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Strong-coupling expansion in quantum field theory

Carl M. Bender
Department of Physics, Washington University, St. Louis, Missouri 63130

Fred Cooper, G. S. Guralnik,* and David H. Sharp
Theoretical Division, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87545

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We derive a simple and general diagrammatic procedure for obtaining the strong-coupling expansion of a d-dimensional quantum field theory, starting from its Euclidean path-integral representation. At intermediate stages we are required to evaluate diagrams on a lattice; the lattice spacing provides a cutoff for the theory. We formulate a simple Padé-type prescription for extrapolating to zero lattice spacing and thereby obtain a series of approximants to the true strong-coupling expansion of the theory. No infinite quantities appear at any stage of the calculation. Moreover, all diagrams are simple to evaluate (unlike the diagrams of the ordinary weak-coupling expansion) because nothing more than algebra is required, and no diagram, no matter how complex, generates any transcendental quantities. We explain our approach in the context of a $g\phi^4$ field theory and calculate the two-point and four-point Green's functions. Then we specialize to $d = 1$ (the anharmonic oscillator) and compare the locations of the poles of the Green's functions with the tabulated numerical values of the energy levels. The agreement is excellent. Finally, we discuss the application of these techniques to other models such as $g\phi^{2N}$, $g(\bar{\psi}\psi)^2$, and quantum electrodynamics.

I. INTRODUCTION

In this paper we formulate a simple diagrammatic prescription for obtaining the strong-coupling expansion of a quantum field theory. We begin by expressing the vacuum functional of a quantum field theory in the presence of an external source as a Euclidean path integral. Then we factor out the kinematical parts of the Lagrangian from the path integral and evaluate the remaining non-Gaussian path integral in closed form. We obtain a formal expansion of the vacuum functional as a series in inverse powers of the coupling constant.

From this expansion we extract a set of simple diagrammatic rules which can be used to compute all of the n-point Green's functions of the theory. (These rules are dual to the ordinary Feynman rules of weak-coupling theory.) The integrals that must be evaluated are formally divergent in any space-time dimension $d>0$. However, if we regularize them by considering them as sums on a lattice, they become finite and simple to evaluate. Having performed all these summations we note that the series which results, a series in inverse powers of the coupling constant, also involves inverse powers of a, the lattice spacing. Thus, every term in this series becomes infinite as the lattice spacing is taken to 0. We identify this series as a high-temperature expansion of the *lattice theory* associated with the quantum field theory we originally set out to solve (it is not the strong-coupling expansion of the quantum field theory). The final step of our calculational procedure is to extrapolate the high-temperature expansion to zero lattice spacing and thereby to obtain the true strong-coupling expansion of the theory.

The procedure we have just described extends to any quantum field theory, regardless of how many fields there are in the Lagrangian, the spins of these fields, or the dimension of space-time. In contrast with ordinary Feynman perturbation theory in powers of g, every stage of the calculation requires nothing more advanced than algebraic manipulation. No transcendental or irrational numbers or functions appear in any finite order as a result of graphical integration.

Our paper is organized as follows: Sections II to V give a very detailed exposition of our prescription applied to a $g\phi^4$ theory in d-dimensional space-time. In Sec. II we perform a formal expansion of the path integral, in Sec. III we show how to evaluate the resulting diagrams on a lattice, and in Sec. IV we discuss various ways to extrapolate to zero lattice spacing and thus to obtain the true strong-coupling expansion. Then in Sec. V we present some of our numerical results. We find that for the case $d=1$ (the anharmonic oscillator), where we can compare with previously published computer calculations, we obtain extremely good results. Finally, in Sec. VI we discuss the strong-coupling calculation for quantum field theories other than $g\phi^4$ theory. For example, we find that the expansion of a $g(\bar{\psi}\psi)^2$ theory in two space-time dimensions is much simpler than that of a $g\phi^4$ theory.

Some of the work reported in this paper is not new. Since completing this work we have discovered many other papers, some of whose intents and approaches overlap to a limited extent with some of ours. For example, Ward[1] was aware that it is possible to expand a $g\phi^4$ theory in inverse powers of $\sqrt{g}$. However, he did not regularize the resulting formal expansion on a lattice nor was he able to eliminate satisfactorily the infinities that appear in the formal expansion. Hori[2] realized that it is possible to expand a Green's function in inverse powers of the free propagator. However, again he was unable to deal with the infinities in his expansion. Another interesting approach to strong-coupling theory, introduced by Caianiello[3] and Caianiello and Scarpetta[4] and advanced by Kainz[5] and Kövesi-Domokos,[6] is called the static ultralocal approximation. This is, in fact, just the first term in the strong-coupling series discussed in this paper. The work by Kövesi-Domokos is the most interesting of this portion of the literature; it recognizes that the ultralocal approximation is the first term of a strong-coupling series (the same kind of series as in this paper), that this series can be derived using functional methods, and that it has a diagrammatic representation.[7] However, again the lattice is not used to regularize the terms in the expansion and the true strong-coupling expansion, which is found by extrapolating to zero lattice spacing, is not found.

There have also been many papers on the behavior of quantum field theory on a lattice. Works by Schiff,[8] Baker,[9] Wilson,[10] Balian, Drouffe, and Itzykson,[11] Drell, Weinstein, and Yankielowicz,[12] Banks, Susskind, and Kogut,[13] and very recently Benzi, Martinelli, and Parisi[14] constitute major advances. Our attitude is somewhat different from that expressed in these papers in the sense that we do not take the lattice to be at all fundamental. Rather, we regard it as a mere computational device for evaluating diagrams. Of all these papers on lattice theories, the one which comes closest to the approach of the present paper is that by Benzi, Martinelli, and Parisi. However, even this paper differs greatly from ours in its approach of setting up the high-temperature expansion on the lattice and in the method used to extrapolate to zero lattice spacing.

II. FORMAL EXPANSION OF $g\phi^4$ THEORY IN INVERSE POWERS OF g

The vacuum persistence functional $Z[J]$ for a d-dimensional $g\phi^4$ quantum field theory in the presence of an external source $J(x)$ can be expressed in Euclidean space as a functional integral:

$$Z[J]=\int D\phi\,\exp\left\{-\int dx[\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}g\phi^4+J\phi]\right\}. \tag{2.1}$$

From this formula one can derive *both* weak- and strong-coupling expansions for $Z[J]$.

A. Weak-coupling expansion

Let us first review the derivation of the weak-coupling expansion. To expand $Z[J]$ in powers of g (weak-coupling expansion) we pull out the interaction term from the functional integral as a functional differential operator:

$$Z[J]=\exp\left[-\tfrac{1}{4}g\int dx\frac{\delta^4}{\delta J(x)^4}\right]\times\int D\phi\,\exp\left\{-\int dx[\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}m^2\phi^2+J\phi]\right\}. \tag{2.2}$$

The remaining functional integral is Gaussian and can therefore be evaluated in closed form,

$$Z[J]=N\exp\left[-\tfrac{1}{4}g\int dx\frac{\delta^4}{\delta J(x)^4}\right]\times\exp\left[\tfrac{1}{2}\int\int dx\,dy\,J(x)G(x,y)J(y)\right], \tag{2.3}$$

where N is a normalization constant which does not depend on J and $G(x,y)$ is the free Euclidean Green's function satisfying

$$(-\partial_x{}^2+m^2)G(x,y)=\delta(x-y)\,. \tag{2.4}$$

If we now expand both exponentials in (2.3) in powers of g and $G(x,y)$, respectively, and multiply these series together, we obtain the formal weak-coupling expansion of the vacuum functional:

$$Z[J]=N\left\{1+\sum_{k=1}^{\infty}g^kA_k[J]\right\}. \tag{2.5}$$

To construct the n-point Green's functions $W_n(x_1,\ldots,x_n)$ of the theory we compute n functional derivatives of $\ln Z[J]$:

$$W_n(x_1,\ldots,x_n)\equiv\frac{\delta^n}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}\ln Z[J]\bigg|_{J\equiv0}. \tag{2.6}$$

Note that the normalization constant N in (2.5) drops out once the functional derivatives in (2.6) are performed. The final result is an expansion for W_n in powers of the unrenormalized coupling constant g; this is the conventional Feynman diagram expansion using connected graphs. Each vertex of the graphs in this expansion is a four-point vertex and contributes one factor of the parameter g; this is because the functional differential operator in (2.3) is a four-point vertex insertion. Each line of the graphs in this expansion is associated with one factor of $G(x,y)$.

B. Strong-coupling expansion

It is just as easy to expand $Z[J]$ in (2.1) in inverse powers of g (strong-coupling expansion). This time we pull out the kinematical term from the functional integral as a functional differential operator:

$$Z[J]=\exp\left[-\frac{1}{2}\int\int dx\,dy\frac{\delta}{\delta J(x)}G^{-1}(x,y)\frac{\delta}{\delta J(y)}\right]\times\int D\phi\exp\left[-\int dx(\tfrac{1}{4}g\phi^4+J\phi)\right],\qquad(2.7)$$

where

$$G^{-1}(x,y)=(-\partial^2+m^2)\delta(x-y).\qquad(2.8)$$

Equation (2.7) is the analog of (2.2). The remaining functional integral in (2.7), even though it is not Gaussian, is very easy to evaluate because it can be viewed as an infinite product of ordinary integrals of the form

$$F(x)\equiv\int_{-\infty}^{\infty}dt\exp(-\tfrac{1}{4}t^4-xt),\qquad(2.9)$$

one for each space-time point y of $\phi(y)$.

