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Light-quark masses and isospin violation

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Isospin-violating effects arising from a nonvanishing quark mass difference $m_d - m_u$ are discussed, in particular for the π^+-π^0 mass-difference question and for $\eta' \to 3\pi$ and $\Sigma \to \Lambda + e + \nu$ decays. The role of the strong anomaly is emphasized, and exploited to bring out certain relations among heavy-quark, light-quark, and gluon operator matrix elements. The anomaly suppresses isospin-violating effects that might otherwise be substantial for the Bjorken electroproduction sum rule.

I. INTRODUCTION

The color gauge theory of strong interactions[1] should provide a complete description of these interactions in terms of a very limited set of parameters: One dimensionless coupling constant and a "current" quark mass for each quark flavor. The coupling constants, via dimensional transmutation,[2] can alternatively be represented by the mass scale M_{st} characteristic of the strong interactions. One can identify M_{st} with the mass of a typical hadron made up of light quarks (up, down, strange), e.g., the nucleon or ρ meson. This is based on the idea that the properties of these hadrons are largely unaffected by the influence of heavy quarks (c, b, t, ...) and, at the same time, insensitive to the values of the light-quark masses [i.e., chiral SU(3)×SU(3) is a good symmetry which is spontaneously broken by the strong interactions]. Alternatively, given our present inability to deal quantitatively with hadron masses, one might identify M_{st} as the renormalization scale parameter, which can be determined experimentally from corrections to scaling in deep-inelastic scattering and which is related in a precise fashion to the strong coupling constant.

The pattern of quark masses determines the (approximate) character of flavor symmetry of the strong interactions. This is the chiral symmetry $SU(N)_L \times SU(N)_R$, where N is the number of quarks with mass significantly smaller than M_{st}. Thus, the observation of approximate $SU(3)_L \times SU(3)_R$ symmetry reflects that the strong interactions are flavor independent and that the up, down, and strange quark masses are much smaller than M_{st}. In the present stage of our ignorance about the unification of strong and weak interactions, the number 3 in SU(3)×SU(3)—i.e., the fact that three of the quarks are light, the others heavy on the hadron mass scale—is without explanation. One could well imagine a world in which all the quarks would be massive on the scale of M_{st}. We simply have no idea at present how to relate M_{st} to the various dimensional parameters which characterize the weak and electromagnetic interactions. The quark masses reflect the pattern of breaking of weak and electromagnetic gauge symmetry. In the present theoretical framework (i.e., the Weinberg-Salam model) there is no understanding of the pattern of quark masses.

As to the actual values of the quark masses, matters are simplest for the heavy quarks. Here one can extract the masses, roughly, from knowledge of the masses of hadrons that contain them (e.g., the charm mass is, roughly, half the mass of J/ψ). Much more difficult is the determination of light-quark masses. Estimates of mass ratios have been attempted on the basis of analysis of pseudoscalar-meson masses, baryon masses, and $\eta \to 3\pi$ decay.[3] The results obtained from these sources are roughly consistent, but considerable theoretical uncertainties are involved in the analyses. Knowledge of light-quark masses is of interest from many points of view. These masses set the scale, at large momenta, for the onset of instanton effects[4]; they are necessary inputs for calculation of, e.g., the p-n mass difference; the possibility that $m_u = 0$ is of interest in connection with the issue of natural P and T invariance in the strong interactions[5]; and perhaps most important, one would like to know these parameters in order to be able to look for interesting patterns in the overall spectrum of quark and lepton masses. Finally, as an immediate issue, one would like to know whether the high degree of validity observed empirically with respect to isospin invariance reflects a true symmetry $[(m_d - m_u)/(m_d + m_u) \ll 1]$; or rather, as in the case of chiral SU(2)×SU(2), an "accidental" symmetry, where

$$(m_d - m_u)/(m_d + m_u) \sim 1$$

but where m_u and m_d are both negligibly small on the scale M_{st} of strong interactions. It is the latter alternative that we would in fact expect within our present understanding of the weak interactions, where isotopic symmetry has no reason to be natural. We have no reason to expect that $(m_d - m_u)/(m_d + m_u)$ is any smaller, say, than $(m_c - m_s)/(m_c + m_s)$. It would be a great surprise indeed if isospin were anything like an exact symmetry.

If isospin is really an accidental symmetry in the above sense, then there might be special situations where the underlying asymmetry contributes noticeable effects.[6] In most circumstances the measure of observable isospin violation will be set by the ratio $(m_d - m_u)/M_{st}$. However, the spectrum of Nambu-Goldstone bosons, where masses arise solely from explicit breaking of SU(3)×SU(3) symmetry, could depend more critically on $(m_d - m_u)/(m_d + m_u)$; correspondingly, processes involving these pseudoscalar mesons might be especially sensitive to the underlying isotopic asymmetry. The situation would be especially striking if axial-vector U(1), in addition to chiral SU(3)×SU(3), were a good symmetry of the strong interactions. In that case we would expect large violations of isotopic-spin symmetry, of order $(m_d - m_u)/(m_d + m_u)$, in the pattern of pseudoscalar-meson masses. In the absence of axial-vector baryon-number conservation, the violations of isospin symmetry become much smaller here, or order $(m_d - m_u)/m_s$. Even so, if $(m_d - m_u)/(m_d + m_u)$ is of order unity, the mass-difference effects can still be larger than those arising from electromagnetic influences—an inevitable contribution to isospin violation—in certain situations.

