Critical behavior of hadronic matter: Critical-point exponents

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(Received 17 May 1978)

In statistical physics, the principle of maximum entropy provides conditions on the singular behavior of thermodynamic functions near a critical point, in the form of inequalities for the critical-point exponents. Hadronic matter, when described by an exponentially rising density of states (dual resonance, statistical bootstrap models), exhibits critical behavior at a finite temperature T_c. We calculate the basic critical-point exponents for such systems and investigate the range of validity of the corresponding inequalities.

I. INTRODUCTION

In statistical physics the principle of maximum entropy, as a basis for equilibrium, is assumed to be valid even in the vicinity of a critical point. By requiring the first derivatives of the entropy $S(E, V)$ with respect to energy E and volume V to vanish, this principle ensures uniform temperature and pressure throughout the system under consideration. For the resulting extremum to be a maximum, the second derivatives must satisfy the inequalities[1]

$$(\partial^2 S/\partial E^2)_V \leq 0\,, \tag{1a}$$

$$(\partial^2 S/\partial E^2)_V (\partial^2 S/\partial V^2)_E - (\partial^2 S/\partial E \partial V)^2 \geq 0 \tag{1b}$$

to make the determinant of the second derivatives negative. In terms of thermodynamic response functions, this leads to the stability conditions

$$C_V = (\partial U/\partial T)_V \geq 0\,, \tag{2a}$$

$$\kappa_T^{-1} = -V(\partial P/\partial V)_T \geq 0 \tag{2b}$$

for the specific heat at constant volume and the isothermal compressibility, respectively; here $U(T, V)$ denotes the internal energy of the system and $P(T, V)$ its pressure.

At a critical point $T = T_c$, the logarithmic derivative of the partition function $Z(T, V)$ becomes singular above a certain order. One can investigate such singularities by studying the behavior of different thermodynamic quantities in terms of

$$t = (T - T_c)/T_c = T/T_c - 1 \tag{3}$$

near $t = 0$. If a thermodynamic function $f(t)$ has for $t \approx 0$ the form

$$f(t) \sim (\pm t)^x\,, \tag{4}$$

then x is the corresponding critical (point) exponent, which is more generally defined by

$$x = \lim_{t \to 0} [\ln f(t)/\ln t]\,. \tag{5}$$

The $\pm$ sign in Eq. (4) is chosen according to whether t approaches zero from above or below. Various possible critical exponents thus provide a scheme for classifying the singular behavior associated with a given critical point. Of particular interest here will be the exponents associated with the specific heat at constant volume

$$C_V \sim (-t)^{-\alpha'}\,, \tag{6}$$

with the order parameter

$$1 - \nu/\nu_c \sim (-t)^{\beta}\,, \tag{7}$$

where $\nu = \overline{N}/V$, the particle density, and ν_c is the most singular part of ν as $T \to T_c$, with the isothermal compressibility

$$\kappa_T \sim (-t)^{-\gamma'}\,, \tag{8}$$

and with the critical isotherm

$$1 - P/P_c \sim |1 - \nu/\nu_c|^{\delta} \tag{9}$$

in terms of the order parameter. We have for definiteness taken $t < 0$ ($T \to T_c$ from below) and used the standard notation for the critical exponents.[2]

The thermodynamic stability conditions (2) can now be used to obtain inequalities for the critical exponents. One thus finds[2] the relations

$$\alpha' + 2\beta + \gamma' \geq 2\,, \tag{10}$$

$$\alpha' + \beta(1 + \delta) \geq 2\,, \tag{11}$$

which are commonly called Rushbrooke and Griffiths inequalities, respectively. We recall that they result from the requirement of maximum entropy, as a prerequisite for a statistical description, in the vicinity of a critical point.

Of special interest for statistical physics is the case when inequalities (10) and (11) become equalities: The *a priori* independent critical exponents can then be expressed in terms of a smaller set of parameters.

In Sec. II, we briefly review the essential features of hadronic matter when governed by an exponentially rising mass spectrum, concentrating, in particular, on the resulting critical behavior. In Sec. III, we then calculate the critical exponents for such systems[3] and investigate the range of validity of the corresponding inequalities.