We argue as follows: On a lattice a trace is a dimensionless sum over space-time points while an integral is a dimensional object. The relation between these two is given by

$$\sum_x=\delta(0)\int dx.\qquad(2.10)$$

By $\delta(0)$ we mean a^{-d}, where a is the lattice spacing and d is the space-time dimension. We thus approximate the functional integral in (2.7), which we denote by $Q[J]$, on a lattice by

$$\begin{aligned}Q[J]&\equiv\int D\phi\exp\left[-\int dx(\tfrac{1}{4}g\phi^4+J\phi)\right]\\&=\int\cdots\int\Big(\prod_x d\phi_x\Big)\exp\left[-\frac{1}{\delta(0)}\sum_x(\tfrac{1}{4}g\phi_x{}^4+J_x\phi_x)\right]\\&=N\prod_x\int d\theta_x\exp\left[-\tfrac{1}{4}\theta_x{}^4-\frac{J_x\theta_x}{g^{1/4}\delta(0)^{3/4}}\right],\end{aligned}\qquad(2.11)$$

where we have made the change of variables $\phi_x=\theta_x g^{-1/4}\delta(0)^{1/4}$ for each lattice point x, and N is a normalization constant which will drop out when we compute the Wightman functions of the theory. [N here is not the same N as in (2.3).] We thus have

$$\begin{aligned}Q[J]&=N\prod_x F[J_xg^{-1/4}\delta(0)^{-3/4}]\\&=N\exp\left\{\sum_x\ln F[J_xg^{-1/4}\delta(0)^{-3/4}]\right\}\\&=N\exp\left\{\delta(0)\int dx\ln F[J(x)g^{-1/4}\delta(0)^{-3/4}]\right\},\end{aligned}\qquad(2.12)$$

where F is the function defined in (2.9) and in the last line we have reverted to continuum language. Of course, the lattice limit cannot really be performed at this stage because $\delta(0)=\infty$; however, the use of continuum language does not introduce ambiguities and greatly simplifies the presentation.

The function $F(x)$ in (2.9) and (2.12) is a transcendental function whose properties are crucial to the structure of the strong-coupling expansion. We therefore enumerate the properties of $F(x)$ below. First, $F(x)$ has a Taylor expansion which converges for all finite x:

$$F(x)=\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{x^{2n}2^n}{(2n)!}\Gamma(\tfrac{1}{2}n+\tfrac{1}{4}).\qquad(2.13)$$

Observe from (2.13) that $F(x)$ is an even function. Second, $F(x)$ satisfies a differential equation which is reminiscent of the Airy equation:

$$F'''(x)=xF(x).\qquad(2.14)$$

Third, $F(x)$ has an asymptotic expansion of the form

$$F(x)\sim2\left(\frac{2\pi}{3}\right)^{1/2}x^{-1/3}\exp(\tfrac{3}{4}x^{4/3})\quad(x\to\infty).\qquad(2.15)$$

This third property can be used to recover the weak-coupling expansion from the strong-coupling expansion. We do not carry out this demonstration here.

Now we return to the result in (2.12). There are many ways to verify this result. The simplest is to demonstrate that $Q[J]$ in (2.11) satisfies the functional differential equation

$$\frac{\delta^3}{\delta J(x)^3}Q[J]=J(x)Q[J]/g,\qquad(2.16)$$

which is analogous to the ordinary differential equation (2.14) satisfied by $F(x)$. To show that $Q[J]$ in (2.12) satisfies (2.16) we functionally differentiate three times:

$$\begin{aligned}\frac{\delta Q[J]}{\delta J(x)}&=\frac{\delta}{\delta J(x)}N\exp\left\{\delta(0)\int dt\ln F[J(t)g^{-1/4}\delta(0)^{-3/4}]\right\}\\&=\frac{\delta(0)^{1/4}g^{-1/4}F'[J(x)g^{-1/4}\delta(0)^{-3/4}]}{F[J(x)g^{-1/4}\delta(0)^{-3/4}]}Q[J],\end{aligned}$$

$$\frac{\delta^2Q[J]}{\delta J(x)^2}=\frac{\delta(0)^{1/2}g^{-1/2}F''[J(x)g^{-1/4}\delta(0)^{-3/4}]}{F[J(x)g^{-1/4}\delta(0)^{-3/4}]}Q[J],$$

where we have used $\delta J(x)/\delta J(x)=\delta(0)$, and

$$\begin{aligned}\frac{\delta^3Q[J]}{\delta J(x)^3}&=\frac{\delta(0)^{3/4}g^{-3/4}F'''[J(x)g^{-1/4}\delta(0)^{-3/4}]}{F[J(x)g^{-1/4}\delta(0)^{-3/4}]}Q[J]\\&=J(x)g^{-1}Q[J],\end{aligned}$$

where we have used (2.14). This verifies (2.16).

Now we have achieved the result that

$$Z[J]=N\exp\left[-\tfrac{1}{2}\int\int dx\,dy\frac{\delta}{\delta J(x)}G^{-1}(x,y)\frac{\delta}{\delta J(y)}\right]\times\exp\left\{\delta(0)\int dx\ln F[J(x)g^{-1/4}\delta(0)^{-3/4}]\right\}. \tag{2.17}$$

This formula is the analog of (2.3).

Next we expand both exponentials in (2.17) and multiply the two series together, taking the indicated functional derivatives with respect to J. [To expand the second exponential, we use the formula in (2.13) for the Taylor expansion for $F(x)$; the Taylor expansion converges for all values of $J(x)g^{-1/4}\delta(0)^{-3/4}$, and we may assume that it converges especially rapidly because $\delta(0)$ is large, g is large, and $J(x)$ is small (it will ultimately be set equal to 0).] The result assumes the general form

$$Z[J]=N\left\{1+\sum_{k=1}^{\infty}g^{-k/2}B_k[J]\right\}, \tag{2.18}$$

in which $B_k[J]$ are integrals over the source function J. This expansion is the analog of the weak-coupling expansion of the vacuum functional in (2.5).

Finally, we expand the logarithm of $Z[J]$ in powers of $g^{-1/2}$. This computation extracts from the series in (2.18) just those terms which are representable as connected diagrams (these are the terms which do not break up into products of integrals). The general form of this series is

$$\ln Z[J]=C_0+\sum_{k=1}^{\infty}g^{-k/2}C_k[J], \tag{2.19}$$

where C_0 is a constant independent of J and

$$C_1[J]=-R\delta(0)^{-1/2}\int dx[J(x)^2+G^{-1}(x,x)],$$

$$C_2[J]=R^2\delta(0)^{-1}2\int\int dx\,dy[J(x)G^{-1}(x,y)J(y)+G^{-1}(x,y)^2]$$
$$+(\tfrac{1}{24}-\tfrac{1}{2}R^2)\delta(0)^{-2}\int dx[J(x)^4+6G^{-1}(x,x)J(x)^2+3G^{-1}(x,x)^2],$$

$$C_3[J]=-R^3\delta(0)^{-3/2}\int\int\int dx\,dy\,dz[4J(x)G^{-1}(x,y)G^{-1}(y,z)J(z)+\tfrac{4}{3}G^{-1}(x,y)G^{-1}(y,z)G^{-1}(z,x)]$$
$$+(R^3-\tfrac{1}{12}R)\delta(0)^{-5/2}\int\int dx\,dy[4J(x)G^{-1}(x,y)J(y)^3+6G^{-1}(x,y)^2J(x)^2$$
$$+12G^{-1}(x,x)J(x)J(y)G^{-1}(x,y)+6G^{-1}(x,x)G^{-1}(x,y)^2]$$
$$+(\tfrac{1}{10}R-R^3)\delta(0)^{-7/2}\int dx[\tfrac{1}{3}J(x)^6+5J(x)^4G^{-1}(x,x)+15J(x)^4G^{-1}(x,x)^2+5G^{-1}(x,x)^3],$$

$$C_4[J]=2R^4\delta(0)^{-2}\int\int\int\int dx\,dy\,dz\,dw[4J(x)G^{-1}(x,y)G^{-1}(y,z)G^{-1}(z,w)J(w)$$
$$+G^{-1}(x,y)G^{-1}(y,z)G^{-1}(z,w)G^{-1}(w,x)]$$
$$+(\tfrac{1}{12}R^2-R^4)\delta(0)^{-3}\int\int\int dx\,dy\,dz[8G^{-1}(x,y)G^{-1}(y,z)J(x)J(z)^3$$
$$+12G^{-1}(x,y)G^{-1}(y,z)J(x)J(y)^2J(z)+24J(x)J(y)G^{-1}(x,y)G^{-1}(x,z)^2$$
$$+12G^{-1}(x,z)G^{-1}(y,z)G^{-1}(x,y)J(x)^2$$
$$+12G^{-1}(x,z)G^{-1}(y,z)G^{-1}(z,z)J(x)J(y)$$
$$+24G^{-1}(x,z)G^{-1}(y,z)G^{-1}(y,y)J(x)J(y)+6G^{-1}(x,y)^2G^{-1}(x,z)^2$$
$$+12G^{-1}(x,z)G^{-1}(y,z)G^{-1}(x,y)G^{-1}(z,z)]$$
$$+(R^4-\tfrac{1}{10}R^2)\delta(0)^{-4}\int\int dx\,dy[4G^{-1}(x,y)J(x)J(y)^5+10G^{-1}(x,y)^2J(x)^4$$
$$+40G^{-1}(x,y)J(x)^3J(y)G^{-1}(x,x)+60G^{-1}(x,y)^2J(y)^2G^{-1}(y,y)$$
$$+60J(x)J(y)G^{-1}(x,y)G^{-1}(x,x)^2+30G^{-1}(x,y)^2G^{-1}(x,x)^2]$$

