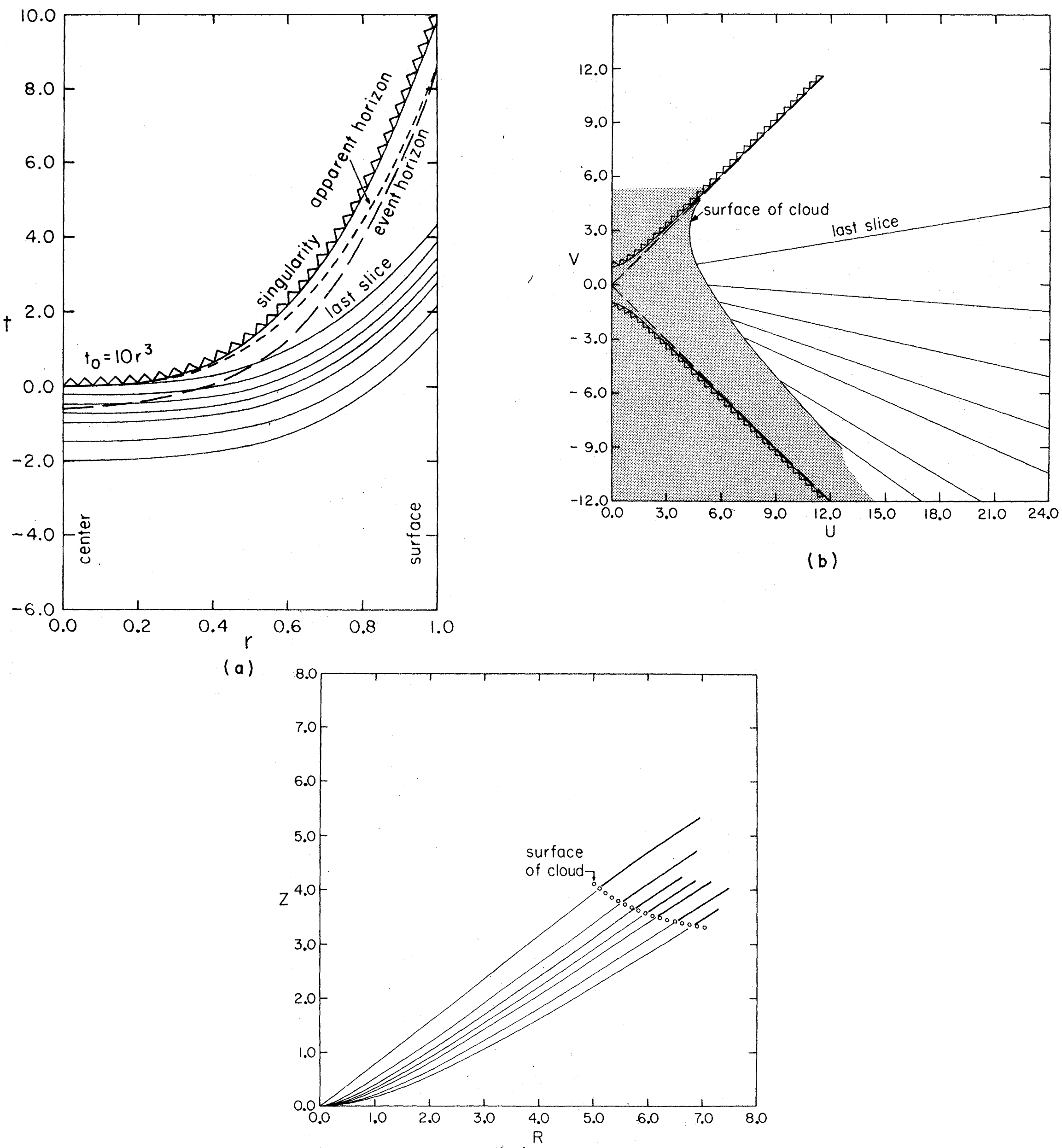

FIG. 3. Maximal slicing fails if the collapse is too inhomogeneous. (a) The interior with the singularity $t=t_0=10r^3$. The slices are integrated out from the center for values of t_{in} approaching $t_{\text{in}}=0$, note that no trapped surfaces exist on any of these slices even though the last slice hits the singularity. (b) Even this last slice $t_{\text{in}}=0$ escapes to spatial infinity. Therefore a large portion of the doc is left uncovered by the maximal slices. Note that the path of the surface of the star is different than in Fig. 2. The causal structure of this spacetime is the same as Oppenheimer-Snyder [Fig. 1(a)]. (c) The isometric embedding diagrams for these slices. Note that the surface of the star is still outside of the black hole ($r_{\text{surf}} \simeq 5M$) when the center of the maximal slice hits the singularity.

avoids the interior singularity by a wide margin ($t_{in} = -1.66304$) and asymptotically approaches the $r = 3M/2$ limit slice of Schwarzschild exterior. The isometric embedding diagrams of these slices are shown in Fig. 2(c). Note that the star always remains concave upward (no "neck" forms)[52] and the exterior approaches the $r = 3M/2$ cylinder discussed by Estabrook *et al.*[3]