In this note we analyze three cases where isospin violation arising from light-quark mass differences might produce significant effects: Two of these cases involve the pseudoscalar mesons directly, namely the $\pi^+ - \pi^0$ mass-difference problem and the decay process $\eta' \to 3\pi$; the third deals with $\Sigma^+ \to \Lambda + e^{\pm} + \nu$ decay. For the pion-mass-difference question we find that light-quark mass difference can contribute significantly, of order 20% of the observed pion-mass difference, if $m_u = 0$ and $m_d/m_s \sim 0.05$. Unfortunately, standard estimates of the electromagnetic contribution may well be uncertain by a similar amount, though, as we shall note, the corrections to the latter are likely to run in such a direction as to rule out too small a value for m_u. As for the $\eta' \to 3\pi$ and $\Sigma \to \Lambda + e + \nu$ process, the tests we discuss here may involve rather demanding experimental measurements.

In the course of our discussion we will take into account the anomaly in the divergence of the axial-vector baryon-number current.[7] It is the existence of this anomaly that destroys U(1) invariance of the strong interactions, thereby reducing isospin violations to order at most $(m_d - m_u)/m_s$. Furthermore, consideration of the anomaly, taken together with Sutherland's theorem,[8] will lead to certain formal identities between gluon and quark operator matrix elements and will show, among other things, that isospin violation for the Bjorken sum rule is only of order $(m_d - m_u)/M_{st}$.

II. INFLUENCE OF THE STRONG ANOMALY

We shall work within the framework of quantum chromodynamics (QCD). Here, with neglect of the masses of the u, d, and s quarks, the strong interactions possess an $SU(3)_L \times SU(3)_R$ chiral symmetry. The apparent axial-vector baryon-number symmetry of QCD is removed by the strong anomaly. Chiral symmetry is spontaneously broken by the dynamics of the strong interactions. We assume, however, that SU(3) is not spontaneously broken, so that the massless quark vacuum is characterized by the vacuum expectation value equalities

$$\langle \bar{u}u \rangle_0 = \langle \bar{d}d \rangle_0 = \langle \bar{s}s \rangle_0 \equiv \sigma \ . \qquad (1)$$

In this situation the theory produces an octet of Nambu-Goldstone bosons, identified with the pseudoscalar-meson octet. In the absence of nonvanishing current-quark masses, the pseudoscalars are strictly massless. Conversely, the actual pseudoscalar masses are therefore sensitive to the values of the current quark masses.[9] Indeed, let us associate the axial-vector current

$$A_\mu^\alpha = \bar{q}\gamma_\mu \gamma_5 \tfrac{1}{2}\lambda^\alpha q, \quad \alpha = 1, 2, \ldots, 8 \qquad (2)$$

to the Goldstone boson states G^β, according to

$$\langle 0 | A_\mu^\alpha | G^\beta(p) \rangle = i f_\pi p_\mu \delta_{\alpha\beta} \ . \qquad (3)$$

The meson mass matrix can then be calculated by standard PCAC (partially conserved axial-vector current) techniques. Ignoring terms of nonleading order in quark masses, one has

$$M_{\alpha\beta}{}^2 = \frac{1}{f_\pi{}^2} \int d^3x\, d^3y \langle [A_0^\alpha(\vec{x}, 0), [A_0^\beta(\vec{y}, 0), \mathcal{H}]] \rangle_0 \ , \qquad (4)$$

where

$$\mathcal{H} = \mathcal{H}_0 + m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s$$

is the Hamiltonian density and $\mathcal{H}_0$ is $SU(3)_R \times SU(3)_L$ symmetric. It then follows that

$$M_{\pi^0\pi^0}{}^2 = M_{\pi^+\pi^+}{}^2 = \frac{1}{f_\pi{}^2}(m_u + m_d)\sigma,$$

$$M_{K^+K^+}{}^2 = \frac{1}{f_\pi{}^2}(m_u + m_s)\sigma, \quad M_{K^0K^0}{}^2 = \frac{1}{f_\pi{}^2}(m_d + m_s)\sigma,$$

$$M_{\pi^0\eta}{}^2 = \frac{1}{f_\pi{}^2}\frac{(m_u - m_d)}{\sqrt{3}}\sigma, \tag{5}$$

$$M_{\eta\eta}{}^2 = \frac{1}{f_\pi{}^2}\frac{m_u + m_d + 4m_s}{3}\sigma.$$

These expressions, valid to lowest order in quark masses, can now be used to estimate quark mass ratios. We neglect the mixing of η^0 and π^0 (a small effect here, to be discussed below). Also, to remove electromagnetic contributions to the pseudoscalar masses, we follow Weinberg's analysis[3,10] by invoking Dashen's theorem. This theorem asserts, again to lowest order in quark masses, that the electromagnetic contributions to $M_{\pi^0}{}^2$, $M_\eta{}^2$, $M_{K^0}{}^2$ and $M_{K^\pm}{}^2 - M_{\pi^\pm}{}^2$ all vanish. One then deduces that

$$\frac{m_d + m_u}{m_s} = \frac{2M_{\pi^0}{}^2}{M_{K^0}{}^2 + M_{K^+}{}^2 - M_{\pi^+}{}^2} = 0.082\ ,$$

$$\frac{m_d - m_u}{m_d + m_u} = \frac{M_{K^0}{}^2 - M_{K^+}{}^2 + M_{\pi^+}{}^2 - M_{\pi^0}{}^2}{M_{\pi^0}{}^2} \simeq 0.29\ . \tag{6}$$