$$
\begin{aligned}
&+\left(R^4-\tfrac{1}{6}R^2+\tfrac{1}{144}\right)\delta(0)^{-4}\int\int dx\,dy\big[2J(x)^3G^{-1}(x,y)J(y)^3+9J(x)^2J(y)^2G^{-1}(x,y)^2\\
&\qquad+12J(x)G^{-1}(x,y)J(y)^3G^{-1}(x,x)+12J(x)J(y)G^{-1}(x,y)^3\\
&\qquad+18J(x)^2G^{-1}(y,y)G^{-1}(x,y)^2\\
&\qquad+18J(x)J(y)G^{-1}(x,y)G^{-1}(x,x)G^{-1}(y,y)+3G^{-1}(x,y)^4\\
&\qquad+9G^{-1}(x,x)G^{-1}(x,y)^2G^{-1}(y,y)\big]\\
&+\left(-\tfrac{1}{1344}+\tfrac{1}{30}R^2-\tfrac{1}{4}R^4\right)\delta(0)^{-5}\int dx\big[J(x)^8+28J(x)^6G^{-1}(x,x)+210J(x)^4G^{-1}(x,x)^2\\
&\qquad+420G^{-1}(x,x)^3J(x)^2+105G^{-1}(x,x)^4\big]\,,
\end{aligned}
$$

where

$$R=\Gamma(3/4)/\Gamma(1/4) \doteq 0.337\,989\,120\,. \tag{2.20}$$

The symbol $\doteq$ means terminated decimal. [The transcendental ratio R originates from the coefficients of the Taylor expansion of $F(x)$ in (2.13). R is the *only* irrational number in all of the expansion coefficients $C_1, C_2, C_3, \ldots$. Numbers such as π, e, or $\sqrt{2}$ never appear in this perturbation expansion.] This expansion is valid in any number of space-time dimensions.

The derivation of the expansion in (2.19) completes the objective of this section. In the next section we calculate the Green's functions of the theory by taking functional derivatives of (2.19).

III. GRAPHICAL EXPANSION OF THE GREEN'S FUNCTIONS

A. Diagrammatic rules

The next step in this calculation is to use (2.6) to generate the n-point Green's functions of the theory from (2.19). Fortunately, we find that the n-point Green's functions obtained by differentiating the complicated-looking expansion in (2.19) can actually be represented as a diagrammatic expansion with a very simple set of accompanying rules. The rules for constructing the $(1/\sqrt{g})^k$ term in the expansion of the n-point Green's function are as follows:

(1) Draw *all connected graphs* having a total of n external lines and k-$n/2$ internal lines; note that we must have $k \geq n/2$. The vertices of these graphs may have any *even* number of lines (external or internal) coming together.

Associate with every $2p$-line vertex the amplitude λ_{2p}. The vertex amplitudes have the form

$$\lambda_{2p}=g^{-p/2}\delta(0)^{1-3p/2}L_{2p}\,,$$

where L_{2p} are the coefficients in the Taylor expansion of $\ln[F(x)/F(0)]$:

$$\ln[F(x)/F(0)]=\sum_{p=1}^{\infty}\frac{x^{2p}L_{2p}}{(2p)!}\,.$$

The first few λ's are given by

$$
\begin{aligned}
\lambda_2&=g^{-1/2}\delta(0)^{-1/2}2R\,,\\
\lambda_4&=g^{-1}\delta(0)^{-2}(1-12R^2)\,,\\
\lambda_6&=g^{-3/2}\delta(0)^{-7/2}(240R^3-24R)\,,\\
\lambda_8&=g^{-2}\delta(0)^{-5}(-10\,080R^4+1344R^2-30)\,,\\
\lambda_{10}&=g^{-5/2}\delta(0)^{-13/2}(725\,760R^5-120\,960R^3\\
&\qquad+4632R)\,.
\end{aligned}
$$

Observe that λ_{2p} is a polynomial in R having integer coefficients whose parity is odd (even) if p is odd (even).

(2) Now for each graph do the following:

(a) Compute the topological symmetry number S for the graph. The symmetry number is the reciprocal of the number of ways the graph can be turned into itself under rotations and reflections of the vertices and lines of the graph.

(b) Label the ith vertex by the space-time coefficient x_i and represent the end of every external line by y_i. Represent every internal line connect-

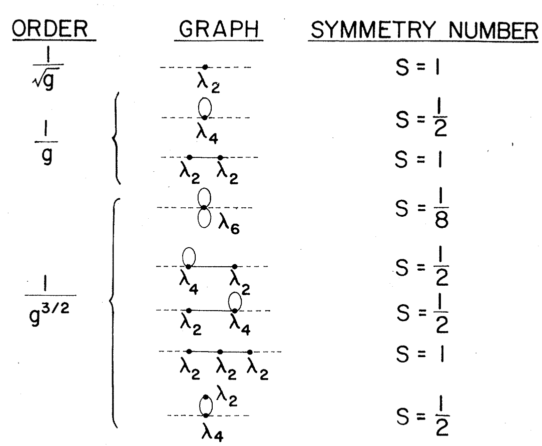

FIG. 1. All graphs contributing to the first three terms in the large-g expansion of the two-point Green's function $W_2(x,y)$ in $g\phi^4/4$ field theory. Internal lines are represented by solid lines and external lines by dashed lines. The symmetry number S of each graph is shown. For each vertex the vertex amplitude λ_{2p} is shown.

ing x_i to x_j by $-G^{-1}(x_i, x_j)$ and every external line connecting x_i to y_j by $-\delta(x_i - y_j)$. Integrate over all vertices x_i.

(3) Multiply together the symmetry number, the coordinate-space integral, and the vertex amplitudes for each graph and sum over all graphs to get the $(1/\sqrt{g})^k$ term in the expansion of the n-point Green's function.

We now illustrate these rules by calculating the two-point Green's function $W_2(x, y)$. The graphs for the first four orders and their accompanying symmetry numbers are given in Figs. 1 and 2. We express the function $W_2(x, y)$ as a series in powers of $1/\sqrt{g}$ as follows:

$$
\begin{aligned}
W_2(x, y) = {} & \lambda_2\delta(x-y)(1/\sqrt{g}\text{ term}) - \left[\tfrac{1}{2}A\lambda_4\delta(x-y) + G^{-1}(x,y)\lambda_2{}^2\right](1/g\text{ term}) \\
& + \left[\tfrac{1}{8}A^2\lambda_6\delta(x-y) + A\lambda_2\lambda_4G^{-1}(x,y) + \lambda_2{}^3H(x,y) + \tfrac{1}{2}B\lambda_2\lambda_4\delta(x-y)\right](1/g^{3/2}\text{ term}) \\
& - \left[\lambda_2{}^4I(x,y) + \tfrac{3}{2}A\lambda_2{}^2\lambda_4H(x,y) + \tfrac{1}{2}C\lambda_2{}^2\lambda_4\delta(x-y) + \tfrac{1}{4}\lambda_4{}^4A^2G^{-1}(x,y)\right. \\
& + \tfrac{1}{4}\lambda_2\lambda_6A^2G^{-1}(x,y) + \lambda_2{}^2\lambda_4BG^{-1}(x,y) + \tfrac{1}{6}L(x,y)\lambda_4{}^2 + \tfrac{1}{48}\lambda_8A^3\delta(x-y) \\
& \left. + \tfrac{1}{4}AB\lambda_2\lambda_6\delta(x-y) + \tfrac{1}{4}AB\lambda_4{}^2\delta(x-y)\right](1/g^2\text{ term}) + \cdots ,
\end{aligned}
\tag{3.1}
$$

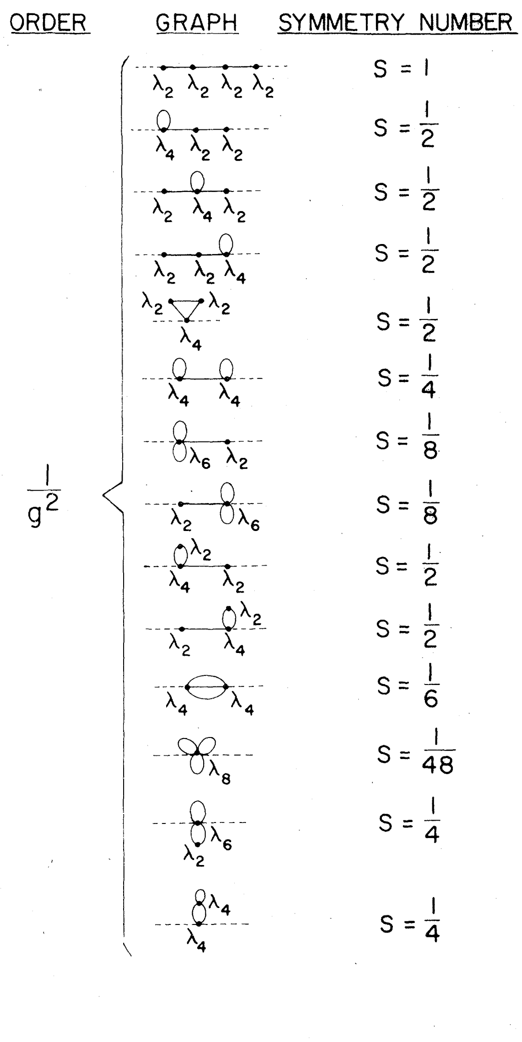

FIG. 2. All graphs contributing to the fourth term in the large-g expansion of the two-point Green's function $W_2(x,y)$ in $g\phi^4/4$ field theory. The format is the same as in Fig. 1.

where $A, B, C, H(x, y), I(x, y), L(x, y)$ are pieces of diagrams that occur frequently in the $1/\sqrt{g}$ expansion. The values of A, B, C, H, I, and L and other pieces of diagrams which commonly occur in this graphical expansion are given in Fig. 3 in terms of $G^{-1}(x, y)$.

