PHYSICAL REVIEW D VOLUME 19, NUMBER 6 15 MARCH 1979

A theory of hadronic structure

Curtis G. Callan, Jr.
Princeton University, Princeton, New Jersey 08540

Roger F. Dashen
The Institute for Advanced Study, Princeton, New Jersey 08540

David J. Gross *
Princeton University, Princeton, New Jersey 08540
(Received 10 August 1978)

We show that the Euclidean vacuum in quantum chromodynamics (QCD) can be regarded as a four-dimensional ensemble of permanent color magnetic dipoles (instantons and meron pairs), with a positive paramagnetic susceptibility. Standard techniques are used to discuss the interactions of this medium for moderate densities. In the presence of color fields (due to quarks), large scale instantons (and other fluctuations) are suppressed, the density is low, and the system is easily treated. Below a critical field strength, this dilute phase is unstable and a first-order phase transition occurs to a dense phase consisting of closely packed instantons and merons, and possibly other things. In this dense phase, we believe that the permeability is infinite (perfect paramagnetism) and thus the normal QCD vacuum cannot tolerate color fields. This leads to a strikingly simple baglike picture of hadrons, as consisting of quarks confined to a region of space-time which is in a very dilute (abnormal) vacuum phase, in equilibrium with the dense vacuum (normal) phase outside the bag. The quarks are confined to the region of dilute phase where their dynamics are simple; and, as we show, they are shielded from the large-scale fluctuations outside the bag. We present a derivation of the static bag for heavy quarks and an estimate (to within a factor of two) of the bag constant. We further discuss some features of the resulting bag model including chiral-symmetry breaking and surface effects.

I. INTRODUCTION

At present, the belief that quantum chromodynamics (QCD) is the true theory of the strong interactions is rather widely held. This is perhaps surprising since no experimentalist has ever seen a quark or a gluon (the fundamental fields of QCD), nor has any theorist succeeded in calculating any hadron parameter from first QCD principles. What persuades the fundamental field theorist, at least, is that, of all renormalizable field theories, only QCD is asymptotically free and only asymptotic freedom appears capable of accounting for scaling phenomena or, more generally, for the tendency of hadrons to behave like collections of free, pointlike constituents when probed at short distances. This sort of evidence, however persuasive, is rather indirect and ought to be superseded eventually by successful QCD calculations of hadron structure, reaction cross sections, etc.

Other theorists, desiring to confront the facts of hadron physics more directly, have chosen to abandon the constraints of renormalizable field theory and construct phenomenological models based on qualitative insights about how hadrons are actually built. The MIT bag model, in particular, is a remarkably successful construction of this type. An energy functional, depending on a small number of parameters, is chosen so as to be consistent with a physical notion that the normal vacuum expels color fields and quarks. When applied to the essentially static problem of low-lying hadron spectroscopy, a few-parameter mass formula emerges which yields a good fit to a surprisingly large number of masses. The disadvantage of this type of treatment is that the starting point is not really a satisfactory quantum theory, and one therefore does not know how to compute quantum corrections or how to extend the picture to nonstatic problems. Similarly, since the model is *not* QCD it will not possess asymptotic freedom and will be unable to incorporate scaling in any natural way. Furthermore, the model contains explicit chiral-symmetry breaking and thus cannot naturally account for the pion or the successes of partial conservation of axial-vector current (PCAC).

To have the best of both worlds, one must demonstrate that the bag model, or some appropriate variant of it, is in fact a good phenomenological approximation of QCD, appropriate to the study of static hadron spectroscopy, but not necessarily to other questions. The status of the bag model would then be something like that of the Landau-Ginzberg approximation to the BCS theory: A phenomenological approximation to an underlying fundamental theory, derivable from it in broad outline if not in precise detail. The problem is to find a phenomenon or mechanism which, on the one hand, emerges naturally from the fundamental QCD equations and, on the other hand, can be exploited to justify the phenomenological approximation.

As long as perturbation theory is the only tool for exploring the content of a field theory, the task is hopeless—qualitatively speaking, what you get out of perturbation theory is what you put in. Something is needed to bridge the gap between the perturbative content of QCD, which can only be directly relevant to short-distance phenomena, and the quite different structures which must govern its behavior on the scale of hadron sizes.

A first step away from perturbation theory is made possible by our recently acquired understanding of instantons and other semiclassical vacuum fluctuations.[1,2,3] Studies[1] of the QCD vacuum from the semiclassical point of view have shown that, while the perturbative picture is correct at the shortest distances because asymptotic freedom drives the coupling strength down, instantons become a significant component of the vacuum fluctuations at scale sizes where the effective coupling is so small that the semiclassical calculational method should be quite accurate.[1,4,5] In fact, we find that as soon as the space-time density of instantons is at all significant, their effects dominate those of ordinary perturbative fluctuations and make the qualitative behavior of the theory completely different from that of simple perturbation theory. This has been spelled out in detail elsewhere, but we might remind the reader of the qualitatively striking effects of instantons on the heavy-quark potential[4] as well as on the flavor symmetries of massless quarks.[1]

Although it is clear that semiclassical effects completely change the qualitative content of QCD, our previous efforts succeeded only in giving a rough notion of those changes. There were two reasons for this. First, our calculations of instanton effects were, in essence, single-body calculations, and we assumed that the net effect of the instantons was simply an incoherent sum of the individual terms. This is correct when the instanton density is low and the integrated effect of instantons small. When the density becomes comparable to one, the instanton-generated effects become very big and change the qualitative behavior of the system, but at the same time, the instantons interact strongly enough so that the single-body picture does not work. Therefore, to get a valid picture of the behavior of the vacuum when instanton density is appreciable, we must include the interactions between instantons in a systematic way. The second shortcoming of our previous work has to do with the properties of vacuum fluctuations on large spatial scales. Since QCD has no intrinsic infrared cutoff, one finds that instantons of increasing scale size are increasingly dense, with the result that beyond a certain critical scale size, ρ_c, there is no reason to believe that the most probable fluctuations have any recognizable structure. To date, we have dealt with this problem by ignoring it—in practice by imposing a cutoff, at approximately ρ_c, on instanton scale size. If we are interested in questions of hadron structure, some version of this assumption must be right: On the one hand, ρ_c is expected to be roughly equal to the hadron scale size; on the other hand, one does not expect fluctuations on scales much larger than the hadron size to have much effect on the hadron itself. The question is, how and in what sense is this expectation realized? Is there a useful way of isolating the fluctuations on a scale smaller than ρ_c, which we may hope to control, from fluctuations on a larger scale whose quantitative understanding will surely be quite difficult?

The present paper represents an approach to these problems which, fortunately, turn out to be interrelated. First, we show how to deal with the multibody effects arising from instanton interactions allowing the instantons to have a nontrivial density, but still ignoring large-scale fluctuations. The key idea is to regard the vacuum as a medium with a paramagnetic susceptibility arising from the presence of a thermodynamic ensemble of permanent color dipoles (the instantons). It turns out that the conditions of the problem are such that well-known techniques for dealing with polar media may be applied to compute the "permeability" of the QCD vacuum even when that permeability is very different from unity (the perturbation theory value). Then the very large effects of instanton interactions are taken into account by an extension to four dimensions of the rules of magnetostatics of permeable media.