The first of these equations we may regard here as reliably fixing the ratio $(m_d + m_u)/m_s$. The second involves small differences of large quantities, so we take it as only indicative of the size of $(m_d - m_u)/(m_d + m_u)$. The further tests in Sec. III deal with alternative ways to get at this quantity. One also finds the modified Gell-Mann–Okubo mass formula

$$M_\eta{}^2 = \frac{4}{3}\left(\frac{M_{K^+}{}^2 + M_{K^0}{}^2}{2} - \frac{2M_{\pi^+}{}^2 - M_{\pi^0}{}^2}{4}\right). \tag{7}$$

This last relation yields $M_\eta = 566$ MeV, to be compared with the experimental value of 548 MeV. The success (to within ~6%) of this formula supports the view that the strange quark mass m_s can indeed be regarded as small compared to M_{st}. In any case it is the above evaluation of quark mass ratios that indicates that $(m_d - m_u)/(m_d + m_u)$ is not so very small, a conclusion that is also indicated by the necessity for a substantial $\bar{u}u - \bar{d}d$ tadpole term in estimates of baryon mass differences[11] and in attempts to account for $\eta \to 3\pi$ decay.

We note that the violation of isospin in the above mass matrix, as measured by

$$(M_{K^+}{}^2 - M_{K^0}{}^2)/(M_{K^+}{}^2 + M_{K^0}{}^2)$$

or by $M_{\pi^0\eta^0}{}^2/M_{\eta\eta}{}^2$ is small, of order $(m_d - m_u)/m_s$. This would not have been the case if the strong interactions were invariant under chiral $U_L(3) \times U_R(3)$. In this situation, for the limit of small quark masses, there would exist a ninth Goldstone boson (η'). Although this extra symmetry is in fact removed by the strong anomaly, it is amusing to pursue a bit further the situation that would prevail if this were not so. Introduce the ninth axial-vector current $A_\mu^0 = \bar{q}\gamma_\mu\gamma_\sigma(\frac{1}{2}\gamma_0)q$ and the corresponding state $G^0(p)$, with

$$\langle 0|A_\mu^0|G^0(p)\rangle = if_0 p_\mu\ .$$

This isoscalar state will mix with η; and in the presence of quark masses the spectrum of neutral pseudoscalar mesons will contain a Goldstone η' and two light Goldstone particles corresponding to π^0 and η. To lowest order in $(m_d + m_u)/m_s$ we find from the nonet mass matrix

$$M_{\eta'}{}^2 = \frac{2m_s}{3}\left(\frac{1}{f_0{}^2} + \frac{2}{f_\pi{}^2}\right)\sigma\ , \tag{8}$$

and

$$M_{\pi^0}{}^2 M_\eta{}^2 = \frac{12\, m_u m_d}{f_\pi{}^2(f_\pi{}^2 + 2f_0{}^2)}\sigma^2\ ,$$

$$M_{\pi^0}{}^2 + M_\eta{}^2 = \frac{1}{f_\pi{}^2}(m_u + m_d)\left(1 + \frac{3f_\pi{}^2}{f_\pi{}^2 + 2f_0{}^2}\right)\sigma \tag{9}$$

$$\leq 4M_{\pi^+}{}^2\ .$$

For the isospin-conserving case $m_u = m_d$, this yields the familiar Weinberg[12] bound $M_\eta{}^2 \leq 3M_\pi{}^2$. If $(m_d - m_u)/(m_d + m_u)$ is not small, one encounters substantial isospin violation as well. For example, with $f_0 = f_\pi$, the isoscalar states would exhibit "magic mixing": The mass eigenstates of the neutrals would be pure, each containing only one quark flavor pair, $\bar{u}u$, $\bar{d}d$, and $\bar{s}s$, with corresponding masses $2m_u\sigma/f_\pi{}^2$, $2m_d\sigma/f_\pi{}^2$, and $2m_s\sigma/f_\pi{}^2$.

Thus, if axial U(1) were a good symmetry and if $(m_d - m_u)/(m_d + m_u)$ were appreciable, one would have not only a light isoscalar particle η but also substantial violations of isotopic spin, the latter even though $(m_d - m_u)/m_s \ll 1$. That is, the $\pi^\pm$-π^0 mass difference would be substantial, and nuclear forces, for example, dominated as they are by light-meson exchange, would show no evidence of isotopic spin symmetry. As it is, however, axial-vector baryon number is not conserved in QCD, owing to the triangle anomaly and the existence of vacuum-tunneling mechanisms that render a nonzero value for the zero-momentum-transfer limit of matrix elements of the anomalous divergence.[13] This eliminates the unwanted fourth light Goldstone meson and reduces the degree of isospin violation from order $(m_d - m_u)/(m_d + m_u)$ to the order $(m_d - m_u)/m_s$. At this latter level, one has mixing between the (isotopically pure) "π^0" and "η" mesons. In the diagonalized mass matrix the physical π^0 and η^0 will respectively dominate the divergences of the axial-vector currents $\cos\lambda\, A_\mu^3 + \sin\lambda A_\mu^8$ and $\cos\lambda A_\mu^8 - \sin\lambda A_\mu^3$, where the mixing angle is given by

$$\sin\lambda = \frac{\sqrt{3}}{4}\left(\frac{m_d - m_u}{m_s}\right). \tag{10}$$

Although this mixing is small, it might nevertheless be significantly larger than the contribution from electromagnetic effects, especially since the electromagnetic contribution to $M_{\eta\pi^0}{}^2$ vanishes in the PCAC approximation. Possible observable consequences of this mixing will be discussed in the following section.