B. Evaluation of diagrams

If we now recall from (2.8) that $G^{-1}(x, y)$ satisfies

$$G^{-1}(x, y) = (-\partial^2 + m^2)\delta(x-y),$$

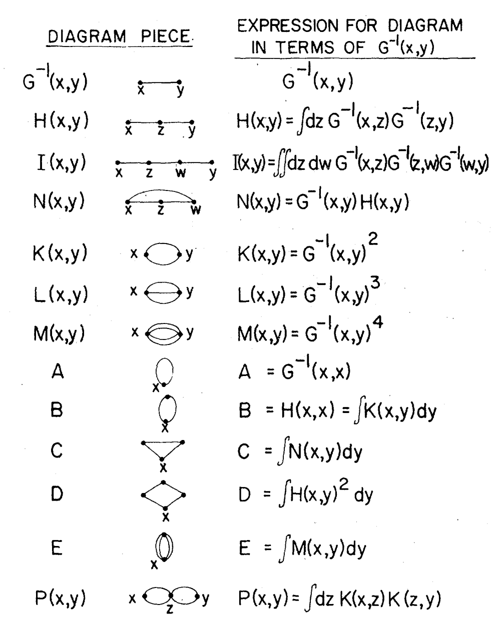

FIG. 3. Pieces of diagrams which commonly occur in the expansion of the Green's functions. The numerical expression for each diagram piece in terms of $G^{-1}(x,y)$ is given next to the diagram. The technique for evaluating each of these diagrams on a lattice is given in Sec. III, part B.

then it is apparent that every one of the terms in (3.1) is singular. For example,

$$A=G^{-1}(x,x)=-\delta''(0)+m^2\delta(0).$$

To make sense of each of the terms in (3.1) we temporarily go onto a space-time lattice. We adopt the following conventions:

a = lattice spacing,

d = dimension of space-time,

$$\delta(x-y)=\frac{1}{a^d}\delta_{i_1,j_1}\delta_{i_2,j_2}\cdots\delta_{i_d,j_d}\equiv\frac{1}{a^d}\delta_0, \qquad (3.2)$$

$$\delta(0)=\frac{1}{a^d},$$

$$\int dx=a^d\sum_{\text{lattice points}}.$$

In one dimension we represent $\partial^2\delta(x-y)$ on a lattice as

$$\frac{1}{a^3}(\delta_{i,j+1}+\delta_{i+1,j}-2\delta_{i,j}).$$

Therefore in d dimensions we formally represent $\partial^2\delta(x-y)$ on a lattice as

$$\frac{1}{a^{2+d}}\left[\sum_{k=1}^{d}(\delta_{+k}+\delta_{-k})-2d\delta_0\right],$$

where the first term δ_{+k} stands for a product of d Kronecker δ functions that takes us one step up the kth axis,

$$\delta_{+k}=\delta_{i_1,j_1}\delta_{i_2,j_2}\cdots\delta_{i_k,j_k+1}\cdots\delta_{i_d,j_d};$$

the second term δ_{-k} is a product of d Kronecker δ functions that takes us one step down the kth axis (to get the second term from the first term we replace δ_{i_k,j_k+1} in the above formula by δ_{i_k+1,j_k}); δ_0 is the product of δ functions in (3.2).

Therefore, the transcription of $G^{-1}(x,y)$ onto the lattice is given by

$$G^{-1}(x,y)=\frac{1}{a^{2+d}}\left[-\sum_{k=1}^{d}(\delta_{+k}+\delta_{-k})+(2d+m^2a^2)\delta_0\right]. \qquad (3.3)$$

Using (3.3) we can easily calculate the diagram pieces in Fig. 3. For example, if we set $x=y$ in (3.3) we get $\delta_+=\delta_-=0$ and $\delta_0=1$. Thus,

$$A=G^{-1}(x,x)=\frac{1}{a^{2+d}}(2d+m^2a^2). \qquad (3.4)$$

To compute $K(x,y)$ (see Fig. 3), we square the expression in (3.3) and use the property of Kronecker δ functions that

$$(\delta_{+k})^2=\delta_{+k},\quad \delta_{+k}\delta_{+l}=0 \text{ if } k\neq l,$$

$$\delta_0^{\,2}=\delta_0,\quad \delta_0\delta_{+k}=0,\text{ etc.}$$

We obtain

$$K(x,y)=G^{-1}(x,y)^2=\frac{1}{a^{4+2d}}\left[\sum_{k=1}^{d}(\delta_{+k}+\delta_{-k})+(2d+m^2a^2)^2\delta_0\right].$$

This expression can be simplified by using (3.3):

$$K(x,y)=-\frac{1}{a^{2+d}}G^{-1}(x,y)+\frac{(2d+m^2a^2)^2+(2d+m^2a^2)}{a^{4+d}}\delta(x-y). \qquad (3.5)$$

We have been able to express the product of two G^{-1}'s as a linear combination of $\delta(x-y)$ and $G^{-1}(x,y)$. This same approach works in our calculation of $L(x,y)$ and $M(x,y)$:

$$L(x,y)=G^{-1}(x,y)^3=\frac{1}{a^{4+2d}}G^{-1}(x,y)+\frac{(2d+m^2a^2)^3-(2d+m^2a^2)}{a^{6+2d}}\delta(x-y), \qquad (3.6)$$

$$M(x,y)=G^{-1}(x,y)^4=-\frac{1}{a^{6+3d}}G^{-1}(x,y)+\frac{(2d+m^2a^2)^4+(2d+m^2a^2)}{a^{8+3d}}\delta(x-y). \qquad (3.7)$$

Next we compute $B=\int K(x,y)dy$. Observe that if we sum $\sum_{k=1}^{d}\delta_{+k}$ over all lattice points each Kronecker δ contributes 1 so the sum $=d$. Thus, from (3.3),

$$\int dy\,G^{-1}(x,y)=a^d\sum_{\substack{\text{all lattice}\\ \text{points } j_k}}G^{-1}(x,y)=\frac{m^2a^2}{a^2}.$$

Thus, we calculate from (3.5) that

$$B=\frac{(2d+m^2a^2)^2+2d}{a^{4+d}}. \qquad (3.8)$$

In a similar way we compute $E=\int M(x,y)dy$ by integrating $M(x,y)$ in (3.7):

$$E=\frac{(2d+m^2a^2)^4+2d}{a^{8+3d}}. \qquad (3.9)$$

It is only slightly more complicated to calculate

$$H(x,y)=\int dz\, G^{-1}(x,z)G^{-1}(z,y)\,.$$

We merely multiply (3.3) by itself and sum over lattice points. The result is

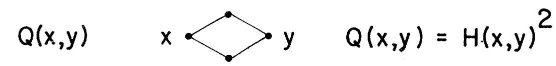

FIG. 4. $Q(x,y)$, the simplest diagram which is not rotationally symmetric on the lattice when $d\geq 2$ [that is, it cannot be expressed as a linear combination of $\delta(x-y)$, $G^{-1}(x,y)$, $H(x,y)$, $I(x,y)$, $\cdots$]. In momentum space this diagram depends on p^2, p^4, and $\sum_{i=1}^d p_i^4$. Fortunately, $\sum_{i=1}^d p_i^4$, which is not rotationally symmetric, can be replaced by $p^4=(p^2)^2$ in the limit of zero lattice spacing.

$$H(x,y)=\frac{1}{a^{4+d}}\left\{\sum_{k=1}^{d}(\delta_{++k}+\delta_{--k})+2\sum_{k=2}^{d}\sum_{l=1}^{k-1}(\delta_{+k}+\delta_{-k})(\delta_{+l}+\delta_{-l})-(4d+2m^2a^2)\sum_{k=1}^{d}(\delta_{+k}+\delta_{-k})+[(2d+m^2a^2)^2+2d]\delta_0\right\}. \tag{3.10}$$

We verify using (3.10) that $B=H(x,x)$.

Next we calculate $N(x,y)=G^{-1}(x,y)H(x,y)$. Multiplying (3.10) and (3.3) gives

$$\begin{aligned} N(x,y)&=\frac{1}{a^{6+2d}}\left\{(4d+2m^2a^2)\sum_{k=1}^{d}(\delta_{+k}+\delta_{-k})+[(2d+m^2a^2)^3+2d(2d+m^2a^2)]\delta_0\right\}\\ &=-\frac{4d+2m^2a^2}{a^{4+d}}\,G^{-1}(x,y)+\frac{(2d+m^2a^2)^3+2(2d+m^2a^2)^2+2d(2d+m^2a^2)}{a^{6+d}}\,\delta(x-y)\,. \end{aligned} \tag{3.11}$$

From this result, we calculate

$$C=\int N(x,y)dy=\frac{(2d+m^2a^2)^3+12d^2+6dm^2a^2}{a^{6+d}}\,. \tag{3.12}$$

To calculate D we square H in (3.10) and integrate over y:

$$\begin{aligned} D=\int H(x,y)^2dy&=a^d\sum_{\substack{\text{lattice}\\ \text{points}}}H(x,y)^2\\ &=\frac{1}{a^{8+d}}[(2d+m^2a^2)^4+12d(2d+m^2a^2)^2+12d^2-6d]\,. \end{aligned} \tag{3.13}$$

We will not bother to calculate $I(x,y)$ in coordinate space because it is very simple in the momentum space:

$$I(p)=G^{-1}(p)^3+(p_{\text{Eucl}}{}^2+m^2)^3=(-p_{\text{Mink}}{}^2+m^2)^3\,. \tag{3.14}$$

In general our objective is always to calculate a piece of a diagram by rewriting it as a linear combination of $\delta(x-y), G^{-1}(x,y)$, $H(x,y), I(x,y),\ldots$. This is because in momentum space these graphs form a complete set of rotationally symmetric objects (functions of p^2):

$$1,\ (p_{\text{Eucl}}{}^2+m^2),\ (p_{\text{Eucl}}{}^2+m^2)^2,\ (p_{\text{Eucl}}{}^2+m^2)^3,\ldots\ .$$

For example, to calculate

$$P(x,y)=\int dz\,K(x,z)K(z,y)$$

we use (3.5) and integrate to get

$$P(x,y)=\frac{1}{a^{4+2d}}\,H(x,y)-\frac{2}{a^{6+2d}}[(2d+m^2a^2)^2+(2d+m^2a^2)]G^{-1}(x,y)+\frac{1}{a^{8+2d}}\,[(2d+m^2a^2)^2+2d+m^2a^2]^2\delta(x-y)\,. \tag{3.15}$$

However, there is one subtlety that must be explained. In Fig. 4 we give the simplest diagram which is *not* a linear combination of $\delta, G^{-1}, H, I,\ldots$. (There are no such diagrams with less than four lines.) Fortunately, the part of the diagram which is not rotationally symmetric vanishes as $a\to 0$.