Next, we consider the behavior of this permeable medium in the presence of external color sources (quarks). We find that the medium responds to the external field in a fashion analogous to magnetostriction: The instanton density is space-time dependent, being large where the external field is small and vice versa. The reason for this behavior is that the local vacuum permeability is determined by the local instanton density. Consequently, the electrostatic energy of a given source configuration can be lowered by adopting a spatially varying instanton density. Detailed study of the thermodynamics of this phenomenon yields the magnetostrictionlike phenomenon mentioned above.

A significant feature of this treatment is that the external field, in effect, acts as an infrared cutoff on instantons and other large-scale fluctuations: For strong enough external field no artificial cutoff on scale sizes is needed because the internal dynamics of the system suppress the unmanageable large-scale fluctuations. In fact, for sufficiently large external field (i.e., sufficiently near the external sources) the vacuum is in an abnormal phase where, as it turns out, the total instanton density is small, and gets smaller with increasing external field. We furthermore find that there is an instability which makes this phase cease to exist below a critical field strength, E_c. At this critical field, there is a first-order phase transition to a phase where large-scale fluctuations are *not* intrinsically suppressed and whose properties we can at best estimate very roughly. The existence of the phase transition can be demonstrated quite convincingly and leads to a strikingly simple picture of static hadron structure: There is a well-defined radius, corresponding to the critical field, at which the two above-mentioned phases coexist; inside that radius and near the color sources, the vacuum is in the simple phase where large-scale fluctuations are suppressed and reliable calculations are possible (abnormal vacuum) while, outside that region, the vacuum is in the phase where large-scale fluctuations are not suppressed and we can at best make rough estimates of vacuum properties (normal vacuum). Quarks turn out to be confined to the abnormal vacuum region where their dynamics can be reliably treated. The overall picture of static hadron structure which emerges is identical in broad outlines to the MIT bag model with the added bonus that since we now see how the various elements of the bag model arise from an underlying field theory, we can rationally try to extend it to new problems.

There are therefore three major results that emerge from a study of the Yang-Mills vacuum as a polar medium: First, for sufficiently strong background external field, the vacuum is in a phase where the large-scale fluctuations are completely under control and nonperturbative effects due to instantons are small and calculable; second, this phase ceases to exist at a critical value of the external field and undergoes an abrupt first-order phase transition to a phase where large-scale fluctuations are not under control and instanton effects are large and not easily calculable; third, when applied to the problem of static hadron structure, the above two facts lead to a phenomenological model very similar to the MIT bag model. At the same time that we have learned how and where we may use the dilute-instanton-gas picture without apologies, we have discovered the mechanism that causes the bag model to emerge as a phenomenological approximation of QCD.

This paper is devoted to a general development of these results. No detailed numerical calculations are attempted since at our present level of understanding we are not in a position to do much better than existing bag calculations, though improvement should be possible. In Sec. II, we remind the reader of various relevant facts about instantons and merons. In Sec. III, we discuss the properties of the QCD vacuum from the point of view of the magnetostatics of a polar medium. In Sec. IV, we compute the permeability of that medium. In Sec. V, we discuss the consequences of the possibility that the instanton density and therefore the vacuum permeability may have reasonably sharp discontinuities. In Sec. VI, we show how variations in the instanton density necessarily arise by virtue of a phenomenon similar to electrostriction and then demonstrate the existence of an instability which must give rise to a first-order phase transition and sharp discontinuities in vacuum properties. In Sec. VII, we use these insights to construct a possibly improved bag model which automatically incorporates chiral-symmetry constraints as any proper descendant of QCD must. In Sec. VIII, we show that the bag has the important property of shielding the quarks inside it from large color field fluctuations in the vacuum. In Sec. IX, we discuss the implications of all this as well as suggestions for future work.

II. ANALOG GAS OF MERONS AND INSTANTONS

The analog gas of instantons and merons, developed in Ref. 1, yields an intuitive physical picture of Euclidean QCD and enables one to use all the machinery of classical statistical mechanics for practical calculations. Let us recall this picture.

The vacuum-to-vacuum amplitude in QCD can be written in the leading semiclassical approximation as

$$Z = \langle \text{vac}|e^{-HT}|\text{vac}\rangle = \int (\mathfrak{D}A_\mu)e^{-S(A)} = \sum_{n_+,n_-} \frac{1}{n_+!}\frac{1}{n_-!}\int\left(\prod_{i=1}^{n_+} dx_i^+\right)\left(\prod_{j=1}^{n_-} dx_j^-\right) \times \prod_i \int \frac{d\rho_i^\pm}{(\rho_i^\pm)^5} D(\rho_i^\pm)\,. \quad (1)$$

This results from evaluating the Euclidean functional integral by saddle-point integration about the (approximate) multiple instanton–anti-instanton saddle points. The saddle points are given by a superposition of Belavin-Polyakov-Schwartz-Tyupkin (BPST) solutions[2] in the singular gauge

$$A_\mu^\alpha(n_+,n_-)=\sum_{i=1}^{n_+}\frac{2R_{\alpha a}^{i+}\bar\eta_{a\mu\nu}(x-x_i^+)^\nu\rho_i^2}{[(x-x_i^+)^2+\rho_i^2](x-x_i^+)^2}+\sum_{i=1}^{n_-}\frac{2R_{\alpha a}^{i-}\eta_{a\mu\nu}(x-x_i^-)^\nu\bar\rho_i^2}{(x-x_i^-)^2[(x-x_i^-)^2+\bar\rho_i^2]},\qquad(2)$$

where each instanton (anti-instanton) has 12 collective coordinates: position x_i^+ (x_i^-), scale size ρ_i ($\bar\rho_i$), and orientation within SU(3) determined by the unitary matrix $R_{\alpha a}$ ($\alpha=1,\ldots,8$; $a=1,\ldots,3$). The density function $D(\rho)$ has been found to be, for an SU(3) gauge group,

$$D(\rho)=C_0x^6e^{-x},\qquad(3)$$

where C_0 is a renormalization scheme dependent constant (values to be given later) and

$$x(\rho)=\frac{8\pi^2}{g^2(\rho)}\approx-11\ln(\rho\mu)\qquad(4)$$

in the small-ρ limit.

Our expression for Z is identical to the grand canonical partition function for a perfect (ideal) gas of instantons and anti-instantons, with equal densities given by $n_0=\int(d\rho/\rho^5)D(\rho)$. In the perfect-gas approximation the dominant configuration consists of an equal density, n_0, of instantons and anti-instantons. This approximation is valid only if n_0 is sufficiently small, since the mean separation between instanton and anti-instanton is proportional to $n_0^{-1/4}$. Knowing the density of instantons we can inquire whether they are indeed far apart. This is the case only if we cut off the ρ integration. Our aim in this paper is to study certain effects which arise precisely because instantons interact and the gas is not perfect. In Ref. 1 we computed the long-range interaction between instantons and anti-instantons. We found that an instanton of size ρ in a weak slowly varying background field, $F_{\alpha,\mu\nu}^{\text{ext}}$, may be assigned an interaction energy

$$S_{\text{int}}=\frac{2\pi^2}{g^2}\rho^2R_{\alpha a}\bar\eta_{a\mu\nu}F_{\alpha,\mu\nu}^{\text{ext}},\qquad(5)$$

and that the interaction energy of an instanton of size ρ with an anti-instanton of size $\bar\rho$ separated by distance R_μ is given by

$$S_{\text{int}}=\frac{8\pi^2}{g^2}\frac{\rho^2\bar\rho^2}{R^4}R_{\alpha a}^{(1)}R_{\alpha b}^{(2)}T_{\mu\mu'}\eta_{a\mu'\nu'}T_{\nu'\nu}\bar\eta_{b\mu\nu},\qquad(6)$$

where $T_{\mu\nu}=g_{\mu\nu}-2\hat R_\mu\hat R_\nu$. The interaction energy of two instantons or two anti-instantons vanishes.