We now turn to a discussion of the matrix elements of the axial-vector currents and their divergences. If m_d/m_u is substantially different from unity, the "naive" divergences will be appreciably asymmetric with respect to isospin, and, at first sight, one might expect corresponding asymmetries in the matrix elements. Thus, for example, the divergence of the formally isoscalar axial-vector current

$$A_\mu^{I=0} = \bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d\,,$$

whose nucleon matrix elements are measurable via the Bjorken sum rule, is naively given by

$$\begin{aligned}\partial_\mu A_\mu^{I=0} &= 2i\,(m_u \bar{u}\gamma_5 u + m_d \bar{d}\gamma_5 d)\\ &= i(m_u + m_d)(\bar{u}\gamma_5 u + \bar{d}\gamma_5 d)\\ &\quad + i\,(m_u - m_d)(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d)\,.\end{aligned} \tag{11}$$

On the other hand, via PCAC the π^0 interpolating field is related to the divergence of the $I=1$ axial-vector current by

$$\begin{aligned}f_\pi M_\pi{}^2 \phi_{\pi^0} = \partial_\mu A_\mu^{I=1} &= i\,(m_u + m_d)(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d)\\ &\quad + i\,(m_u - m_d)(\bar{u}\gamma_5 u + \bar{d}\gamma_5 d)\,.\end{aligned} \tag{12}$$

For the nucleon matrix elements, if we neglect the contribution of the η pole and retain only the π^0 contribution, we would then find that

$$\langle N|\partial_\mu A_\mu^{I=0}|N\rangle_{I=0} = \left(\frac{m_u - m_d}{m_u + m_d}\right)\langle N|\partial_\mu A_\mu^{I=1}|N\rangle_{I=1}\,,$$

where the subscript $I=1$ on the nucleon matrix elements denotes the difference between proton and neutron matrix elements. In the Bjorken sum rule for spin-dependent electroproduction, one encounters the nucleon matrix elements of $A_\mu^{I=1}$ and the above result would imply a substantial degree of isospin violation in this sum rule if $(m_d - m_u)/(m_d + m_u)$ were non-negligible.

However, the above analysis has not allowed for effects of the anomaly in divergences of the axial-vector currents. Indeed, in the absence of the anomaly we would have full $U_L(2)\times U_R(2)$ symmetry; the η would be about as light as π^0, it would mix with π^0, and it could no longer be neglected. Let us therefore reconsider the above analysis, in a generalized version, with allowance for the anomaly. We will find that the naive divergences indeed exhibit substantial isospin violation, as above, but that the anomaly $(g^2/8\pi^2)\,\mathrm{Tr}G\tilde{G}$ does likewise, in such a way that the combined effects conspire to render all matrix elements of the axial-vector currents free of large isospin violation.

Consider the axial-vector current $\bar{q}(x)\gamma_\mu\gamma_5 q(x)$, where q is some quark field. In addition to its naive part, the divergence also contains an anomaly term:

$$\partial^\mu[\bar{q}\gamma_\mu\gamma_5 q] = 2i\,m_q \bar{q}\gamma_5 q + (g^2/8\pi^2)\,\mathrm{Tr}G\tilde{G}\,.$$

Let us now consider the matrix element of this divergence, taken between the vacuum and a state containing an isoscalar (γ_S) and isovector (γ_V) photon, the current carrying momentum p (see Fig. 1):

$$\int dx\, e^{ip\cdot x}\langle \gamma_V\gamma_S|\,\partial^\mu[\bar{q}(x)\gamma_\mu\gamma_5 q(x)]\,|0\rangle\,. \tag{13}$$

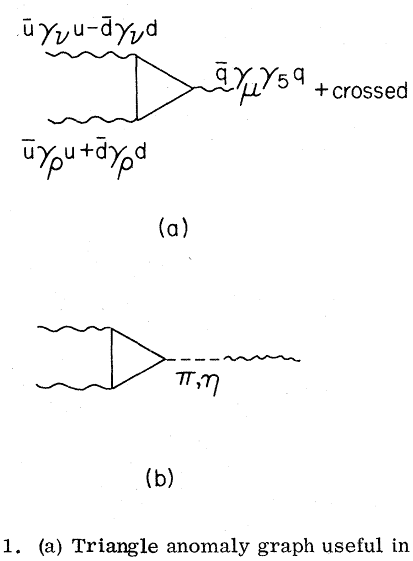

FIG. 1. (a) Triangle anomaly graph useful in analyzing symmetry breaking. (b) Saturation of triangle graph by Nambu-Goldstone bosons.