To explain the appearance of nonrotational symmetry on a lattice consider first the one-dimensional expressions

$$\delta(x-y)=\frac{1}{a}[\delta_{i,j}],$$

$$g(x,y)=\partial^2\delta(x-y)=\frac{1}{a^3}[\delta_{i,j+1}+\delta_{j,i+1}-2\delta_{i,j}],$$

$$h(x,y)=\int g(x,z)g(z,y)dz$$

$$=\partial^4\delta(x,y)=\frac{1}{a^5}[\delta_{i,j+2}-4\delta_{i,j+1}+6\delta_{i,j}-4\delta_{i+1,j}+\delta_{i+2,j}].$$

These expressions can be rewritten using simpler notation in the form of row vectors:

$$\delta(x-y)=\frac{1}{a}[1],$$

$$g(x,y)=\frac{1}{a^3}[1\quad -2\quad 1],$$

$$h(x,y)=\frac{1}{a^5}[1\quad -4\quad 6\quad -4\quad 1].$$

Now we compute $q=h(x,y)^2$:

$$q(x,y)=\frac{1}{a^{10}}[1\quad 16\quad 36\quad 16\quad 1].$$

In one dimension this can be reexpressed as a linear combination of the row vectors δ, g, and h:

$$q(x,y)=\frac{1}{a^5}h(x,y)+\frac{20}{a^7}g(x,y)+\frac{70}{a^9}\delta(x-y).$$

Now consider the same expressions $\delta(x-y)$, $g(x,y)$, and $h(x,y)$ in two dimensions. In matrix form

$$\delta(x-y)=\frac{1}{a^2}[1],\qquad g(x,y)=\frac{1}{a^4}\begin{bmatrix} & 1 & \\ 1 & -4 & 1\\ & 1 & \end{bmatrix},\qquad h(x,y)=\frac{1}{a^6}\begin{bmatrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1\\ & 2 & -8 & 2 & \\ & & 1 & & \end{bmatrix}.$$

Now $q(x,y)=h(x,y)^2$ is the matrix

$$q(x,y)=\frac{1}{a^{12}}\begin{bmatrix} & & 1 & & \\ & 4 & 64 & 4 & \\ 1 & 64 & 400 & 64 & 1\\ & 4 & 64 & 4 & \\ & & 1 & & \end{bmatrix},$$

which can be reexpressed as

$$q(x,y)=\frac{2}{a^{12}}\begin{bmatrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1\\ & 2 & -8 & 2 & \\ & & 1 & & \end{bmatrix}-\frac{1}{a^{12}}\begin{bmatrix}1\\ -4\\ 6\\ -4\\ 1\end{bmatrix}-\frac{1}{a^{12}}[1\quad -4\quad 6\quad -4\quad 1]+\frac{76}{a^{12}}\begin{bmatrix} & 1 & \\ 1 & -4 & 1\\ & 1 & \end{bmatrix}+\frac{676}{a^{12}}[1]$$

$$=\frac{2}{a^6}h(x,y)-\frac{1}{a^6}(\partial_1{}^4+\partial_2{}^4)\delta(x-y)+\frac{76}{a^8}g(x,y)+\frac{676}{a^{10}}\delta(x-y).$$

In momentum space this expression is

$$q(p)=\frac{2}{a^6}p_{\text{Eucl}}{}^4-\frac{1}{a^6}(p_1{}^4+p_2{}^4)-\frac{76}{a^8}p_{\text{Eucl}}{}^2+\frac{676}{a^{10}}\,,$$

which is *not* rotationally symmetric. The simplest way to interpret this expression is to argue that in the lattice limit rotational invariance must be recovered. Thus, we examine $q(p)$ in that frame of reference for which $p_2=0$,

$$q(p)=\frac{p_1{}^4}{a^6}-\frac{76p_1{}^2}{a^8}+\frac{676}{a^{10}}\,,$$

and then argue that

$$q(x,y)=\frac{1}{a^6}h(x,y)+\frac{76}{a^8}g(x,y)+\frac{676}{a^{10}}\delta(x-y)$$

must be the correct expansion for $q(x,y)$ because it has the correct rotationally invariant lattice limit. We have verified these assertions by performing lattice Fourier transforms.

Using the approach we have just described we compute in d dimensions that (if $m=0$)

$$Q(p^2)=\frac{1}{a^{4+d}}p_{\text{Mink}}{}^4+\frac{16d^2+8d-4}{a^{6+d}}p_{\text{Mink}}{}^2+\frac{16d^4+48d^3+12d^2-6d}{a^{8+d}}\,. \tag{3.16}$$

We use the result in (3.16) later to calculate the four-point Green's function.

This completes our discussion of the techniques required to calculate integrals on a lattice.

C. The two-point Green's function W_2

We now return to the evaluation of the two-point Green's function that was begun in part A. Using the methods of part B we can easily evaluate the graphs contributing to the two-point function.

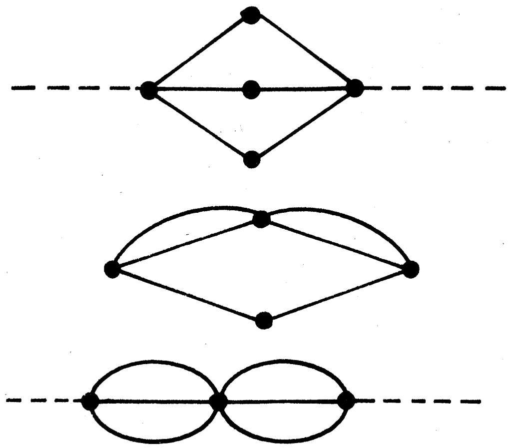

FIG. 5. Sixth-order one-particle irreducible graphs which give rise to p^4 terms in the denominator of the two-point Green's function [see (3.17)]. Until sixth order the highest power of p^2 is 1.

After converting to momentum space, we find that the diagram expansion geometrizes. That is, the function $W_2(p)$ takes the form

$$W_2(p)=\frac{\lambda_2}{1+[\text{1-particle-irreducible graphs}]}\,.$$

In this form, graphs such as H and I (see Fig. 3) no longer contribute because they are the iterations of simpler graphs.

After performing the computations to fourth order (graphs with four internal lines), we find that $W_2(p)$ has the explicit form

$$W_2(p)=\frac{\lambda_2}{1+(1/\sqrt{g})\,\alpha+(1/g)\beta+(1/g^{3/2})\gamma+(1/g^2)\delta}\,, \tag{3.17}$$

where λ_2 is given at the beginning of Sec. III and

$$\alpha=a^{d/2-2}\left[-2Rp_{\text{Mink}}{}^2a^2-6dR+\frac{d}{2R}+m^2a^2\left(\frac{1}{4R}-R\right)\right],$$

$$\beta=a^{d-4}\left[12dR^2+\frac{d^2}{4R^2}-2d^2-d+m^2a^2\left(-2d+\frac{d}{4R^2}\right)+m^4a^4\left(\frac{1}{16R^2}-\frac{1}{2}\right)\right],$$

$$\begin{aligned}\gamma=a^{3d/2-6}\Bigg[&p_{\text{Mink}}{}^2a^2\left(2R-12R^3-\frac{1}{12R}\right)+24(d^2-d)R^3+(2d^3+4d)R\\&-\left(\frac{4d^3}{3R}+\frac{d^2}{2R}+\frac{d}{6R}\right)+\frac{d^3}{8R^3}+a^2m^2\left(12dR^3+3d^2R-\frac{d}{4R}-\frac{2d^2}{R}+\frac{3d^2}{16R^3}\right)\\&+a^4m^4\left(\frac{3dR}{2}-\frac{d}{R}+\frac{3d}{32R^3}\right)+a^6m^6\left(\frac{R}{4}-\frac{1}{6R}+\frac{1}{64R^3}\right)\Bigg],\end{aligned}$$

$$\delta=a^{4d-8}\left[p_{\text{Mink}}{}^2a^2\left(-48R^4d+4dR^2+d-\frac{d}{12R^2}\right)+p_{\text{Mink}}{}^2m^2a^2\left(-24R^4+2R^2+\tfrac{1}{2}-\frac{1}{24R^2}\right)+\cdots\right],$$

where to this order we need only retain in δ the terms containing $p_{\text{Mink}}{}^2$.