The interaction energy at large distances falls off like $1/R^4$, which is what one expects of a dipole-dipole interaction in four dimensions. In the following we shall pursue this analogy and show that the long-range behavior of the instanton gas is precisely that of a gas of colored magnetic dipoles in four spatial dimensions. The instantons are not *pure* dipoles—it turns out that underneath the R^{-4} term of (6) there is an R^{-8} term which may presumably be neglected unless the instanton gas is extremely dense.

For very small scales only instanton and Gaussian fluctuations need to considered. However, at large scales, where the coupling becomes bigger, we must also consider merons.[6] The gas of meron (and antimeron) pairs can be developed exactly as in the case of instantons. The meron pair solution has the form

$$A_\mu^\alpha=R_{\alpha a}\eta_{a\mu\nu}\left[\frac{(x-x_1)^\nu}{(x-x_1)^2}+\frac{(x-x_2)^\nu}{(x-x_2)^2}\right]\qquad(7)$$

corresponding to a meron singularity at x_1 and x_2. These singularities need to be smoothed out. This can be done in a way that does not introduce an arbitrary scale (or cutoff) by inserting constraints (and their associated Jacobians) into the functional integral and integrating over all meron core radii. The Gaussian functional integral about the constrained solution is then perfectly well-defined and, up to one overall constant, has been performed.[1] In a gas of meron pairs the density of pairs of core sizes r_1 and r_2, small compared to the meron separation Δ, is equal to[1]

$$\int C\frac{dr_1}{r_1^5}\frac{dr_2}{r_2^5}d^4\Delta\exp[-\tfrac12x(r_1)-\tfrac12x(r_2)-S_m(r_1,r_2;\Delta)],\qquad(8)$$

where C is a constant containing some powers of $\bar g$, but otherwise independent of r_1, r_2, and Δ. The terms $\frac12x(r_i)$ are simply the action of half an instanton in each core, and S_m is the renormalized interaction of two merons, its dependence on r_1, r_2, and Δ given by

$$\frac{\partial S_m}{\partial\ln\Delta}=\tfrac34\hat x(\Delta),\quad\frac{\partial}{\partial\ln r_i}S_m=-\tfrac38\hat x(r_i),\qquad \hat x=x-6.55.\qquad(9)$$

The derivatives are the same as those of the classical action,

$$S=\frac{6\pi^2}{g_0^2}\ln\frac{\Delta}{(r_1r_2)^{1/2}},$$

with the bare coupling, g_0, replaced by the indicated running couplings. With the Gaussian integral now evaluated, merons are on the same footing as instantons.

Meron pairs, just like instantons, interact as four-dimensional dipoles. Equations (5) and (6) can be taken over for meron pairs except that one must replace $2\rho^2\eta_{a\mu\nu}$ by

$$\tfrac{1}{2}\Delta^2[\eta_{a\mu\nu} - \hat{\Delta}^\lambda\hat{\Delta}^\nu\eta_{a\mu\lambda} - \hat{\Delta}^\lambda\hat{\Delta}^\mu\eta_{a\lambda\nu}].$$

We shall see below that a meron pair of separation Δ can be regarded as a dipole with dipole moment proportional to Δ^2. Large meron pairs are more important than instantons simply because the density of such pairs is proportional to $\Delta^3 d\Delta e^{-S_m}$, whereas an instanton of equal size has density proportional to $d\Delta\,\Delta^{-5}D(\Delta)$. Once $\hat{x}(\Delta) \leq \frac{32}{3}$, or $x(\Delta) \leq 17.22$, meron pairs dominate. In what follows, we will not make heavy explicit use of the meron pairs, but it is important to recognize that they have exactly the same kind of long-range interactions as do instantons and will participate in all the physical effects we are about to describe.

III. FOUR-DIMENSIONAL MAGNETOSTATICS

In the previous section we have seen that the Euclidean functional integral, evaluated in the semiclassical approximation, can be regarded as the grand canonical partition function of a gas of dipolelike objects—instantons and meron pairs. Here we shall develop this analogy further, showing that instantons and meron pairs behave exactly like four-dimensional permanent dipoles and that computing their contribution to the Euclidean vacuum functional reduces to a study of the properties of a four-dimensional polar medium. We will find that techniques and concepts developed in the study of three-dimensional polar media carry over very nicely to our four-dimensional problem and provide helpful new insights.

First, we note that the Euclidean vacuum-to-vacuum amplitude

$$Z = \int [dA_\mu] \exp\left[-\frac{1}{4g^2}\int \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})d^4x\right],$$

is the partition function of (colored) *static magnetism* in four dimensions. In other words, we can regard $F^\alpha_{\mu\nu}$ ($\mu, \nu = 1, \ldots, 4$; $\alpha = 1, \ldots, 8$) as the spatial components of a five- (four space, one time) dimensional non-Abelian gauge theory, whose equilibrium thermodynamics is determined by Z. The electric field, $F_{0\mu}$, integrates out of the classical partition function and the energy density is simply

$$\mathcal{E}(x) = \frac{1}{4g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} = \frac{1}{4g^2}\sum_{\alpha=1}^{8} B^\alpha_{\mu\nu}B^\alpha_{\mu\nu}, \tag{10}$$

where the magnetic field, $B^\alpha_{\mu\nu}$, is defined as

$$B^\alpha_{\mu\nu} = \tilde{F}^\alpha_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\gamma\beta}F^\alpha_{\gamma\beta}. \tag{11}$$

This picture is, of course, valid in general for Euclidean Yang-Mills theory. However, when Z is evaluated by saddle-point integration about the instanton and meron pair saddle points, the picture simplifies. As long as the density of instantons and meron pairs, as well as the effective coupling, are reasonably small, expanding the functional integral about the instanton and meron pair saddle point amounts to linearizing the Yang-Mills equations. We can then regard Z as the thermodynamic partition function for a linear, Abelian, five-dimensional static gauge theory, in a medium of instanton and meron pair dipoles (whose statistical mechanics must also be done, including their interactions with each other and the linearized gauge field). But this is just a four-dimensional version of the magnetostatics of a polar medium, a system about which a great deal is known. In what follows we will work out four-dimensional magnetostatics and show how it applies to our system of instantons and meron pairs.