II. THE STATISTICAL DESCRIPTION OF HADRONIC MATTER

The behavior of a system of many strongly interacting particles is at present far from solved. In accordance with dual resonance and statistical-bootstrap models, we shall assume that the dominant feature of hadron physics is the formation of resonances whose number $\tau(M)$ grows for large mass M linearly exponential in M (Refs. 4, 5):

$$\tau(M) \simeq cM^a e^{bM}, \quad a, b, c = \text{constants}. \tag{12}$$

The parameter b is of a rather universal nature in both models mentioned. In the dual resonance model, it is essentially determined by the slope of the Regge trajectories governing resonance distributions; in the statistical-bootstrap model, it is fixed by the range of hadronic forces. Both cases lead to $b^{-1} \sim 150$ MeV, i.e., b is roughly an inverse pion mass. The constant a, on the other hand, is somewhat more dependent on the version of the model used. Statistical-bootstrap schemes give $a \leqslant -\frac{5}{2}$ (Ref. 6); the most stringent version has $a = -3$.[7] In the dual resonance model, a depends on the dimension d of the corresponding oscillator space; for $d = 4$, one has $a = -\frac{5}{2}$, while higher-dimensional spaces lead to larger values of $-a$.[8]

Empirically, one has obtained b from transverse-momentum spectra in a great variety of collision experiments,[9] and the remarkable agreement found with both an exponential form and $b^{-1} \sim 150$ MeV provides strong support for a density of states (12). However, present models and data have not yet led to any significant information on a.

The essential input for the form (12) is a hadronic scale parameter (Regge slope, interaction range). If hadrons are composed of pointlike constituents, one may expect such a scale to lose its significance at sufficiently high energy density, at which hadronic matter would turn into quark matter. Such transitions are currently under intensive investigation, starting either from a constituent interaction theory[10] or from a Van der Waals type approach.[11] Here we want to study the aspects of critical behavior already inherent in the hadronic picture itself.

To obtain a statistical description of hadronic matter, we follow Beth and Uhlenbeck[12] and Belenkij[13] and consider an *ideal gas* containing the "basic" hadrons (for simplicity pions) and all possible resonances as noninteractive constituents. Such a system, in a volume V, then has the level density

$$\sigma(E, V, \lambda) = \sum_N \frac{(\lambda V)^N}{N!} \int \prod_1^N [dm_i \tau(m_i) d^3p_i] \times \delta^{(4)}\left(\sum_1^N p_i - P\right), \tag{13}$$

with λ denoting the (relativistic) fugacity and $E^2 = P_\mu P^\mu$ the squared total c.m. energy of the system. Note that for

$$\tau(m) = \delta(m-\mu), \quad \mu = \text{pion mass}, \tag{14}$$

relation (13) would reduce to the level density of an ideal relativistic pion gas. Quantum statistics will not be explicitly considered here.

For the grand canonical partition function

$$Z(\beta, V, \lambda) \equiv \int d^4P\, e^{-\beta_\mu P^\mu} \sigma(E, V, \lambda), \tag{15}$$

with $\beta \equiv (\beta_\mu \beta^\mu)^{1/2} = 1/T$ as inverse temperature, we have from Eq. (13)

$$\ln Z(\beta, V, \lambda) = \lambda V \varphi_\tau(\beta), \tag{16}$$

with

$$\varphi_\tau(\beta) \equiv \int_\mu^\infty dm\, \tau(m) \int d^3p\, e^{-\beta_\mu p^\mu}. \tag{17}$$

The virial equations for the pressure P and the particle density ν are then

$$P = (\lambda/\beta V) \ln Z(\beta, V, \lambda) = (\lambda/\beta)\varphi_\tau(\beta), \tag{18}$$

$$\nu = (\lambda/V)\partial \ln Z/\partial\lambda = \lambda\varphi_\tau(\beta). \tag{19}$$

Eliminating the fugacity λ in Eqs. (18) and (19) yields

$$P = \nu/\beta, \tag{20}$$

which is the equation of state of an *ideal gas*. The presence of strong interactions is reflected only in the fact that $\tilde{N} = \nu V$ is the mean number of pions and resonances, not the average pion number.