Notice that $p_{\text{Mink}}{}^2$ appears only to the zeroth and first powers in these coefficients. Eventually, of course, all powers of $p_{\text{Mink}}{}^2$ must appear. However, $p_{\text{Mink}}{}^4$ does not appear until sixth order. (In Fig. 5

there are examples of sixth-order one-particle irreducible graphs which give rise to $p_{\rm Mink}{}^4$ in the denominator of the two-point Green's function.) Thus, until sixth order there is only *one* pole in the two-point function. Solving for $p_{\rm Mink}{}^2=M^2$ gives the mass^2 of the pole as a power series in $1/\sqrt{g}$:

$$\begin{aligned}p_{\rm Mink}{}^2=M^2=\frac{1}{2Ra^2}\Bigg\{1&+\Bigg[-\frac{(12R^2-1)d}{2Ra^{2-d/2}\sqrt{g}}+\frac{(288d-144)R^4+(24-24d-48d^2)R^2+6d^2-1}{24R^2a^{4-d}g}\\&+\frac{(1152d^2-576d)R^6+(96d^3-144d)R^4+(52d-24d^2-64d^3)R^2+6d^3-3d}{a^{6-3d/2}48R^3g^{3/2}}+\cdots\Bigg]\\&+m^2a^2\Bigg[-\frac{4R^2-1}{4Ra^{2-d/2}\sqrt{g}}+\frac{d-8dR^2}{4R^2a^{4-d}g}\\&+\frac{(1152d-576)R^6+(288d^2-144)R^4+(52-24d-192d^2)R^2+18d^2-3}{96R^3a^{6-3d/2}g^{3/2}}+\cdots\Bigg]\\&+m^4a^4\left(-\frac{8R^2-1}{16R^2a^{4-d}g}+\frac{48dR^4-32dR^2+3d}{32R^3a^{6-3d/2}g^{3/2}}+\cdots\right)+\cdots\Bigg\}.\end{aligned}\tag{3.18}$$

We presume that as we go to higher order more and more poles appear and that these bunch up to form the cut of the Lehmann-Symanzik-Zimmermann (LSZ) representation. We find that up to fourth order, Z_3, the residue at the pole is 1, which is consistent with there being only one excitation to this order in perturbation theory.

Observe that when $d<4$ the formula in (3.18) does not have a lattice limit, even when $d=1$ (the anharmonic oscillator) where it is known that M^2 is a *finite* quantity expressible in terms of the bare parameters g and m. Indeed as $a\to 0$ each successive term in the series is more and more singular. In the next section we discuss various procedures for extrapolating to zero lattice spacing.

D. The vacuum energy density

Because we have not taken care to define precisely the normalization factor N in (2.18), the energy density E of the vacuum, which is given by

$$\langle 0_+|0_-\rangle=\exp\left(iE\int d^dx\right),$$

cannot be determined up to a constant. However, the derivative of the energy density with respect to g can be computed using the formal identity

$$4g\frac{dE}{dg}=\delta(0)-\int d^dx\,W_2(x,y)G^{-1}(x,y).\tag{3.19}$$

To evaluate the right-hand side of this equation we take all of the diagrams contributing to the two-point function $W_2(x,y)$ and join the two external lines with a free inverse propagator.

We have calculated the right side of (3.19) to fifth order in powers of $1/\sqrt{g}$ for a zero bare mass theory ($m=0$):

$$\begin{aligned}4g\frac{dE}{dg}=\frac{1}{a^d}\Bigg[1&-\frac{4dR}{\sqrt{g}a^{2-d/2}}-\frac{8d^2R^2-8dR^2-2d^2}{ga^{4-d}}-\frac{16d^3R^3-48d^2R^3+12d^2R}{g^{3/2}a^{6-3d/2}}\\&-\frac{(32d^4-192d^3+48d)R^4+(32d^3-16d^2+8d)R^2-2d^4/3-4d^3-d/3}{g^2a^{8-2d}}\\&-\frac{(64d^5-640d^4+480d^2)R^5+(80d^4-120d^2)R^3-(2d^5/3-20d^4/3-20d^3+20d^2/3)R}{g^{5/2}a^{10-5d/2}}-\cdots\Bigg].\end{aligned}\tag{3.20}$$

Again, as in (3.18), this formula is finite term by term to all orders in perturbation theory when $a\neq 0$. However, as $a\to 0$ the terms in this series become more and more singular when $d<4$ and the series ceases to exist, even when $d=1$, where it is known that E, the ground-state energy of the anharmonic oscillator, is a finite function of the bare parameters m and g. The extrapolation of this series to zero lattice spacing is discussed in the next section.

E. Two-particle bound states

If the Green's function

$$B(x,y)=\langle 0\,|[\phi(x)^2\phi(y)^2]_+|0\rangle-\langle 0|\phi(x)^2|0\rangle\langle 0|\phi(y)^2|0\rangle$$

in momentum space has a pole, we can interpret this as a two-particle bound state. We express $B(x,y)$ in terms of the ordinary Green's functions as

$$B(x,y)=W_4(x,x,y,y)-2W_2(x,y)^2\,.$$

In Fig. 6 we illustrate the diagrams contributing to the first three orders of W_4. In our calculation we also included the next two orders of diagrams (up to diagrams having 4 internal lines).

We find that the pole of $B(x,y)$ in momentum space in a zero-bare-mass theory ($m=0$) is given by

$$p_{\rm Mink}{}^2=M^2=\frac{2}{a^2(1-4R^2)}\left[1+\frac{16dR^3}{(1-4R^2)\sqrt{g}\,a^{2-d/2}}+\frac{(192d^2-960d+1344)R^6+(144d^2+528d-768)R^4+(12d^2-60d+104)R^2-3d^2-3d}{(48R^4-24R^2+3)ga^{4-d}}+\cdots\right].\tag{3.21}$$

We examine the lattice limit of this equation in the next section.

IV. EXTRAPOLATION TO ZERO LATTICE SPACING

In the preceding section we derived three expansions: (3.18) for the location of the pole of the two-point Green's function, (3.20) for the derivative of the ground-state energy density with respect to g, and (3.21) for the mass of the two-particle bound state. All three expressions exist and are finite term by term as a power series in $1/\sqrt{g}$ so long as $a\neq 0$. However, as $a\to 0$ we see that every term in the series becomes singular when $d<4$ and that the coefficients of increasingly higher powers of $1/\sqrt{g}$ become increasingly singular as $a\to 0$. Before we can develop a method to extrapolate to zero lattice spacing, we must understand the origin of these singularities. We claim that these singularities occur because the true large-g expansions of the Green's functions of the continuum theory are *not* series in powers of $1/\sqrt{g}$.

As an example, consider the function

$$f(x)=\sqrt{x}\,.$$

We wish to expand this function as a Taylor series in powers of x. Observing that this expansion does not exist, we introduce the "lattice spacing" a:

$$f_a(x)=(x+a)^{1/2}\,.$$

Now we *can* expand in powers of x:

$$f_a(x)=\sqrt{a}+\frac{x}{2\sqrt{a}}-\frac{x^2}{8a^{3/2}}+\cdots\,.\tag{4.1}$$

However, just as with the expansions in (3.18), (3.20), and (3.21), this expansion ceases to exist as $a\to 0$, and this is because $f(x)$ has no Taylor expansion at the origin.

There is, of course, a simple way to make sense out of the series in (4.1). If we square (4.1) then we observe that the series truncates:

$$f_a(x)^2=a+x+0+0+0+\cdots\,.$$

Now we can perform the "lattice limit" $a\to 0$. We thus obtain $f(x)^2=x$.

A. Unsuccessful approaches

This simple example suggests that we should try to find functional relations that the series (3.18), (3.20), and (3.21) satisfy. For example, Bender, Guralnik, Keener, and Olaussen[15] showed that the lowest-energy pole M of the two-point function

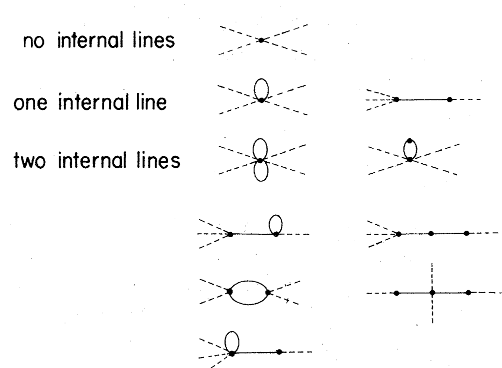

FIG. 6. Graphs contributing to the four-point Green's function $W_4(x,y,z,w)$. All graphs having 0, 1, and 2 internal lines are shown. The result in (3.21) also uses graphs having 3 and 4 internal lines.

for $d=1$ (the anharmonic oscillator) satisfies an approximate cubic relation of the form

$$M^3+\alpha m^2 M+\beta g=0\,, \tag{4.2}$$

where α and β are numbers. Ginsburg and Montroll[16] then showed that (4.2) is actually the first of an infinite sequence of polynomial functional relations of the form

$$\begin{aligned} &E^3+a_1E+b_1=0\,,\\ &E^6+a_2E^4+b_2E^3+c_2E^2+d_2=0\,,\\ &E^9+a_3E^7+b_3E^6+c_3E^5+d_3E^4+e_3E^3+f_3=0\,, \end{aligned} \tag{4.3}$$

and so on.

It would be very nice if we could use (3.18) to establish a sequence of functional relations (one for each new order in powers of $1/\sqrt{g}$) such as those in (4.3) whose coefficients are well behaved as $a\to 0$. Unfortunately, we have not been able to find such a sequence of functional relations.

A second possible approach involves the use of higher-order WKB analysis. Bender, Olaussen, and Wang[17] showed that when $m=0$ the eigenvalues E_n of the form

$$\epsilon^{3/4}\sum_{k=0}^{\infty}\epsilon^{-3k/2}a_n=(n+\tfrac{1}{2})\sqrt{\pi}\,,\quad E=g^{1/3}2^{-4/3}\epsilon\,,$$

and

$$\begin{aligned} &a_0=(3R)^{-1}\,,\quad a_1=-R/4\,,\quad a_2=11/(3\times 2^9R)\,,\\ &a_3=7\times 11\times 61R/(3\times 5\times 2^{11})\,,\ldots\,, \end{aligned} \tag{4.4}$$

where R is given in (2.20). The remarkable similarity in structure between (4.4) and (3.20) is very suggestive. In fact, there really is a deep connection between the WKB approximation and the ultralocal approximation.[18] However, we do not see how to deduce the functional relation in (4.4) from the series in (3.20).