Magnetostatics in four spatial dimensions can be treated in complete analogy to conventional electrodynamics. In a vacuum, the energy density is given by the Abelian version of Eq. (10). The dipolar medium may have a net dipole moment density which will be specified by a bulk magnetization tensor, $M_{\mu\nu}$ (eventually we will derive an expression for $M_{\mu\nu}$ in terms of the properties of elementary dipoles). With the help of $M_{\mu\nu}$ we may define two useful field variables: $B_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$, the microscopic magnetic field (whose sources are the dipoles and any possible truly external sources), and second,

$$H_{\mu\nu} = B_{\mu\nu} - 4\pi^2 M_{\mu\nu}. \tag{12}$$

It will turn out that *only* the external sources act as sources for H. In the absence of external sources, $H=0$ and the net magnetization is zero. Therefore, for weak fields we should be able to set

$$M_{\mu\nu} = \chi H_{\mu\nu}, \quad B_{\mu\nu} = \mu H_{\mu\nu}, \tag{13}$$

where $\mu = 1 + 4\pi^2\chi$ is the permeability of the medium (χ is called the susceptibility). One can further show that the energy density, including the orientation energy of the dipoles, in the presence of an external source, J_μ, is

$$\mathcal{E} = \tfrac{1}{4}B_{\mu\nu}H^{\mu\nu} + J_\mu A^\mu = \tfrac{1}{4\mu}B_{\mu\nu}B^{\mu\nu} + J_\mu A^\mu. \tag{14}$$

There is, of course, a vector potential for $B_{\mu\nu}$,

$$F_{\mu\nu} = \tilde{B}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}B_{\alpha\beta} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{15}$$

which guarantees that B is divergenceless

$$\partial^\mu B_{\mu\nu} = \tfrac{1}{2}\partial^\mu \epsilon_{\mu\nu\alpha\beta}\tilde{B}_{\alpha\beta} = 0\,. \tag{16a}$$

The usual Maxwell equation which says that the source of $F_{\mu\nu}$ is the total current density may be converted to a statement that the source of $H_{\mu\nu}$ is the *external* current

$$\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\mu H_{\alpha\beta} = \partial^\mu \tilde{H}_{\mu\nu} = J_\nu\,. \tag{16b}$$

In the Lorentz gauge, $\partial^\mu A_\mu = 0$, and we have (for constant μ), a simple equation for A_μ:

$$\Box A_\mu = \mu J_\mu\,. \tag{17}$$

We may of course amuse ourselves by solving Eqs. (16) and (17) with various boundary conditions. A sample problem, whose solution will be needed in order to compute μ itself, is the determination of the field $B^C_{\mu\nu}$ inside an empty spherical cavity in a medium of permeability μ, under the condition that the external field approach a constant value $B^E_{\mu\nu}$ far awar from the cavity. Textbook manipulations, given in Appendix I, show that

$$B^C_{\mu\nu} = \frac{2}{\mu+1}B^E_{\mu\nu}\,,$$

i.e., that the field in the cavity is reduced if $\mu > 1$ (as will be the case).

Since permanent dipoles play such a large role here, we shall want to study the field of a single dipole located at a particular point, x_0. The relevant solution of $\Box A = 0$ is

$$A_\mu = D_{\mu\nu}\partial^\nu \frac{1}{(x-x_0)^2}\,, \tag{18}$$

where $D_{\mu\nu} = -D_{\nu\mu}$ which has dimensions of length squared, is the "dipole moment." The source of this field, in the sense of Eq. (17), is easily seen to be

$$J_\mu = -D_{\mu\nu}\partial^\nu 4\pi^2 \delta^{(4)}(x-x_0)\,. \tag{19}$$

Note that dipoles in four space dimensions are characterized by antisymmetric tensors. This is also the case in three dimensions, but there the moment tensor, $\int d^3x(x_a J_b - x_b J_a)$, is equivalent to a single vector. In four dimensions, however, there are *two* types of dipole according to whether $D_{\mu\nu}$ is self-dual or anti-self-dual, and in general ($\tilde{D}_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}D_{\alpha\beta}$)

$$D_{\mu\nu} = D^+_{\mu\nu} + D^-_{\mu\nu}\,,$$
$$D^\pm_{\mu\nu} = \tfrac{1}{2}(D_{\mu\nu} \pm \tilde{D}_{\mu\nu}) = \pm\tilde{D}^\pm_{\mu\nu}\,. \tag{20}$$

The magnetic field, $B_{\mu\nu}$, that arises from a dipole located at the origin is easily found to be

$$F_{\mu\nu} = \tilde{B}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}B_{\alpha\beta} = \frac{4}{x^4}T_{\mu\alpha}(x)D_{\alpha\beta}T_{\beta\nu}(x)\,, \tag{21}$$

where

$$T_{\alpha\beta}(x) = \delta_{\alpha\beta} - 2\hat{x}_\alpha \hat{x}_\beta\,. \tag{22}$$

The tensor $T_{\alpha\beta}$ has the important properties

$$\det(T) = -1\,,$$
$$T_{\alpha\beta}T_{\beta\gamma} = \delta_{\alpha\gamma}\,, \tag{23}$$
$$x^\alpha T_{\alpha\beta} = -x^\beta\,,$$

from which it follows that

$$B_{\mu\nu} = -\frac{4}{x^4}T_{\mu\alpha}(x)\tilde{D}_{\alpha\beta}T_{\beta\nu}(x)\,. \tag{24}$$

This means that if $\tilde{D}_{\mu\nu}$ is self-dual (anti-self-dual), then $B_{\mu\nu}$ is anti-self-dual (self-dual): A dipole moment of given duality creates a magnetic field of opposite duality.

Let us evaluate the interaction energy of a dipole, $D_{\mu\nu}$, with a background magnetic field, $B_{\mu\nu}$,

$$\begin{aligned}\mathcal{E} &= \int d^4x A_\mu(x) J_\mu(x) \\ &= -\int d^4x A_\mu(x) 4\pi^2 D_{\mu\nu}\partial^\nu \delta^4(x) \\ &= +\int d^4x 4\pi^2 \partial^\nu A_\mu(x) D_{\mu\nu}\delta^4(x) \\ &= -2\pi^2 \tilde{B}_{\mu\nu}D_{\mu\nu}\,.\end{aligned} \tag{25}$$

If the dipole moment, $D_{\mu\nu}$, is self-dual (anti-self-dual) there will be no interaction with an anti-self-dual (dual) magnetic field. Therefore, since the duality of a dipole magnetic field is opposite to that of the dipole that produces it, two self-dual (or anti-self-dual) dipoles will not interact. This is the underlying reason why instantons do not interact with instantons.

Finally, we record the expression for the interaction energy of two dipoles, $D^1_{\mu\nu}$ and $D^2_{\mu\nu}$ separated by vector distance R_μ. Using Eqs. (23), (25) we find

$$\mathcal{E} = -\frac{8\pi^2}{R^4}T_{\mu\alpha}(\hat{R})D^1_{\alpha\beta}T_{\beta\nu}(\hat{R})D^2_{\mu\nu} \tag{26}$$

which, of course, vanishes if $D^1_{\mu\nu}$ and $D^2_{\mu\nu}$ are both either self-dual or anti-self-dual.

Let us now return to our analog gas of instantons and meron pairs. Comparing Eqs. (2) and (18), we see that the long-range field of an instanton is simply that of a (colored) permanent magnetic dipole, with dipole moment

$$D^\alpha_{\mu\nu} = -\rho^2 R_{\alpha a}\bar{\eta}_{a\mu\nu}\,. \tag{27}$$

The color index, α, indicates that there is an independent moment for each component of color. Furthermore, the interaction energy of an instanton and an anti-instanton, given in Eq. (6), is precisely that of Eq. (26), provided we sum over the independent interaction energies of each color component. Note that the non-Abelian aspect of color plays no role here, except of course insofar as it is responsible for the existence of elementary dipoles in the first place. (It should be borne in mind that this is due to our decision to linearize the theory about a certain class of saddle points, an approximation with a limited, but useful, range of validity.)