We now insert the mass spectrum (12) into the generating function (17) to find

$$\varphi_\tau(\beta) = (4\pi c/\beta) \int_\mu^\infty dm\, m^{a+2} e^{bm} K_2(m\beta), \tag{21}$$

where $K_2(x)$ is a modified Hankel function, which for large argument becomes

$$K_2(x) \approx (\pi/2x)^{1/2} e^{-x}[1 + O(x^{-1})]. \tag{22}$$

Near $\beta = b$, this yields

$$\varphi_\tau(\beta) \simeq c(2\pi/b)^{3/2}(\beta - b)^{-(a+5/2)} \times \Gamma(a + \tfrac{5}{2}, \mu(\beta - b)), \tag{23}$$

with $\Gamma(x, y)$ denoting the incomplete Γ function. For $a + \frac{5}{2} \neq 0, -1, -2, \ldots,$ we have

$$\Gamma(a+\tfrac{5}{2},\mu(\beta-b))(\beta-b)^{-(a+5/2)} \equiv F_a(\beta-b)$$
$$\simeq (\beta-b)^{-(a+5/2)}\Gamma(a+\tfrac{5}{2}) - \mu^{(a+5/2)}/(a+\tfrac{5}{2}) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}\mu^{k+a+5/2}}{(k+a+\tfrac{5}{2})k!}(\beta-b)^k\,, \quad (24)$$

while for $a=-\frac{5}{2}$,

$$F_{-5/2}(\beta-b) \simeq -\ln(\beta-b) - \mu - C + O(\beta-b) \quad (25)$$

and

$$F_a(\beta-b) \simeq (-1)^{n+1}(1/n!)\left\{\sum_{k=0}^{n-1}(-1)^k k!\,[\mu(\beta-b)]^{n-k-1} + [\mu(\beta-b)]^n \ln[\mu(\beta-b)] + O((\beta-b)^n)\right\} \quad (26)$$

for $a+\frac{5}{2}=-n=-1,-2,\ldots$.

From Eqs. (23)–(26) we see that $T=T_c \simeq 1/b$ constitutes a critical temperature of the system[14,15]: Above a certain order, all derivatives of $\phi_T(\beta)$ with respect to β [or already $\phi_T(\beta)$ itself] become singilar as $\beta \to b$ ($T \to T_c$). The nature of this critical point becomes more transparent if we consider the energy density

$$\epsilon = U/V = (-1/V)\partial \ln Z/\partial\beta$$
$$= -\lambda\,\partial\varphi_T(\beta)/\partial\beta\,, \quad (27)$$

where U denotes the internal energy of the system. From Eqs. (23)–(26) we find

$$\epsilon \sim \begin{cases}(\beta-b)^{-(a+7/2)}, & a>-\frac{7}{2}\\ -\ln(\beta-b), & a=\frac{7}{2}\\ \text{const}, & a<-\frac{7}{2}\end{cases} \quad (28)$$

as $\beta \to b$. Thus for $a \geqslant -\frac{7}{2}$, the temperature of the system approaches its critical value only as the energy density becomes infinite: $T_c = 1/b$ in this case is a *limiting temperature*.[4] For $a<-\frac{7}{2}$, $T=T_c$ is reached at a finite energy density ϵ_c: the system thus exists for $\epsilon > \epsilon_c$, and we have, in general, some type of phase transition.[14,15]

III. THE CRITICAL EXPONENTS OF HADRONIC MATTER

For the specific heat at constant volume, we have from Eq. (27)

$$C_V \equiv (-V/T^2)(\partial\epsilon/\partial\beta)_V$$
$$= (\lambda V/T^2)(\partial^2\varphi_T(\beta)/\partial\beta^2)_V\,. \quad (29)$$

From the definition of $\phi_T(\beta)$ it is clear that the stability condition (2a) is always satisfied. Inserting Eqs. (23)–(26) in Eq. (29) yields for $\beta \simeq b$

$$C_V \simeq C(2\pi/b)^{3/2} \times \begin{cases}(\beta-b)^{-(a+9/2)}\Gamma(a+\frac{9}{2}), & a>-\frac{9}{2}\\ -\ln(\beta-b), & a=-\frac{9}{2}\\ \text{const}, & a<-\frac{9}{2}\,.\end{cases} \quad (30)$$

Hence the critical exponent α' is given by

$$\alpha' = \begin{cases}a+\frac{9}{2}, & a \geqslant -\frac{9}{2}\\ 0, & a \leqslant -\frac{9}{2}\,.\end{cases} \quad (31)$$