B. Successful approaches

Fortunately, there are several simple and natural techniques for extrapolating to zero lattice spacing. They all begin by observing that

$$x=\frac{1}{\sqrt{g}a^{2-d/2}} \tag{4.5}$$

is a dimensionless parameter for all dimensions d. Using (4.5) we eliminate all reference to the lattice spacing a in favor of the parameters x and g. For example, (3.18) now has the form

$$\begin{aligned} M^2=&g^{2/(4-d)}x^{4/(4-d)}\ \text{(power series in } x)\\ &+m^2\ \text{(power series in } x)\\ &+m^4g^{2/(d-4)}x^{4/(d-4)}\ \text{(power series in } x)\\ &+\cdots\,, \end{aligned} \tag{4.6}$$

(3.20) has the form

$$4g\frac{dE}{dg}=g^{d/(4-d)}x^{2d/(4-d)}\ \text{(power series in } x)\,, \tag{4.7}$$

and (3.21) has the form

$$M^2=g^{2/(4-d)}x^{4/(4-d)}\ \text{(power series in } x)\,. \tag{4.8}$$

In the above formulas, by a power series in x, we mean a formal Taylor series whose first term is a constant.

Our objective now is to show that in the lattice limit $a\to 0$, $x\to\infty$, expressions such as $x^{4/(4-d)}$ (power series in x) can be sensibly extrapolated to a number. If this can be done, then it becomes evident why the original expansion in powers of $1/\sqrt{g}$ did not make sense in the limit $a\to 0$. Apparently, the true strong-coupling expansion of M^2 is actually not a series in powers of $1/\sqrt{g}$, but rather by dimensional analysis alone is a series in powers of $m^2g^{2/(d-4)}$. From here on we refer to series in powers of $1/\sqrt{g}$ as *high-temperature expansions* and to series in powers of $m^2g^{2/(d-4)}$ as *strong-coupling expansions*.

We propose the following simple Padé-type extrapolation technique for making sense out of any series of the form

$$s=x^{\alpha}\ \text{(power series in } x)\quad (\alpha\neq 0) \tag{4.9}$$

as $x\to\infty$ (the lattice limit). First, raise (4.9) to the $1/\alpha$ power:

$$\begin{aligned} s^{1/\alpha}&=x\ \text{(new power series in } x)\\ &=\frac{x}{\sum_{n=0}^{\infty}a_nx^n}\,. \end{aligned} \tag{4.10}$$

Now raise (4.10) to the second, third, fourth,... powers:

$$\begin{aligned} s^{2/\alpha}&=\frac{x^2}{\sum_{n=0}^{\infty}b_nx^n}\,,\\ s^{3/\alpha}&=\frac{x^3}{\sum_{n=0}^{\infty}c_nx^n}\,,\\ s^{4/\alpha}&=\frac{x^4}{\sum_{n=0}^{\infty}d_nx^n}\,, \end{aligned} \tag{4.11}$$

and so on.

Define a set of extrapolants of the form

$$\begin{aligned} &s_1=\left(\frac{1}{a_1}\right)^{\alpha}, &&s_2=\left(\frac{1}{b_2}\right)^{\alpha/2},\\ &s_3=\left(\frac{1}{c_3}\right)^{\alpha/3}, &&s_4=\left(\frac{1}{d_4}\right)^{\alpha/4}, \end{aligned} \tag{4.12}$$

and so on. The numbers $s_1,s_2,s_3,\ldots$ exist and are finite for all n. Moreover, for the models we have examined, the numbers s_n converge rapidly (usually monotonically) to a limit, which we define to be the sum of the series s.

Let us review the philosophy we have used. Until the very last line of the calculation, we have taken the coupling g to be large and the lattice spacing a to be fixed. Thus, we have treated x, as defined in (4.5), as a *small* parameter. This justifies the manipulations in going from (4.9) to (4.10) to (4.11). At the end of the calculation we fix g and take $a \to 0$. Thus, we perform the limit $x \to \infty$ and derive (4.12). However, this limit is not taken until *after* (4.11) has been established; it is the last step of the calculation.

Of course, we expect that when d is larger than 1 the quantity that we are calculating may in fact be infinite. For example, when $d=2$ or 3 the ground-state energy density in (4.7) is expected to be infinite. Therefore, we expect that the sequence of approximants $s_1, s_2, s_3, \ldots$, even though each term is finite, should diverge. This is what we actually observe. However, we can calculate by trial and error what the (finite) coefficient of this divergence is: We divide the right side of (4.7) by x to some power P or $\ln(1+x)$ to some power Q (or both) and determine the values of P and Q for which the new sequence of approximants converges. Specific examples of this approach are given in the next section.

We also point out that the conventional Green's function renormalization schemes can be carried out in the strong-coupling limit. Specifically, we can define the renormalized mass as the pole of the two-point Green's function, the wave-function renormalization Z_3 as the residue at the mass pole, and the renormalized charge as the value of the renormalized four-point Green's function at, say, zero four-momentum on the external legs. Then we can replace the bare parameters m and g with the physical masses and charges. The zero-lattice-spacing limit can be then taken at the end of the calculation. Although we do not know if this approach consistently removes all infinities from the theory, we see no reason it should fail.

Finally we mention the special case $d=4$. For this case g is dimensionless and we need not introduce the variable x as in (4.5). Instead we propose to use $y=m^2a^2$ as the dimensionless expansion parameter in expansions such as that in (3.18). There are three possibilities to consider. If we hold the renormalized quantities, mass and charge, fixed and allow $a \to 0$ then (a) $y \to 0$, (b) $y \to \infty$, or (c) $y \to$ a constant. We have not yet investigated these possibilities in detail.

V. NUMERICAL RESULTS

In this section we examine the lattice limit $a \to 0$ of the three formulas (3.18), (3.20), and (3.21) derived in Sec. III.

A. The case $d=1$ (anharmonic oscillator)

First, M in (3.18) and (4.6) is the pole of the two-point Green's function and is therefore the difference between the first excited level and the ground-state energy of the anharmonic oscillator. The large-coupling expansion of M is a series in powers of $m^2g^{-2/3}$:

$$M = \alpha g^{1/3} + \beta m^2 g^{-1/3} + \gamma m^4 g^{-1} + \delta m^6 g^{-5/3} + \cdots .$$

Using the tabulated computer results of Hioe and Montroll[19] we find that the values of the coefficients in this series are

$$\alpha \approx 1.0808 ,$$

$$\beta \approx 0.3399 ,$$

and so on.

How closely can we predict these numbers? We set $d=1$ in (4.6) and compute the square root of this series as a series in powers of x. Then we extrapolate the coefficients of $g^{1/3}$ and $m^2g^{-1/3}$ to zero lattice spacing using the method described in Sec. IV. We obtain the following approximants to α and β:

$$\begin{aligned} &\alpha_1 = 1.1194 \quad (3.57\%\ \text{error}), \\ &\alpha_2 = 1.1021 \quad (1.97\%\ \text{error}), \\ &\alpha_3 = 1.0973 \quad (1.53\%\ \text{error}) \end{aligned} \tag{5.1}$$

and

$$\beta_1 = 0.3134 \quad (7.80\%\ \text{error}),$$

$$\beta_2 = 0.3227 \quad (5.06\%\ \text{error}).$$

Observe the monotonic approach to the limiting values of α and β.

Second, in the anharmonic oscillator the derivative of the ground-state energy $4g(dE/dg)$ in (3.20) and (4.7) has the form $\alpha g^{1/3}$ when $m=0$. Hioe and Montroll[19] give the value of α as

$$\alpha \approx 0.569\,473.$$

There are enough terms in (3.20) to supply five approximants to α:

$$\begin{aligned} &\alpha_1 = 0.6242 \quad (9.61\%\ \text{error}), \\ &\alpha_2 = 0.5861 \quad (2.92\%\ \text{error}), \\ &\alpha_3 = 0.5754 \quad (1.04\%\ \text{error}), \\ &\alpha_4 = 0.5717 \quad (0.39\%\ \text{error}), \\ &\alpha_5 = 0.5707 \quad (0.22\%\ \text{error}). \end{aligned} \tag{5.2}$$

Again observe the monotonic convergence of the sequence of approximants.

Third, in the anharmonic oscillator the two-particle bound-state pole M of the four-point Green's function is the difference between the second excited level and the ground-state energy. When $m=0$, M has the form

$$M=\alpha g^{1/3}.$$

Hioe and Montroll[19] give the value of α as

$$\alpha\approx 2.5317$$

Using (3.21) and (4.8), setting $d=1$, and taking the square root to solve for M, we obtain two successive approximants to α:

$$\alpha_1=2.2933 \quad (9.42\% \text{ error}),$$

$$\alpha_2=2.4070 \quad (4.93\% \text{ error}).$$

These approximants all give excellent numerical results. However, there are many techniques for accelerating the convergence of sequences of approximants which we could apply here to obtain even better results. Two such techniques are known as Shanks transformation and Richardson extrapolation.[20] Shanks transformation, the simpler of the two, consists of defining a new sequence α_n' in terms of the original sequence

$$\alpha_n' \equiv \frac{\alpha_{n+1}\alpha_{n-1}-\alpha_n^{\,2}}{\alpha_{n+1}+\alpha_{n-1}-2\alpha_n},$$

which often converges more rapidly to the limit α than the original sequence does. Even though Shanks transformation is most useful for sequences which oscillate about a limit, it even appears to work well on the monotone sequences which we have derived here. For example, α_2', the Shanks transform of α_1, α_2, and α_3 in (5.1), is 1.0955, which reduces the relative error from 1.53% to 1.36%, and α_3', the Shanks transform of α_2, α_3, and α_4 in (5.2), is 0.5697, which reduces the relative error from 0.39% to 0.048%.