In ordinary electrodynamics permanent dipoles exist only due to the quantization of spin. In QCD they exist due to the quantization of topological charge—a unique feature of four-dimensional non-Abelian gauge theory. The other special non-Abelian feature of our theory is the existence of merons, which we shall see are magnetic *null poles*. One might have thought that, since instantons are magnetic dipoles, they should have as constituents a pair of magnetic monopoles. Recall, however, that a monopole field in four dimensions falls off as $1/r^2$ (if $A_\mu \simeq 1/r^{1+l}$ and $F_{\mu\nu} \simeq 1/r^{2+l}$, we say that we have an l pole.) Thus for such fields, as for dipole fields, the equations of motion linearize asymptotically and are easily analyzed. However, there are no solutions of the monopole type. Alternatively, we can argue that to conserve topological charge the constituents of the instanton must have fractional topological charge. But this requires fields that fall off as $1/r$, i.e., null poles. Merons, which are solutions (albeit singular) of the nonlinear equations of motion, are precisely such fields—magnetic null poles with one-half unit of topological charge.[7]

Since merons are inherently nonlinear we are not able to treat a gas of ionized merons by the linear techniques developed above. This the main impediment to a full understanding of "meron plasma." Isolated meron pairs, however, are simply distorted instantons and, like instantons, have a dipole moment. Indeed we may deduce from Eq. (7) (transformed to singular gauge) that the dipole moment of a meron pair, separated by vector distance Δ_μ is

$$(D^{\alpha}_{\mu\nu})_{\text{meron pair}} = \tfrac{1}{8}\Delta^2 R_{\alpha a}[\eta_{a\mu\nu} + T_{\mu\alpha}(\hat{\Delta})T_{\nu\beta}(\hat{\Delta})\eta_{a\alpha\beta}]\,. \quad (28)$$

By virtue of the remarks following Eq. (24), this is the sum of a self-dual and an anti-self-dual piece of equal magnitude.

Finally, we should indicate how quarks or other color-carrying external fields enter this magnetostatic picture. Recall that quark-antiquark time histories are represented in Euclidean field theory by current loops which contribute to the partition function a term $\mathrm{Tr}P\exp(i\int A^\mu dx_\mu)$. (Quark dynamics are accounted for by integrating over all possible loops.) When one does the ordinary statistical mechanics of a loop carrying a fixed current, the Boltzmann factor of the magnetic field is weighted by the same factor to take account of the work necessary to maintain the fixed current. [The reader who is surprised that i appears in classical statistical mechanics may note that the interaction of a charged particle with a magnetic field, $\int A_\mu(dx^\mu/dt)dt$, remains real upon passing to imaginary time. Since A_μ and $-A_\mu$ contribute equally, the net contribution of the phase factor is real.]

With the aid of the picture developed above, many of the physical effects of instantons are easily understood. A gas of permanent magnetic dipoles gives rise to a *positive* susceptibility and therefore to a permeability, μ, greater than one. This is paramagnetism and corresponds to antiscreening behavior.

Placing a current loop in such a paramagnetic medium will increase its self-energy, the opposite of what one would expect from Lenz's law. As this antiscreening is of fundamental importance in QCD, we will discuss it in some detail, starting with the more familiar case of screening. In classical statistical mechanics there is no paramagnetism (which only arises from quantization of angular momentum). This is a consequence of Lenz's law: A current loop responds to an applied field in such a way as to reduce the net field. This phenomenon is easily understood in terms of the functional integral

$$Z = \int dA_\mu \exp\left(-\tfrac{1}{2}\int B^2_{\mu\nu}d^4x + i\oint A^\mu dx_\mu\right) \quad (29)$$

for a current loop. If the loop is small, it may be characterized by an area element, $S_{\mu\nu}$, normal in four dimensions to the loop surface. Then $i\int A^\mu dx_\mu \simeq B_{\mu\nu}(x_0)D_{\mu\nu}$, where x_0 is the center of the loop and $D_{\mu\nu} = iS_{\mu\nu}$ is an *imaginary* effective dipole moment. Since the susceptibility is, as we shall see, always proportional to $\langle D^2_{\mu\nu}\rangle$, χ will be negative and μ less than 1.

Thus in a functional integral formalism, Lenz's law is the statement that dipole moments of current loops are imaginary. This carries over to field theory. The action for a scalar field, $(D_\mu\psi)^2$, coupled to only a gauge field A_μ, is bilinear in ψ and the integration over ψ can be performed explicitly yielding $(\det D^2)^{-1/2}$. If the determinant is expanded in the number of closed loops, each term can be represented as a path integral over closed loops $x_\mu(\tau)$. In the Euclidean formalism all factors are real and positive *except* for a path ordered exponential $P\exp(i\oint A^\mu dx_\mu)$. For a small loop and slowly varying A_μ, we have

$i \oint A^{\mu} dx_{\mu} = F^{\mu\nu} M_{\mu\nu}$, where $M_{\mu\nu}$ is an imaginary dipole moment. Thus scalar fields coupled to gauge fields obey Lenz's law and screen rather than antiscreen. In the weak-coupling limit this is just the familiar statement that scalar fields make a positive contribution to the β function. Instantons and meron pairs, on the other hand, are permanent dipoles. They have real dipole moments due to the quantization of topological charge and antiscreen, or cause the coupling, to increase.

The effect of instantons on quark interactions is then analogous to the behavior of an electric current loop in a paramagnetic gas of magnetic dipoles. The dipoles antiscreen and cause self-energy of the loop to increase. Scalar fields (or fermion fields) work in the opposite way: They screen an external current. The limit of complete screening is what one would obtain from the Higgs mechanism.

Meron pairs work in the same direction as instantons since they, too, carry a permanent dipole moment. Moreover, since the moment increases rapidly with pair size, the susceptibility would diverge if it were possible to integrate over arbitrarily large separations. Divergent susceptibility, a property which one might call "perfect paramagnetism," would imply confinement. Unfortunately, because merons of large separation become dense and impossible to treat pairwise, we are unable to say whether the susceptibility actually diverges, but can say that it must grow rapidly as we move into scales dominated by merons. In what follows we will see that we can actually compute the susceptibility in situations where it is large (albeit finite) and that by studying the possibility of spatially varying susceptibility, we can see a plausible way of understanding low-mass hadrons.

IV. PERMEABILITY OF THE VACUUM

We have demonstrated that the four-dimensional gas of instantons and meron pairs is a gas of permanent magnetic dipoles which, by analogy with three-dimensional magnetic systems, should give rise to a positive vacuum susceptibility and antiscreen colored fields. In this section we shall show how one can reliably compute the vacuum permeability, even in circumstances when it is large. The calculation we shall do is exact insofar as we can neglect the short-range interactions of the dipoles, even if the resulting permeability is large. So, once again, the basic approximation is to consider configurations with integrated instanton and meron pair density less than (but not necessarily very much less than) one.

This is a big improvement over Ref. 1, where we found that the basic hadron scale had to be the scale where integrated density became unity, but could not actually compute in that regime because we did not see how to treat the cooperative effects of instantons on each other. Let us now review the discussion of permeability given in Ref. 1. We start with a computation of the response of a single dipole to a weak external field, ignoring the interaction with other dipoles. Consider then a particular instanton, located at the origin, of size ρ and group orientation $R_{\alpha a}$. It has a dipole moment $D^{\alpha}_{\mu\nu} = -\rho^2 R_{\alpha a}\eta_{a\mu\nu}$. If we apply a weak external field, $B^{\alpha}_{\mu\nu}$, the energy of the instanton in this field is, according to Eq. (25), given by

$$E(\rho, R) = -\frac{2\pi^2}{g^2(\rho)} \tilde{B}^{\alpha}_{\mu\nu} D^{\alpha}_{\mu\nu}. \tag{30}$$