Using Eq. (20), we next obtain the isothermal compressibility of an ideal gas,

$$\kappa_T^{-1} = -V(\partial P/\partial V)_{\overline{N},\beta} = \nu/\beta\,, \quad (32)$$

which, with Eq. (19), yields

$$\kappa_T^{-1} = \lambda\varphi_T(\beta)/\beta\,. \quad (33)$$

Inserting the hadronic form (23)–(26) leads to the critical exponent γ':

$$\gamma' = \begin{cases}-(a+\frac{5}{2}), & a \geqslant -\frac{5}{2}\\ 0, & a \leqslant -\frac{5}{2}\,.\end{cases} \quad (34)$$

The critical part of the particle density is found to be

$$\nu_c \sim \begin{cases}(\beta-b)^{-(a+5/2)}, & a>-\frac{5}{2}\\ -\ln(\beta-b), & a=-\frac{5}{2}\\ \text{const}, & a<-\frac{5}{2}\end{cases} \quad (35)$$

TABLE I. Critical exponents.

	$a \leq -\frac{9}{2}$	$-\frac{9}{2} \leq a \leq -\frac{7}{2}$	$-\frac{7}{2} \leq a \leq -\frac{5}{2}$	$-\frac{5}{2} \leq a$
α'	0	$a+\frac{9}{2}$	$a+\frac{9}{2}$	$a+\frac{9}{2}$
β	1	1	$-(a+\frac{5}{2})$	$a+\frac{5}{2}$
γ'	0	0	0	$-(a+\frac{5}{2})$
δ	1	1	1	1

TABLE II. Rushbrooke and Griffiths inequalities.

	$a \le -\frac{9}{2}$	$-\frac{9}{2} \le a \le -\frac{7}{2}$	$-\frac{7}{2} \le a \le -\frac{5}{2}$	$-\frac{5}{2} \le a$
Δ_R	0	$(a+\frac{9}{2}) \ge 0$	$-(a+\frac{5}{2}) \ge 0$	$2(a+\frac{5}{2}) \ge 0$
Δ_G	0	$(a+\frac{9}{2}) \ge 0$	$-(a+\frac{5}{2}) \ge 0$	$3(a+\frac{5}{2}) \ge 0$

yielding for the critical behavior of the order parameter

$$1-\nu/\nu_c \sim \begin{cases} (\beta-b)^{a+5/2}, & a > -\frac{5}{2} \\ -1/\ln(\beta-b), & a = -\frac{5}{2} \\ (\beta-b)^{-(a+5/2)}, & -\frac{5}{2} > a > -\frac{7}{2} \\ -(\beta-b)\ln(\beta-b), & a = -\frac{7}{2} \\ (\beta-b), & a < -\frac{7}{2}. \end{cases} \tag{36}$$

For the exponent β we thus have

$$\beta = \begin{cases} a+\frac{5}{2}, & a \geqslant -\frac{5}{2} \\ -(a+\frac{5}{2}), & -\frac{5}{2} \geqslant a \geqslant -\frac{7}{2} \\ 1, & a \leqslant -\frac{7}{2}. \end{cases} \tag{37}$$

Finally, we consider the critical isotherm as a function of the order parameter, obtaining

$$1-P/P_c \sim 1-\nu/\nu_c \tag{38}$$

from Eq. (20), so that

$$\delta = 1 \tag{39}$$

for all a, as a consequence of the ideal gas character of the systems considered here.

Our results for the critical exponents are summarized in Table I. We note here already that their values are determined if one parameter, the exponent a in the spectrum (12), is fixed.

Consider now the quantities

$$\Delta_R = \alpha' + 2\beta + \gamma' - 2, \tag{40}$$

$$\Delta_G = \alpha' + \beta(1+\delta) - 2; \tag{41}$$

they must be greater than, or equal to, zero, whenever the Rushbrooke and Griffiths inequalities hold. In Table II the values of Δ_K and Δ_G are shown for the various intervals of a. We see that both inequalities are satisfied for all a, and that for $a = -\frac{5}{2}$ and $a \leqslant -\frac{9}{2}$ the inequalities become equalities.