Sutherland's theorem implies that this matrix element (with the kinematical factor removed) vanishes as $p \to 0$. This is typically used, in connection with the electromagnetic anomaly, to derive a chiral-symmetry theorem for the $\pi^0 \to 2\gamma$ amplitude. Here we use it to derive a theorem for the matrix element of the strong anomaly $\mathrm{Tr}G\tilde{G}$. First, suppose that q is neither an up nor down quark field. Saturating the above matrix element with the π^0 pole, we then find that

$$\langle 0|\, 2i m_q \bar{q}\gamma_5 q |\, \pi^0\rangle \langle 0|\, (g^2/8\pi^2)\mathrm{Tr}G\tilde{G}|\, \pi^0\rangle = 0 \ . \tag{14}$$

On the other hand, applying the above reasoning to the current $\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d$, we derive

$$i\langle 0|\, (m_u \bar{u}\gamma_5 u + m_d \bar{d}\gamma_5 d)|\, \pi^0\rangle + \langle 0|\, (g^2/8\pi^2)\,\mathrm{Tr}G\tilde{G}|\, \pi^0\rangle = 0 \ . \tag{15}$$

Applying the PCAC reasoning of Eqs. (11) and (12) to the first term of Eq. (15), we have

$$\begin{aligned} 2i\langle 0|\, (m_u \bar{u}\gamma_5 u + m_d \bar{d}\gamma_5 d)|\, \pi^0\rangle &= \left(\frac{m_u - m_d}{m_u + m_d}\right) f_\pi M_\pi^{\,2} \\ &= -2\langle 0|\, (g^2/8\pi^2)\mathrm{Tr}G\tilde{G}|\, \pi^0\rangle \\ &= i\langle 0|\, m_s \bar{s}\gamma_5 s|\, \pi^0\rangle = i\langle 0|\, m_c \bar{c}\gamma_5 c|\, \pi^0\rangle \\ &= i\langle 0|\, m_b \bar{b}\gamma_5 b|\, \pi^0\rangle \ldots \ . \end{aligned} \tag{16}$$

We have ignored the contributions of the η pole in this discussion. Its contribution in Eq. (13) is again proportional to $m_u - m_d$; but whereas the π^0 pole is located at a value proportional to $m_u + m_d$, the η mass is proportional to $4m_s + m_u + m_d$: The η contribution has a different analytic form and must vanish separately. Returning to the main point, we have learned from Eq. (15) that in the chiral limit

$$\langle 0|\, \partial^\mu(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d)|\, \pi^0\rangle = 0 \ .$$

Therefore, even if $(m_d - m_u)/(m_d + m_u) \sim 1$, the isospin-violating corrections to Bjorken's sum rule, as more generally to all matrix elements of the axial-vector currents, are of order $(m_d - m_u)/M_{st}$ and thus very small. We *have* learned that the nucleon matrix elements of the anomaly term $(g^2/8\pi^2)\mathrm{Tr}G\tilde{G}$ are far from isoscalar, and we have learned the value of the one-pion matrix element of this operator. These extra pieces of information appear to be of little value, however, since we have no independent way of measuring these matrix elements.

Furthermore, Eq. (16) shows that the π^0 is "contaminated" by heavier quark components, in violation of a naive application of Zweig's rule. In particular, the strange-quark contribution

$$\partial i\langle 0|\, m_s \bar{s}\gamma_5 s|\, \pi^0\rangle = \left(\frac{m_u - m_d}{m_u + m_d}\right) f_\pi M_\pi^{\,2}$$

provides an alternative derivation of the mixing of π^0 and η^0 given in Eq. (10). This is useful, since the previous derivation involved the use of PCAC for the η meson.

To summarize, we have seen that the anomaly eliminates the U(1) symmetry that would have produced major isospin violation in the pseudoscalar mass spectrum. Moreover, it is just the large isospin violation of the anomaly operator that serves to essentially cancel corresponding isospin violations for naive divergences, thereby restoring isospin symmetry to matrix elements of the axial-vector currents. Thus, in QCD the largest source of nonelectromagnetic isospin violation that can be expected arises from the mixing of states, say π^0 and η^0, that would otherwise be unmixed; in amplitude these effects are of order $(m_d - m_u)/m_s$.

III. APPLICATIONS

A. The $\pi^\pm$-π^0 mass difference

The effect of a nonvanishing light-quark mass difference $m_d - m_u$ on the pion mass difference $M_{\pi^+} - M_{\pi^0}$ can be read off directly from Eq. (5). The π^0-η^0 mixing term in the mass matrix serves to reduce the π^0 mass relative to that of $\pi^\pm$; to leading order in $(m_u + m_d)/m_s$

$$M_{\pi^+}{}^2 - M_{\pi^0}{}^2 = \frac{(m_d - m_u)^2}{4m_s(m_d + m_u)}\, M_{\pi^+}{}^2 \ . \tag{18}$$

One sees here a formally interesting point. In the chiral limit, where all the quark masses, hence the pion masses, go to zero, the *ratio* of $\pi^\pm$ and π^0 masses remains sensitive to the quark mass ratio and sensitive therefore to isospin breaking at the quark mass level. Of course we do not know the quark mass ratios; but, for example, with $m_u = 0$, one would have $\Delta M_\pi \equiv M_{\pi^\pm} - M_{\pi^0} \approx 1.4$ MeV, a 30% contribution to the observed mass difference $(\Delta M_\pi)_{\mathrm{exp}} = 4.6$ MeV. For the quark mass ratios of Eq. (5)—which rely on the use of PCAC not only for pions but also for kaons—the contribution reduces to $\Delta M_\pi \approx 0.12$ MeV.