Thus the orientation-dependent part of the Boltzmann factor for the instanton (neglecting its interaction with the anti-instantons in the gas) is

$$P(\rho, R) = \frac{\exp\{-[2\pi^2/g^2(\rho)]\tilde{B}^{\alpha}_{\mu\nu}\rho^2 R_{\alpha a}\bar{\eta}_{a\mu\nu}\}}{\int [dR] \exp\{-[2\pi^2/g^2(\rho)]\tilde{B}^{\alpha}_{\mu\nu}\rho^2 R_{\alpha a}\bar{\eta}_{\mu\nu}\}}. \tag{31}$$

The mean dipole moment of the instanton in the background field, $B_{\mu\nu}$, is

$$\langle D^{\alpha}_{\mu\nu}\rangle = \int [dR_{\alpha a}] P(\rho, R) D^{\alpha}_{\mu\nu}. \tag{32}$$

$dR_{\alpha a}$ is normalized so that

$$\int [dR] R_{\alpha a} R_{\beta b} = \tfrac{1}{8}\delta_{\alpha\beta}\delta_{ab}, \qquad \int [dR] = 1, \quad \text{and} \quad \int [dR] R_{\alpha a} = 0. \tag{33}$$

For a weak external field, $\tilde{B}$, this is easily calculated by expanding the exponent in Eq. (31) and keeping the first nonvanishing term:

$$\langle D^{\alpha}_{\mu\nu}\rangle = \int [dR]\, \rho^2 R_{\alpha a}\bar{\eta}_{a\mu\nu}\left(+\frac{2\pi^2}{g^2(\rho)}\rho^2 R_{\beta b}\bar{\eta}_{b\lambda\epsilon}\tilde{B}^{\beta}_{\lambda\epsilon}\right) = \frac{2\pi^2}{g^2(\rho)}\frac{\rho^4}{8}\bar{\eta}_{a\mu\nu}\bar{\eta}_{a\lambda\epsilon}\tilde{B}^{\alpha}_{\lambda\epsilon}. \tag{34}$$

But $\bar{\eta}_{a\mu\nu}\bar{\eta}_{a\lambda\epsilon} = \delta_{\mu\lambda}\delta_{\nu\epsilon} - \delta_{\mu\epsilon}\delta_{\nu\lambda} - \epsilon_{\mu\nu\lambda\epsilon}$, and therefore

$$\langle D^{\alpha}_{\mu\nu}\rangle = \frac{4\pi^2}{g^2(\rho)}\frac{\rho^4}{8}(B^{\alpha}_{\mu\nu} - \tilde{B}^{\alpha}_{\mu\nu}). \tag{35}$$

To evaluate the net magnetization of the instanton gas, we must add the mean dipole moments of all the instantons and anti-instantons (for which we obtain the same expression with $B - \tilde{B}$ replaced by $B + \tilde{B}$). Since the vacuum density of these is equal, and given by $n(\rho)$ [see Eq. (4) for the perfect gas], we find that the magnetization, $M_{\mu\nu}$, is

$$M_{\mu\nu}=\int\frac{d\rho}{\rho}n(\rho)\frac{8\pi^2}{g^2(\rho)}\frac{\rho^4}{8}B_{\mu\nu}\,. \tag{36}$$

This result is of general validity so long as B is sufficiently small and interpreted as the local field at the position of the dipole (not necessarily the same as the average B field in the medium). We could in fact, calculate the exact nonlinear dependence of M on B by not expanding the integrand in Eq. (31). In this paper, however, we will work in the weak-field approximation since it produces linear equations and relatively transparent physics. In a sufficiently dilute-gas configuration, μ is close to one and distinctions between local and average fields and between B and H are not important. In that case, following Ref. 1, we may convert Eq. (36) to a dilute-gas (DG) expression for the permeability,

$$\begin{aligned}\mu_{DG}&=1+4\pi^2\chi_{DG}\\&=1+4\pi^2\int\frac{d\rho}{\rho5}D(\rho)\frac{8\pi^2}{g^2(\rho)}\frac{\rho^4}{8}\,,\end{aligned} \tag{37}$$

where we take $n(\rho)=n_0(\rho)=(1/\rho^4)D(\rho)$.

This expression for μ can be written as

$$\mu_{DG}=1+\frac{\pi^2}{2}\int\frac{d\rho}{\rho}C_0x^7(\rho)\exp[-x(\rho)]\,, \tag{38}$$

or, if we take the asymptotic-freedom result, $x(\rho)=11\ln(1/\rho\mu)$, as

$$\mu_{DG}=1+\frac{\pi^2}{22}C_0\int_{x_C}^{\infty}dx\,x^7e^{-x}\,. \tag{39}$$

This expression is not defined until the lower limit is specified or, equivalently, until we have decided on the precise set of configurations we propose to integrate over. We will eventually be interested in configurations for which the dilute-gas picture is reasonable, yet μ_{DG}, as defined by Eq. (39), is substantially greater than 1. In other words, the instanton gas is such a strongly polar medium that a simple linear approximation to μ is likely to be inadequate.

To circumvent this difficulty we shall adapt Onsager's[8] treatment of polar dielectrics to four-dimensional magnetostatics. This method involves no approximations except the neglect of all but dipolar interactions.[9] We consider once again a single dipole or instanton and divide the space into two regions: a spherical region which contains the single dipole and the rest of space. The dipoles in the region outside the sphere will be treated as a macroscopic continuous medium with permeability μ. We then calculate the mean magnetic dipole moment of the instanton under consideration in a slowly varying, weak external field $H_{\mu\nu}^{\alpha\,\mathrm{ext}}$. The calculation is similar to the one described above except that we must now distinguish between the external field $H_{\mu\nu}^{\alpha\,\mathrm{ext}}$ and the local field in the spherical cavity, $(B_{\mu\nu}^{\alpha})_{\mathrm{local}}$, which differs from $B_{\mu\nu}^{\alpha\,\mathrm{ext}}$. (See Fig. 1.) The field inside the cavity can easily be calculated by standard magnetostatic techniques. The calculation is performed in Appendix I where we show that

$$(B_{\mu\nu}^{\alpha})_{\mathrm{local}}=\left(\frac{2}{1+\mu}\right)B_{\mu\nu}^{\alpha\,\mathrm{ext}}=\frac{2\mu}{1+\mu}H_{\mu\nu}^{\alpha\,\mathrm{ext}}\,. \tag{40}$$

There is in addition a reaction field on the dipole due to the fact that is polarizes the medium. This field is, however, proportional to $D_{\mu\nu}$ itself and will not affect the orientation. On the other hand, it does affect the scalar density n and in the absence of B^{ext} actually accounts for all the instanton–anti-instanton interactions (in the dipole approximation). We will return to this in Sec. VI. Having calculated $(B_{\mu\nu}^{\alpha})_{\mathrm{local}}$ we can evaluate the approximation as in Eq. (31), and calculate the mean moment of the instanton. It is clear that Eqs. (34), (35) need to be multiplied by $2/(1+\mu)$, and Eq. (37) by $2\mu/(1+\mu)$. Therefore,

$$\mu=1+4\pi^2\frac{2\mu}{1+\mu}\int\frac{d\rho}{\rho}n(\rho)\frac{8\pi^2}{g^2(\rho)}\frac{\rho^4}{8}\,, \tag{41}$$

or if we define

$$\eta\equiv4\pi^2\int\frac{d\rho}{\rho}n(\rho)\frac{8\pi^2}{g^2(\rho)}\frac{\rho^4}{8}=\mu_{DG}-1\,, \tag{42}$$

then

$$\mu=\eta+(\eta^2+1)^{1/2}\,. \tag{43}$$

This expression should be a very good approximation to the permeability. We have, of course, made some approximations. For example, the polarizability of instantons has been ignored. Classically instantons are nonpolarizable permanent dipoles, but, due to quantum fluctuations