IV. CONCLUSIONS

We have shown that the statistical description of hadronic matter, as obtained with an exponentially rising resonance spectrum, satisfies the Rushbrooke and Griffiths inequalities for critical-point exponents. Such a description thus fulfills the prerequisites of a statistical approach even in the vicinity of the critical point implied by the spectrum.

The results obtained here suggest two further directions of investigation. In statistical physics, equalities in the relations between critical exponents are obtained if scale invariance is assumed for suitable thermodynamic quantities.[2] One then finds expressions for the four exponents α', β, γ', and δ in terms of two parameters x_1 and x_2:

$$\alpha' = 2 - x_1, \tag{42a}$$

$$\beta = (1-x_2)x_1, \tag{42b}$$

$$\gamma' = (2x_2 - 1)x_1, \tag{42c}$$

$$\delta = x_2/(1-x_2). \tag{42d}$$

Do these relations hold for our problem? From Table I we see that

$$x_1 = 2, \quad x_2 = \tfrac{1}{2}, \quad a \leqslant -\tfrac{9}{2} \tag{43}$$

and

$$x_1 = 0, \quad x_2 = \tfrac{1}{2}, \quad a = -\tfrac{5}{2} \tag{44}$$

indeed lead to our exponents in that range of a, where Δ_R and Δ_G both vanish. Moreover, such a parametrization is possible only there; for example, for $-\frac{7}{2} < a < -\frac{5}{2}$, relations (42a) and (42d) imply $x_1 = -(a+\frac{5}{2}), x_2 = \frac{1}{2}$, which disagrees with both β and γ' from Table I.

Our results are thus in accord with the predictions of thermodynamic scaling, and it would be of great interest to study the relation between the statistical-bootstrap approach (as a means of obtaining the hadron thermodynamics of Sec. II) and the scaling-law hypothesis or considerations leading to such scaling laws (Kadanoff construction, renormalization group).

Another line of investigation appears if one considers further critical-point inequalities. By making additional assumptions of physical plausibility and/or mathematical simplicity, and also by considering the exponents associated with the critical behavior of other thermodynamic quantities than the ones above, many more inequalities can be derived.[2] Examples are the "further" Griffiths inequalities

$$\gamma' \geqslant \beta(\delta - 1), \tag{45}$$

$$\gamma(\delta+1) \geqslant (2-\alpha)(\delta-1), \tag{46}$$

which both break down here for $a > -\frac{5}{2}$ (above a

$= -\frac{5}{2}$, both P and ν diverge as $T \to T_c$). An investigation of such further relations, their underlying assumptions, and their range of validity, should provide further information about the critical behavior of hadronic matter.

ACKNOWLEDGMENTS

Stimulating discussions with R. Hagedorn, F. Karsch, and D. Miller are gratefully acknowledged.

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[1]See, e.g., L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1958), p. 60.

[2]See, e.g., H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Clarendon, Oxford, 1971); M. E. Fisher, Rep. Prog. Phys. 30, 615 (1967).

[3]For previous work on this, see D. E. Miller, Phys. Lett. 64B, 193 (1976).

[4]R. Hagedorn, Nuovo Cimento Suppl. 3, 147 (1965).

[5]S. Fubini and G. Veneziano, Nuovo Cimento 64A, 811 (1969); K. Bardakçi and S. Mandelstam, Phys. Rev. 184, 1640 (1969).

[6]See e.g., J. Letessier *et al.*, Nuovo Cimento 45A, 580 (1978).

[7]S. Frautschi, Phys. Rev. D 3, 2821 (1971).

[8]K. Huang and S. Weinberg, Phys. Rev. Lett. 25, 895 (1970).

[9]See survey in H. Satz, Report No. CERN-TH 2532, 1978 (unpublished) and Proceedings of the Symposium on Jets in High Energy Collision, Copenhagen, 1978, Phys. Scr. (to be published).

[10]See V. Baluni, Phys. Rev. D 17, 2092 (1978), and references therein.

[11]See T. Celik and H. Satz, Z. Phys. C (to be published).

[12]E. Beth and G. E. Uhlenbeck, Physica 4, 915 (1937).

[13]S. Z. Belenkij, Nucl. Phys. 2, 259 (1956).

[14]R. D. Carlitz, Phys. Rev. D 5, 3231 (1972).

[15]N. Cabibbo and G. Parisi, Phys. Lett. 59B, 67 (1974).