The bulk of the pion mass difference effect presumably arises from isospin-violating electromagnetic contributions. Insofar as these could be brought under theoretical control, Eq. (18)—which describes the quark-mass-difference contribution—would provide a measure of the quark mass ratios of interest here. Since the $\pi^{\pm}$-π^0 mass difference is a $\Delta I = 2$ effect, one expects that the electromagnetic contributions are dominated by low-mass intermediate states. The celebrated PCAC analysis of Das *et al.*[14] expresses $(\Delta M_\pi)_{em}$ in terms of an integral involving the difference of vector and axial-vector spectral functions. Saturating with the ρ and A_1 meson resonances, with parameters determined by appeal to the Weinberg spectral sum rules,[15] they find

$$(\Delta M_\pi)_{em} \simeq 5.0 \text{ MeV}.$$

This result was based further on the use of a narrow-width resonance approximation and the use of the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin formula[16] for g_ρ, the coupling of the ρ meson to the electromagnetic current. On using the present experimental determination of g_ρ and allowing for finite width of the ρ resonance (but otherwise following Das *et al.* with respect to ρ, A_1 saturation and use of the Weinberg sum rules), we find that the PCAC prediction should be corrected upward to $(\Delta M_\pi)_{em} \approx 6.1 \pm 0.8$ MeV. This is not an authoritative best-fit analysis; but it does suggest an uncomfortable discrepancy with experiment. Moreover, estimates of the corrections to PCAC, as analyzed by Langacker and Pagels,[17] indicate that the corrections are of order 15% of the observed mass difference, in a direction that increases the discrepancy; and finally any contribution from a quark mass difference, see Eq. (18), can only further increase the troubles. It may be, for the electromagnetic contributions, that one has to go beyond saturation with ρ and A_1, allowing for non-negligible contributions from other perhaps fairly low, mass resonances (asymptotic freedom guarantees that the spectral sum rules converge rapidly once one gets to the high-mass region).

B. $\eta' \to 3\pi$ decay

This reaction is interesting because it is forbidden by isospin (G-parity) invariance and is thus sensitive to symmetry violation. The dominant decay mode for η' is $\eta' \to \eta + 2\pi$. This process feeds the $\eta' \to 3\pi$ reaction via η^0-π^0 mixing. The mixing angle has already been given, in Eq. (10). We allowed there only for effects arising from the light-quark mass difference $m_d - m_u$. In the PCAC limit electromagnetism cannot contribute to η^0-π^0 mixing and so must enter into $\eta' \to 3\pi$ decay only in a more indirect fashion. Insofar as η^0-π^0 mixing is the dominant mechanism, as it might well be if $(m_d - m_u)/(m_d + m_u)$ is appreciable, it may be reasonable to neglect electromagnetic contributions althogether. One can then easily relate the $\eta' \to 3\pi$ and $\eta' \to \eta + 2\pi$ amplitudes. Supposing that the amplitudes are essentially constant over phase space (as seems the case experimentally for $\eta' \to \eta + 2\pi$ decay), one finds

$$r \equiv \frac{\Gamma(\eta' \to \pi^0\pi^+\pi^-)}{\Gamma(\eta' \to \eta\pi^+\pi^-)} \approx (16.8)\frac{3}{16}\left(\frac{m_d - m_u}{m_s}\right)^2, \quad (19)$$

where the numerical factor (16.8) is the ratio of phase-space volumes. For $m_u = 0$, $r \approx 2.0\%$; for the quark mass ratios of Eq. (6), $r \approx 0.18\%$. These are small effects but not obviously beyond experimental detection. The present experimental upper limit in r is ~5%.

C. $\Sigma^{\pm} \to \Lambda + e^{\pm} + \nu\ \beta$ decay

In this subsection we are interested in a different kind of mixing and its implications, namely the mixing between Σ^0 and Λ^0. This will be induced by a light-quark mass difference $m_d - m_u$, as of course also by electromagnetism. The former effect is dealt with easily. The SU(3)-symmetry-breaking terms in the Hamiltonian can be written $\epsilon_8 u_8 + \epsilon_3 u_3$, where ϵ_8 and ϵ_3 are expressible in terms of quark masses and where u_8 and u_3 are scalar quark densities, members of the same octet. For the effective baryon mass matrix one finds the off-diagonal terms

$$M_{\Sigma^0\Lambda^0} = M_{\Lambda^0\Sigma^0} = \frac{\sqrt{3}}{4}\left(\frac{m_d - m_u}{m_d + m_u}\right)\left(\frac{m_d + m_u}{m_s}\right)(M_{\Sigma^0} - M_{\Lambda^0})$$

$$\simeq 2.5\left(\frac{m_d - m_u}{m_d + m_u}\right)\text{MeV}, \quad (20)$$

where, in the last equality, we have used the first of Eqs. (6). This off-diagonal effect implies that the physical Σ^0 and Λ^0 states are mixtures of the "ideal" (i.e., isotopically pure) states Σ_i^0, Λ_i^0:

$$\Sigma^0 = \cos\rho\, \Sigma_i^0 - \sin\rho\, \Lambda_i,$$

$$\Lambda^0 = \sin\rho\, \Sigma_i^0 + \cos\rho\, \Lambda_i,$$

where

$$\sin\rho \simeq 0.033\left(\frac{m_d - m_u}{m_d + m_u}\right). \quad (21)$$

Mixing between Σ^0 and Λ^0 is of course also generated by electromagnetism. But we believe this is a much smaller effect [provided $(m_d - m_u)/(m_d + m_u)$ is not too small], for an interesting reason. Once the high-frequency parts of one-photon exchange are absorbed in quark mass renormalization, the remaining effects are presumably well described by the Born terms (Fig. 2). This is the basis of the successful calculation of electromagnetic mass splittings carried out by Coleman