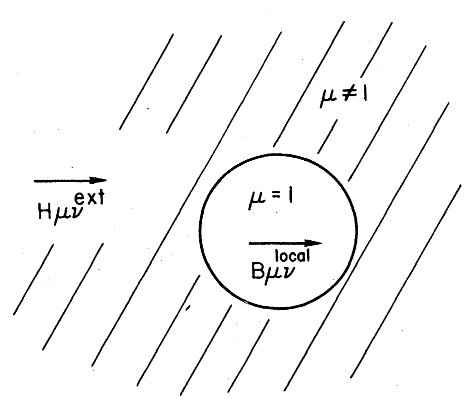

FIG. 1. A spherical cavity with $\mu=1$ in a paramagnetic medium with $\mu\neq1$.

they acquire a finite polarizability. On the other hand, this effect should be of order $g^2/8\pi^2$ and so truly negligible. We have also neglected all but dipolar interactions. This too should be an excellent approximation since the next term in the instanton–anti-instanton interaction energy is an octopole-octopole term which falls off like $1/R^8$. Given these approximations Eq. (43) is exact. One can show that the result is independent of the size of the spherical region which was chosen and holds even when this region is macroscopic.[9] We also note that the analogous treatment of the ordinary dielectric constant works very well even for substances with very large susceptibility (such as water where $\epsilon \approx 78$ at room temperature). Thus, we may trust Eq. (42) even when η is very large and are now in a position to discuss the physics of the regime of scale sizes where instanton effects are large.

We must still, however, calculate the density, $n(\rho)$, of instantons of size ρ. Once the gas is not dilute it is not a good approximation to use for $n(\rho)$ the perfect gas density $n_0(\rho) = (1/\rho^4)D(\rho)$. Instead, we must include the effect of instanton interactions which will change the density. In Sec. VI, where we calculate the structure of the vacuum in the presence of color fields, such calculations are presented.

Finally, one must conclude that the susceptibility of the vacuum is large. The calculation of η, including instantons up to an integrated density of one, is precisely the same as described in the discussion of Eq. (39), and gives the large value $\eta \simeq 10$. According to Eq. (43), when η is large, $\mu \simeq 2\eta$ and we conclude that the vacuum permeability is at least $\mu \simeq 20$. While this is not yet perfect paramagnetism, it is very strong paramagnetism. We shall see in the next few sections how it can be exploited in some very interesting ways having a bearing on the problem of hadron structure. In fact, we believe that the breakup of meron pairs will actually make μ diverge.[1] [From Eqs. (8), (9), and (28) it follows that the contribution of meron pairs to η is $\int \Delta^4 d^4\Delta \exp(-S_m)$; this integral, which we have encountered elsewhere,[1] begins to diverge when the upper limit is extended beyond $\hat{x}(D) = x(D) - 6.55 = \frac{32}{3}$.] The detailed way in which this happens is not understood, but since instantons by themselves produce a large permeability ($\gtrsim 20$) the precise manner in which μ diverges may not be important.

V. BOUNDARY CONDITIONS AND BAG FORMATION

We have seen in the preceding sections that the naturally occurring density of instanton and meron pairs in Euclidean space-time makes the vacuum behave like a medium of large (probably infinite) permeability, μ. It is implicit in our whole approach that the permeability at a point depends on the instanton and meron pair density, n, at that point. In the undisturbed vacuum, it is obvious that this density, and therefore the permeability, is uniform. In the presence of external sources (i.e., quarks) it is almost certain that the most probable configuration is one in which n (and, therefore, μ) varies from point to point. The thermodynamic reason for this is that while any spatial variation of n raises the free energy, the attendant variation of μ may lower the electrostatic energy of the color field attached to the quarks by an even greater amount. A limiting case of this possibility is the creation of a finite-size region around the quarks where $n \sim 0$ and $\mu \sim 1$ with an abrupt transition to normal vacuum values at the boundary of the region. (See Fig. 2.) This limiting case is obviously very closely related to the MIT bag picture of hadron structure and raises the tantalizing possibility that one may be able to derive the bag, at least as a phenomenological approximation, from QCD first principles. In the next section, we will study the thermodynamics of the QCD vacuum in the presence of external sources and isolate a mechanism which favors the formation of sharp bag boundaries. In the present section, we will study the easier problem of the behavior of the color field in the presence of such boundaries.

Let us therefore assume that the density, and also μ, have baglike space-time variation and see what this implies about the fluctuating; lin-

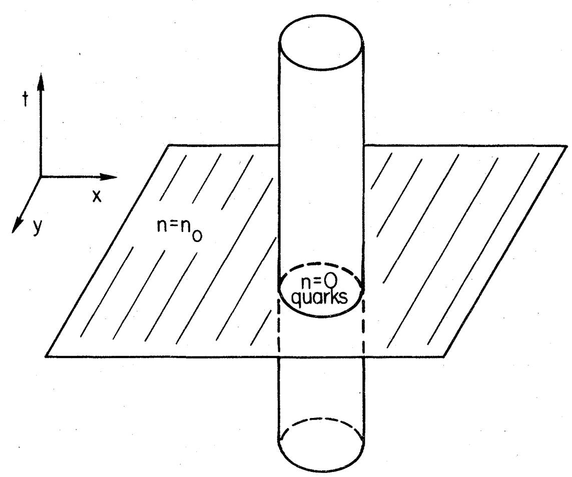

FIG. 2. A static bubble or bag in the QCD vacuum. The cylindrical region corresponds to a region of space whose time histories have no instanton or meron pair tunneling events.

earized Yang-Mills field. If we choose the boundaries of the bag to be time independent, then we can translate our Euclidean picture to a Minkowski description. In Minkowski space, the field tensor $F^{\alpha}_{\mu\nu}$ is related to the usual microscopic average fields by $E^{\alpha}_{i}=F^{\alpha}_{0i}$ and $B^{\alpha}_{i}=\frac{1}{2}\epsilon_{ijk}F^{\alpha}_{jk}$, while the intensity tensor $G^{\alpha}_{\mu\nu}$, composed of D^{α}_{i} and H^{α}_{i}, is related to $F^{\alpha}_{\mu\nu}$ by[10]

$$F^{\alpha}_{\mu\nu}=(1+4\pi^2\chi)G^{\alpha}_{\mu\nu}=\mu G^{\alpha}_{\mu\nu}. \quad (44)$$

In terms of μ the usual dielectric constant ϵ is, as a direct consequence of Lorentz invariance, just $\epsilon=\mu^{-1}$. Note that, for $\chi>0$, $\epsilon=\mu^{-1}<1$ and the interaction energy between real time static charges, $qq'/\epsilon r$, is increased. This is the Minkowski space version of the antiscreening of current loops in four Euclidean dimensions.

Consider now a static bag, with sharp boundaries, inside of which $n\simeq 0$ and $\epsilon=1/\mu\simeq 1$ and outside of which n and μ take on their normal, large, vacuum values. Just as in ordinary electro- or magnetostatistics, in the presence of sharp boundaries there will be boundary conditions on $F_{\mu\nu}$. Equations (16) and (17) imply that in the static case, the curls of E^{α}_{i} and $H^{\alpha}_{i}=(1/\mu)B^{\alpha}_{i}$, and (in the absence of sources on the boundary) the divergences of $D^{\alpha}_{i}=(1/\mu)E^{\alpha}_{i}$ and B^{α}_{i} vanish. Therefore, across the boundary between two regions with differing μ, the normal components, D_n and B_n, of D and B and the tangential components, E_t and H_t, of E and H are continuous.