That maximal slicing can fail is shown in Fig. 3. Here $t_0 = 10r^3$ and the causal structure is as in Fig. 1(a). Spacetime is globally hyperbolic, the singularity is totally spacelike, the strong-energy condition holds, and still maximal slicing fails. As seen from Fig. 4(c), the surface of the star is at $r_{sch} \cong 5M$ when the failure occurs and the event horizon is still inside the star [Fig. 3(a)]. This example indicates theorems on avoidance of singularities by maximal slicing are going to be very difficult to prove without further hypotheses, such as our hypothesis of crushing singularities.

We turn to $K = K_0 < 0$ slicing. Here the slices should be asymptotically null and the limit slice should be even further from the singularity than for maximal slices. In Fig. 4, a $K_0 = -1$ slicing of Oppenheimer-Snyder we see that indeed this is the case. The limit slice has $t_{in} = -2.8$. As one increases the inhomogeneity of the collapse the limit surfaces of $K = K_0$ get progressively pulled into the singularity. As long as the causal structure is as in Fig. 1(a) or 1(b), one can turn $|K_0|$ up high enough that the associated $K = K_0$ slicing will fill the doc. However, once the collapse is so inhomogeneous that a globally naked singularity forms, then even $K_0 \to -\infty$ will not avoid the singularity. This is because for a globally naked singularity a Cauchy horizon forms which hits $\mathcal{I}^+$ at a finite value of null time. Thus there is no way to extend a $K = K_0$ slicing to arbitrarily large null time without crossing the Cauchy horizon. This is demonstrated in Fig. 5 where a $K_0 = -10$ slicing of a $t_0 = 10r^2$ spacetime is shown. From the diagrams one sees that the surfaces are almost null. Even so, the *last slice*, which hits the singularity at $r = 0$, escapes in the doc. As mentioned above, this failure can be turned around to be a new test to use in numerical relativity to show that a globally naked singularity has formed.

Finally, we consider $K = K_0 > 0$ slicings. Since these slicings are asymptotically null to $\mathcal{I}^-$ and have limit surfaces closer to the singularity than maximal slices, it would seem that these slices are useless for gravitational waves. However, it is hoped they will prove useful for using numerical relativity to study singularity structure. We calculated a $K = -2/t_{in}$ slicing[53] of Oppenheimer-Snyder collapse. This particular form of K was chosen because it yields the slices of constant

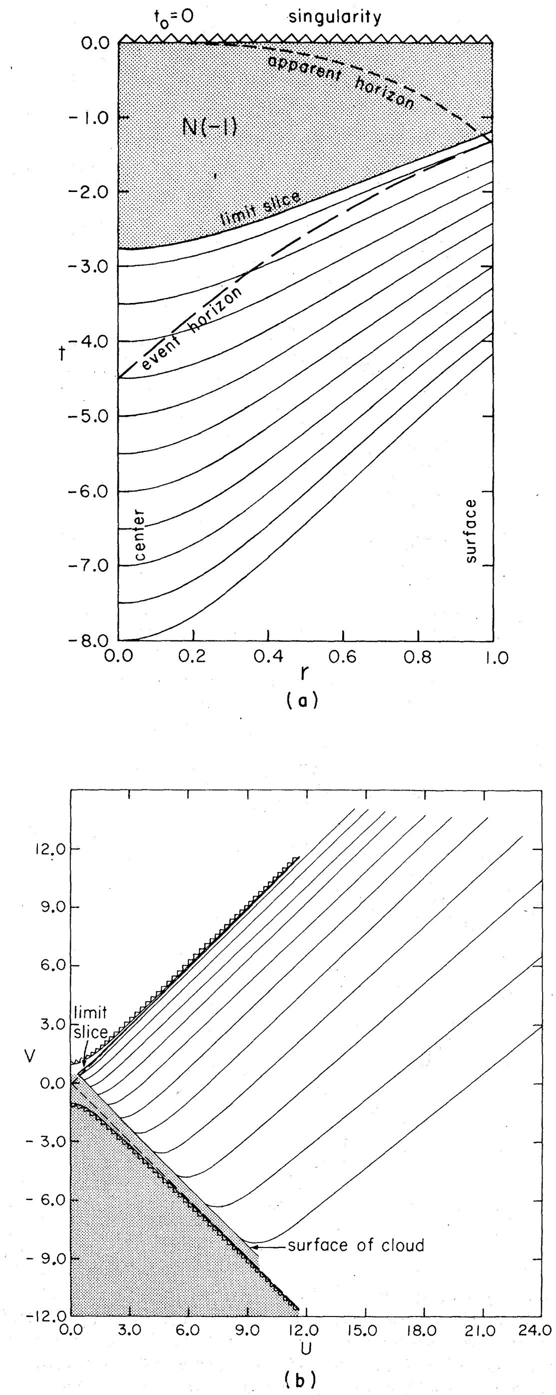

FIG. 4. $K_0 = -1$ slicing of oppenheimer-snyder. (a) The singularity, apparent and event horizons are as in Fig. 2. Note that the $K_0 = -1$ slices are much steeper inside the cloud than maximal slices. The neighborhood of avoidance, $N(-1)$, is larger than $N(0)$. (b) In the vacuum exterior the slices become asymptotically null very quickly. The $K_0 = -1$ slices approach $\mathcal{I}^+$ while the maximal slices (Fig. 2) approach I^0.