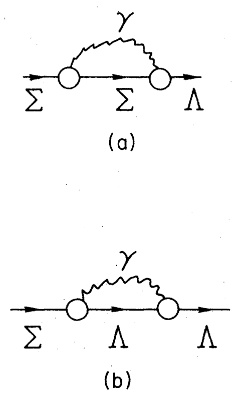

FIG. 2. Contributions of the Σ^0 and Λ intermediate states to Σ^0-Λ mixing through electromagnetism.

and Schnitzer.[11] In Σ^0-Λ^0 mixing there are two graphs to be considered, involving, respectively, intermediate Σ^0 and Λ^0 particles. In the SU(3) limit the two contributions cancel, since the Σ^0 and Λ^0 form factors are equal and opposite in this limit.[18] Since, in general, one-photon exchange generates self-masses of order 1 MeV, this added suppression suggests that the electromagnetic contribution to Σ^0-Λ^0 mixing is very small. We shall neglect it, therefore, and adopt the mixing angle of Eq. (21).

Dalitz and Van Hippel[19] have discussed the physical implications of Σ^0, -Λ^0 mixing for hypernuclear forces. The effects are rather difficult to separate from the other, symmetry-conserving forces. Mixing also has implications for $\Sigma^\pm \to \Lambda + e^\pm + \nu$ β decay. This reaction proceeds through coupling of the leptons to the weak vector and axial-vector currents, V_μ and A_μ; and one encounters (say for Σ^+ decay)

$$\begin{aligned} \langle \Sigma^+ | V_\mu | \Lambda \rangle &= \bar{u}(\Sigma)\left[F_1(q^2)\gamma_\mu + F_2(q^2)\frac{\sigma_{\mu\nu}q_\nu}{M_\Sigma + M_\Lambda} + F_3(q^2)\frac{q_\mu}{M_\Sigma + M_\Lambda}\right]u(\Lambda)\,, \\ \langle \Sigma^+ | A_\mu | \Lambda \rangle &= \bar{u}(\Sigma)\left[G_1(q^2)\gamma_\mu + G_2(q^2)\frac{\sigma_{\mu\nu}q_\nu}{M_\Sigma + M_\Lambda} + G_3(q^2)\frac{q_\mu}{M_\Sigma + M_\Lambda}\right]\gamma_5(\Lambda)\,. \end{aligned} \tag{22}$$

In the overall amplitude (after contraction with the lepton current matrix element), the form factors F_3 and G_3 are multiplied by the electron mass and so make negligible contributions. As for the form factor F_1, if isospin were an exact symmetry this would have to vanish (linearly in q^2) as $q^2 \to 0$. A nonvanishing value of $F_1(0)$ therefore signals and is sensitive to the isospin-violating effects of mixing. One finds

$$g_V \equiv F_1(0) = \sqrt{2}\ \sin\rho\,. \tag{23}$$

For the remaining form factors, as also for $F_1(q^2) - F_1(0) \simeq q^2(\partial F_1/\partial q^2)_0$, we can ignore the small mixing effects and appeal to standard SU(3) estimates, or experiment. The "charge radius" term $(\partial F_1/\partial q^2)_0$, which might obscure the mixing effect, can be related by SU(3) considerations to the electromagnetic charge radius of the neutron. The latter is known to be very small. In good approximation, therefore, we can set $F_1 = \sqrt{2}\ \sin\rho$, independent of q^2. This is the term of interest. The form factor analogous to G_2 for ordinary neutron β decay can arise there only from second-class currents, which are not part of the theoretical framework under discussion or indicated by experiment. In $\Sigma \to \Lambda + e + V$ decay, a G_2 term can nevertheless arise, in principle, through SU(3)-symmetry-breaking effects; but it seems reasonable nonetheless to suppose that it is negligible. For the remaining form factors F_2 and G_1 we may, for rough purposes, ignore their q^2 dependence. Since F_2 appears with a cofactor of order q/M it is kinematically suppressed, so that the decay process is dominated by $G_1 \simeq G_1(0) \equiv g_A$. From the $\Sigma^+ \to \Lambda + e^\pm + \nu$ decay rates one infers[20]

$$G_1(0) \equiv g_A \simeq 0.61\,. \tag{24}$$

The "magnetic" transition quantity $F_2(0)$ is not known experimentally, but SU(3) considerations relate it to the magnetic moment μ_n of the neutron and thus serve to indicate the expected magnitude

$$F_2(0) \equiv \mu \simeq -\left(\frac{3}{2}\right)^{1/2}\mu_n \approx 2.3\,. \tag{25}$$

The quantity of interest, g_V, is small compared to the dominant parameter g_A; but not hopelessly so if $(m_d - m_u)/(m_d + m_u)$ is of order unity. In this extreme limit g_V/g_A is about 8%.