One can accelerate the convergence even further by repeatedly Shanks transforming and obtaining α_n'', α_n''', and α_n'''', ... However, we do not wish to pursue any of these numerical questions further until more terms in the sequence of approximants have been found.

B. The case $d>1$

We expect that the three quantities in (4.6)–(4.8) are infinite in higher space-time dimensions because they are expressed in terms of bare parameters. Nevertheless, the approximants to these quantities are finite. Therefore, we expect the approximants to exhibit some characteristic form of nonconvergent behavior.

Let us consider just the series for the derivative of the ground-state energy density in (4.8) because five approximants to α are known. [In general, $4g(dE/dg)=g^{d/(4-d)}\alpha$, where α is a number.] We first examine what happens in two dimensions. We set $d=2$ and compute α_n using the procedure in Sec. IV. The first three α's might well belong to a converging sequence:

$$\alpha_1=0.5471, \quad \alpha_2=0.8780, \quad \alpha_3=0.8446.$$

However, $\alpha_4=2.1720i$, an imaginary number. The first indication that something is wrong is that the sequence in (4.12) involves taking fractional roots of *negative* numbers.

This observation suggests some intriguing numerical experimentation. We can assume that the sequence of approximants is trying to approach ∞ and look for a sequence of approximants which converge to the coefficient of ∞. We do this by guessing that $g(dE/dg)$ is logarithmically divergent and finding the approximants for

$$\lim_{x\to\infty}\frac{4gdE/dg}{\ln(1+x)}=g^{1/2}\frac{\alpha}{\ln(1+x)}.$$

We are happy to discover that while there *are* negative coefficients in the inverse series for $\alpha/\ln(1+x)$ in powers of x, as we raise this series to higher and higher powers, more and more of the first few terms are positive. Specifically, without the cutoff term $\ln(1+x)$, the series

$$\sum_{n=0}^{\infty}a_n x^n, \ \sum_{n=0}^{\infty}b_n x^n, \ \sum_{n=0}^{\infty}c_n x^n, \ldots$$

in (4.11) are

$$1+1.352x-0.344x^2+0.537x^3-1.356x^4+3.599x^5+\cdots,$$

$$1+2.704x+1.139x^2+0.143x^3-1.141x^4+3.161x^5+\cdots,$$

$$1+4.056x+4.450x^2+1.288x^3-1.244x^4+2.114x^5+\cdots,$$

$$1+5.408x+9.589x^2+6.445x^3-0.212x^4+0.479x^5+\cdots,$$

$$1+6.760x+16.556x^2+18.083x^3+6.747x^4-0.611x^5+\cdots.$$

Observe that while the coefficients of x, x^2, and x^3 in the first, second, and third series are positive, the x^4 term in the fourth series is negative and the x^5 term in the fifth series is negative. This indicates that the extrapolants to α do not converge.

On the other hand, when we include the cutoff term $\ln(1+x)$, these same five series become

$$1+2.204x+0.120x^2+0.225x^3-1.309x^4+3.869x^5+\cdots,$$

$$1+4.408x+5.098x^2+0.980x^3-1.612x^4+2.024x^5+\cdots,$$

$$1+6.612x+14.933x^2+12.970x^3+0.842x^4-2.165x^5+\cdots,$$

$$1+8.816x+29.625x^2+46.900x^3+31.400x^4-0.177x^5+\cdots,$$

$$1+11.020x+49.174x^2+113.475x^3+139.000x^4+73.656x^5+\cdots.$$

Now observe that the coefficient of x^n in the nth series is *positive*; in fact, the character of the nth series changes abruptly after the x^n term. Now $[\alpha/\ln(1+x)]_n$, the nth extrapolant to $\alpha/\ln(1+x)$, gotten by raising the coefficient of x^n in the nth series to the $-1/n$ power, appears to form a converging (but no longer monotonic) sequence:

$$[\alpha/\ln(1+x)]_1=0.453\,7,$$

$$[\alpha/\ln(1+x)]_2=0.442\,9,$$

$$[\alpha/\ln(1+x)]_3=0.425\,6,$$

$$[\alpha/\ln(1+x)]_4=0.422\,4,$$

$$[\alpha/\ln(1+x)]_5=0.423\,2.$$

Next we examine what happens in three dimensions. Again we find that the extrapolants are fractional roots of both positive *and* negative numbers. (The coefficient of x^2 in the second series is negative.) Moreover, we quickly determine that $\alpha/\ln(1+x)$ is still divergent.

We therefore assume that α is an algebraically divergent quantity and that it diverges like x^P as $x\to\infty$, where P is a positive number to be determined. The condition that the coefficient of x^n in the nth series be positive is a sequence of increasingly tight inequality constraints: $P<6, P>2.17, P>3.45, \ldots$. As a rough guess we take $P=4$. We find that now the coefficient of x^n in the nth series is *positive*. The extrapolants to α/x^P are

$$4.113, 3.933, 4.743, 4.541, 5.076.$$

Unfortunately there are not enough terms to claim that this is a converging sequence. However, we feel that there is enough numerical evidence here to warrant an extremely detailed numerical study of $g\phi^4$ quantum field theory to very high order in powers of $1/\sqrt{g}$ hoping that it might elucidate the nature of the parametric dependence upon d.

VI. OTHER QUANTUM FIELD THEORIES

Although we have not yet completed any numerical studies of quantum field theories other than the $g\phi^4$ theory discussed in Secs. II–V, we believe that the methods used here are completely general and easy to carry over to any theory.

A. $g\phi^{2N}$ Theory

To generalize from a $g\phi^4/4$ interaction to a $g\phi^{2N}/2N$ interaction, one need only generalize the function $F(x)$ in (2.9) to $F_{2N}(x)$ defined by

$$F_{2N}(x)\equiv\int_{-\infty}^{\infty}dt\,\exp\left(-\frac{1}{2N}t^{2N}-xt\right). \tag{6.1}$$

The diagrams of this $g\phi^{2N}$ theory look exactly the same as those of the $g\phi^4$ theory except that the vertices L_{2p} defined in Sec. III must be replaced with $L_{2p}(2N)$ defined by

$$\ln\left[\frac{F_{2N}(x)}{F_{2N}(0)}\right]=\sum_{p=1}^{\infty}\frac{x^{2p}L_{2p}(2N)}{(2p)!}. \tag{6.2}$$

B. Self-interacting Fermi theory

Consider a self-interacting Fermi theory defined by the Lagrangian

$$L(x)=\bar\psi\not\partial\psi+m\bar\psi\psi+g(\bar\psi\psi)^2+\bar\eta\psi+\bar\psi\eta, \tag{6.3}$$

where η and $\bar\eta$ are anticommuting c-number sources. Then the vacuum persistence functional $Z[\eta,\bar\eta]$ can be expressed in Euclidean space as a functional integral analogous to that in (2.1):

$$Z[\eta,\bar\eta]=\int D\bar\psi D\psi\exp\left[\int L(x)dx\right]. \tag{6.4}$$

Following the approach used to derive (2.17), we define the inverse Fermi propagator S_F^{-1} by

$$S_F^{-1}(x,y)=(\not\partial+m)\delta(x-y). \tag{6.5}$$

Then we simplify the integral representation (6.4) by factoring out the kinematic term as the functional derivative operator

$$\exp\left[\int\int dx\,dy\frac{\delta}{\delta\eta(x)}S_F^{-1}(x,y)\frac{\delta}{\delta\eta(y)}\right]$$

and evaluate the remaining functional integral. The result of this procedure is much simpler than that in (2.17) for the corresponding four-point interaction Bose theory. For example, in two-dimensional space, where η and $\bar\eta$ are two-component spinors, we have

$$Z[\eta,\bar{\eta}]=\exp\left[\int\int dx\,dy\frac{\delta}{\delta\bar{\eta}(x)}S_F{}^{-1}(x,y)\frac{\delta}{\delta\eta(y)}\right]\times\exp\left[-\frac{1}{g\delta(0)^2}\int\eta_5(x)dx\right], \tag{6.6}$$

where $\eta_5(x)=\frac{1}{2}(\bar{\eta}\eta)^2$. Thus, there is just one four-point interaction vertex, rather than an infinite number of vertices as in the case of Bose theories. Diagrams consist of $S_F{}^{-1}$ lines and η_5 vertices.

Things are almost as simple in four-dimensional space-time. The only change is that the exponential function

$$\exp\left[-\frac{1}{g\delta(0)^2}\int\eta_5(x)dx\right]$$

is replaced by an s-wave Coulomb wave function.

C. Quantum electrodynamics

It is just as easy to expand quantum electrodynamics as a series in powers of $1/e$. In two-dimensional space-time, vertices are (1) any even number of inverse photon lines, (2) any odd number of inverse photon lines and two inverse electron lines, and (3) any odd number of inverse photon lines and four inverse electron lines. In four-dimensional space-time there are, in addition to the above three types of vertices, (4) any odd number of inverse photon lines and six inverse electron lines, and (5) any odd number of inverse photon lines and eight inverse electron lines. We find that the relative strengths of the vertices do not involve transcendental numbers such as R as in $g\phi^4$ field theory (see Sec. III), but rather are all rational numbers satisfying a simple difference equation (similar to the one satisfied by the Fibonacci numbers):

$$a_{n+2}+a_{n+1}+a_n/6=0\,.$$

Because this equation is so simple (there is no simple corresponding equation in $g\phi^4$ theory), we are able to sum up infinite classes of diagrams in closed form. Our results are now very preliminary, but we hope to submit a manuscript on these theories shortly.[21]

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*Permanent address: Physics Department, Brown University, Providence, Rhode Island 02912.

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