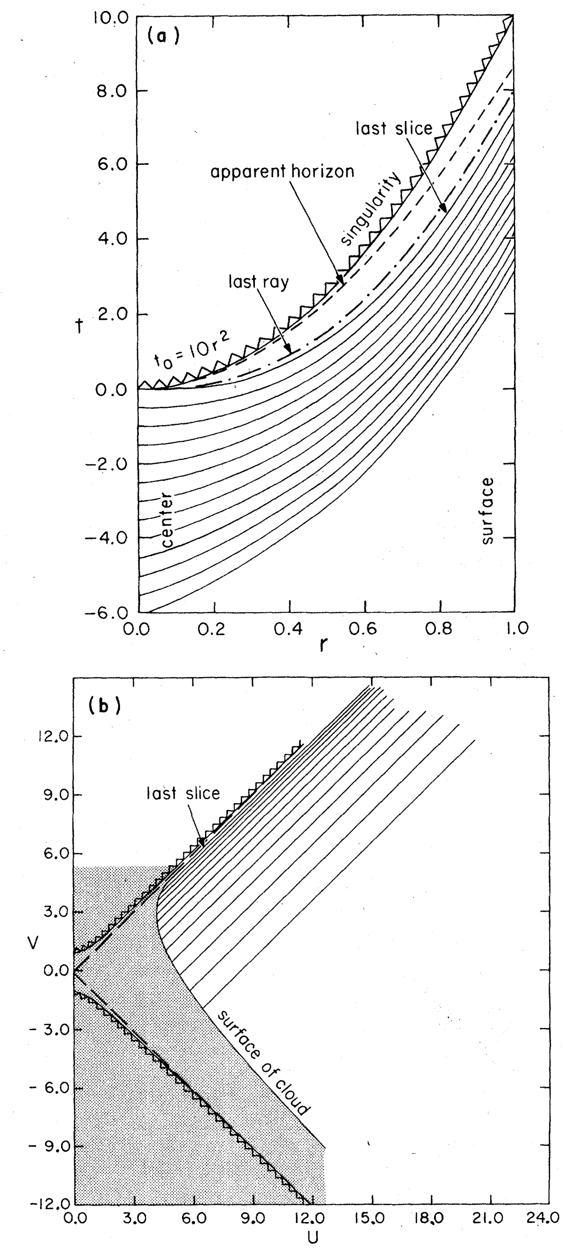

FIG. 5. $K_0 = -10$ slicing of a global naked singularity. (a) By choosing the singularity $t = t_0 = 10r^2$ the causal structure becomes Fig. 1(c). As in Fig. 3(a), the slices can be extended until $t_{in} = 0$. We also show the path (dot-dash line) of a light ray emitted at (t, r) 1(0, 0). Note that the $K_0 = -10$ slices are very close to null. The event horison is singular and is not shown. (b) The exterior region is covered up to the last slice which is outside of the vacuum event horizon.

curvature inside the homogeneous interior (i.e., it coincides with the TB time coordinate). While it uniformly approaches the singularity inside $r = 1$, it does not wrap up around any of the Schwarzschild singularity for $r > 1$. Through a different choice of boundary conditions, the slices could be made to wrap up around the Schwarzschild singularity also; we hope to return to this question in a future paper.

V. CONCLUSIONS

Under what conditions is maximal, or constant curvature, slicing guaranteed to work? Our examples indicate that merely imposing one or another of the usual energy conditions is far too weak. It is a tempting conjecture that imposing the vacuum Einstein equations is strong enough. However, to prove this conjecture would require far more detailed information about the nature of the Cauchy evolution process than is available in currently known theorems.

As an intermediate step in analyzing this difficult problem, which partakes of much of the entire singularity problem in general relativity, we have proposed the definition of the "crushing singularity." This definition and its corollaries have proved extremely useful to us in making sense out of the widely assorted collection of examples and counterexamples that we have presented.

Our advice to numerical relativists is to go ahead and use maximal slicing, but to be aware that there is no theoretical guarantee that it will not fail.

Note added in proof. As a further example in favor of conjecture (2.15), we have shown that all Gowdy T^3 cosmologies [cf. B. K. Berger, Ann. Phys. (N.Y.) 83, 458 (1974)] have crushing singularities; these models are vacuum and spatially compact.

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[23]This assertion can be proven from Theorem 2.17.

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