Our essential prediction is for the quantity $F_1(0) \equiv g_V = \sqrt{2}\ \sin\rho$, where $\sin\rho$ is given in Eq. (21), together with the observation that $F_1(q^2) \approx F_1(0)$ should be essentially constant over the small range of q^2 involved in $\Sigma \to \Lambda + e + \nu$ β decay (q^2 is the square of the four-momentum transfer to the leptons). For the rest, a full experimental analysis should leave open the remaining form factors as functions of q^2. The full decay spectrum in all the variables of the process (electron and neutrino momenta, parent and daughter hyperon polarization) is very lengthy but can be straightforwardly worked out when the time comes. Here, only to indicate the kinds of effects one will have to deal with experimentally to get at g_V, we shall make several reasonable approximations. We ignore the form factors F_3, G_2 and G_3 and treat the others as constants, independent of q^2, with the notation $F_1(0)=g_V$, $G_1(0)=g_A$, and $F_2(0)=\mu$. Moreover, we treat g_V as small, as it surely is, so we retain g_V only where it interferes with the dominant g_A; and we ignore terms quadratic in μ since these are accompanied by kinematic suppression factors. Finally, we work only to first order in E/M, where E is the electron energy and $M=\frac{1}{2}(M_\Sigma+M_A)$; and we neglect the electron mass.

For unpolarized Σ hyperons, the decay spectrum in the Σ rest frame has the structure[21]

$$d\Gamma=\frac{G_F^2\cos^2\theta_C}{(2\pi)^5}W(E_0-E)^2E^2\,dE\,d\Omega_e\,d\Omega_\nu,$$

where

$$\begin{aligned}W=&f_1(E)+f_2(E)\hat p\cdot\hat k+f_3(E)[(\hat p\cdot\hat k)^2-\tfrac{1}{3}]\\&+f_4(E)\vec\sigma_\Lambda\cdot\hat p+f_5(E)\hat p\cdot\hat k\,\vec\sigma_\Lambda\cdot\hat p\\&+f_6(E)\vec\sigma_\Lambda\cdot\hat k+f_7(E)\,\hat p\cdot\hat k\,\vec\sigma_\Lambda\cdot\hat k\,.\end{aligned}\qquad(26)$$

Here E is the electron energy, E_0 its maximum value, $\hat p$ and $\hat k$ are unit vectors along the momenta of electron and neutrino respectively, and $\vec\sigma_\Lambda$ is the Λ particle spin operator. The spectrum summed over Λ spin is just $\frac{1}{2}\mathrm{Tr}W$ (i.e., the above expression without f_5, f_6, f_7); the expectation value of Λ polarization, $\vec{\mathrm{P}}$ is

$$\vec{\mathrm{P}}=\frac{\mathrm{Tr}\,W\vec\sigma_\Lambda}{\mathrm{Tr}W}\,.$$

With the approximations noted above, one finds

$$f_1=3g_A^2-\frac{2E_0}{M}(g_A^2\pm g_A\mu)+\frac{2E}{M}(5g_A^2\pm 2g_A\mu)\,,$$

$$f_2=-g_A^2+\frac{2E_0}{M}(g_A^2\pm g_A\mu)-\frac{4E}{M}(3g_A^2\pm g_A\mu)\,,$$

$$f_3=\frac{3E}{M}g_A^2\,,$$

$$f_4=2g_Ag_V\pm 2\left[g_A^2-\frac{E_0}{2M}(g_A^2\pm g_A\mu)+\frac{E}{2M}(5g_A^2\pm 3g_A\mu)\right]\,,$$

$$f_5=\pm\frac{E}{M}(5g_A^2\pm g_A\mu)\,,$$

$$f_6=2g_Ag_V\mp 2\left[g_A^2-\frac{E_0}{2M}(2g_A^2\pm 2g_A\mu)+\frac{E}{2M}(7g_A^2\pm 3g_A\mu)\right]\,,$$

$$f_7=\pm 2\left[\frac{E_0}{2M}(g_A^2\pm g_A\mu)-\frac{E}{2M}(7g_A^2\pm g_A\mu)\right]\,.$$

The upper (lower) signs refer to Σ^- (Σ^+) β decay.

The wanted g_Vg_A interference term appears in the correlations described by f_4 and f_6. Each is dominated by a kinematically unsupressed g_A^2 contribution. However, this dominant effect cancels in the sum f_4+f_6: In the sum, in addition to the g_Vg_A term, one meets g_A^2 and $g_A\mu$ both multiplied by the small factors E/M or E_0/M. The same thing happens for f_4 and f_6 separately if one sums either for both Σ^- and Σ^+ decay. One sees also that f_3 depends only on g_A^2 and that g_A^2 and $g_A\mu$ enter on a comparable footing (same kinematic suppression factors) in f_5.

It is of course a pity that none of the above correlations arises exclusively from g_Vg_A interference. With polarized Σ's, a much richer structure arises and there is now one correlation in particular that *does* depend exclusively on g_Vg_A interference: namely, among the complete set of additional correlations, one encounters the term in W

$$+2g_Vg_A\vec\sigma_\Lambda\cdot[\vec{\mathrm{P}}_\Sigma\times(\hat k\times\hat p)]\,,$$

where $\vec{\mathrm{P}}_\Sigma$ is the Σ polarization. This correlation is distinguished from all other spin-spin correlations in that it is antisymmetric under $\hat p\longleftrightarrow\hat k$.

It is evident from all of the above that experimental detection of the (at best) small vector coupling coefficient g_V, by one means or another, will be very demanding.

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