Let us now use these continuity conditions to express the field energy density $\mathcal{E}$ just *outside* the bag in terms of the fields $\vec{E}^{\alpha}$ and $\vec{B}^{\alpha}$ just *inside* the bag

$$\mathcal{E}=\frac{1}{2}\left[\mu(E^{\alpha}_{n})^2+\frac{1}{\mu}(E^{\alpha}_{t})^2+\frac{1}{\mu}(B^{\alpha}_{n})^2+\mu(B^{\alpha}_{t})^2\right]. \quad (45)$$

In the limit $\mu\to\infty$ (we have so far only succeeded in making μ *large* outside the bag, but it is so large that $\mu=\infty$ is probably a good approximation for discussing boundary conditions), it is clear that on the interior surface of a bag we must have [see Fig. 3(a)]

$$E^{\alpha}_{n}=0, \quad B^{\alpha}_{t}=0. \quad (46)$$

The covariant form of this boundary condition is simply $F^{\mu\nu}n_{\nu}=0$. n_{ν} is the (spacelike) normal to the bag surface. These are, in fact, the boundary conditions of the MIT bag model.[11]

For this picture to make sense, it is necessary to establish two things. First, it must be the case, as we have argued above, that the normal state of the QCD vacuum is indeed a very good and possibly perfect paramagnet with large or infinite permeability μ. Second, it must be the case that

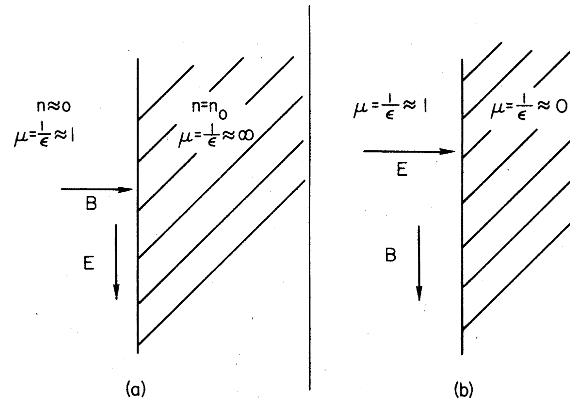

FIG. 3. (a) The boundary conditions at the surface of a perfect paramagnet (QCD vacuum). (b) The boundary conditions at the surface of a perfect diamagnet (superconductor or Higgs vacuum).

in the presence of color sources (quarks), a first-order phase transition will occur. Indeed, the above bag picture is that of two phases in equilibrium, the normal vacuum phase outside the bag with large instanton density and large μ, and an abnormal (although quite familiar) phase inside the bag with small instanton density and small μ. These phases can be in equilibrium due to the existence of the color field produced by the quarks, so that the pressure of the vacuum outside the bag (which will tend to collapse the bag) is balanced by the chromodynamic pressure (plus the kinetic pressure of the quarks) inside the bag.

Note that the boundary conditions $E^{\alpha}_{n}=B^{\alpha}_{t}=0$ are just the *opposite* of those at the surface of a superconductor where $B_n=0$ (the Meissner effect) and $E_t=0$ (since it is a conductor) as in Fig. 3(b). In other words, a superconductor is a perfect diamagnet (i.e., has $\mu=\epsilon^{-1}=0$) and manifests perfect screening. The QCD vacuum, on the other hand, is at least a very good and possibly a perfect paramagnet with $\mu=\epsilon^{-1}=\infty$, manifesting perfect antiscreening (to produce confinement).

It is well known that the field-theoretic analog of perfect diamagnetism (possessed by superconductors) is the Higgs mechanism. If we represent the gauge field propagator as follows,

$$D_{\mu\nu}(k)=\frac{d(k^2)}{k^2}(g_{\mu\nu}+\text{gauge terms}), \quad (47)$$

then the quantity one would normally call μ is just $d(0)$. Obviously, if the Higgs mechanism is active, $D_{\mu\nu}$ has no pole at $k^2=0$ and $\mu=d(0)=0$. Many people have suggested that the QCD vacuum should possess a "magnetic Higgs mechanism"—

i.e., that $\epsilon = \mu^{-1}$ should vanish and that the boundary conditions on $\overline{E}$ and $\overline{B}$ should be reversed from the electric Higgs, or superconductor, case as in Fig. 3(a). We think we have demonstrated that this happens automatically in QCD by virtue of straightforward mechanisms generated by the theory itself.

These boundary conditions are of direct interest only if rapid spatial variations in μ actually occur in the QCD vacuum. Showing that they do is the business of the next section.

In the following section, we shall show that there is indeed such a first-order phase transition in QCD. What we can establish with reliability is that there exist two phases—a gaseous phase with very low density of instantons and small μ and a much denser phase with large density and large μ, which may be in equilibrium in the presence of an external E field of critical value E_c. Although the dilute (gaseous) phase is easily treated, we cannot reliably calculate the properties of the dense (normal vacuum) phase. Thus we cannot, as yet, accurately calculate E_c, which determines the bag constant or prove that $\mu = \infty$ in the dense phase.

If, however, we assume that $\mu = \infty$ (or is at least very large in the dense phase) then a bag will surely be formed if quarks are inserted into the QCD vacuum. Clearly, if such quarks are inserted into the normal (dense) vacuum phase [Fig. 4(a)], they will have a very large energy (infinite if μ is actually infinite). Therefore, a bag, or flux tube, will be created, inside of which the vacuum is in the dilute phase. For this to happen, the field E, created by the quarks, must be tangential to the surface of the bag and equal to E_c at the surface, so that the dense and dilute phase are in equilibrium (recall that the tangential component of E is continuous across the boundary). This will determine the size of the bag. If the quarks are very far apart, a flux tube will be created of transverse size L [see Fig. 4(b)] so that roughly $E_c L^2 = \text{flux}$, and the flux tube will have an energy $\approx E_c^2 R L^2$.

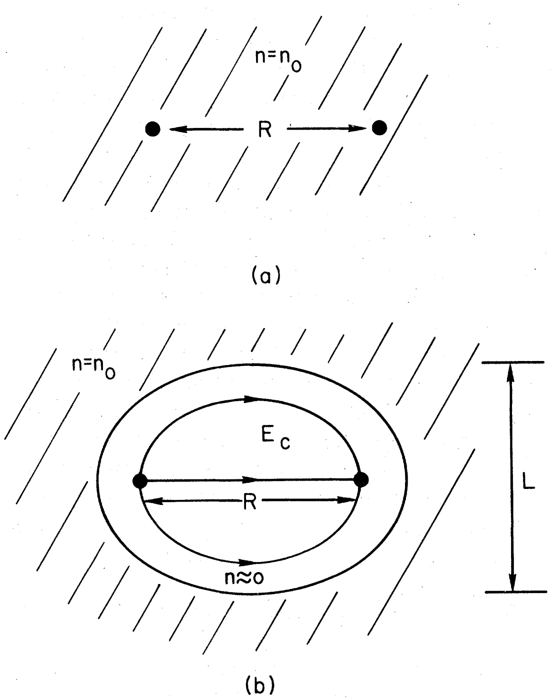

FIG. 4. (a) Quark in a uniform sea of instantons and meron pairs. (b) Quarks in a bubble or bag in the shape of a flux tube.

VI. MAGNETOSTRICTION AND THE FIRST-ORDER PHASE TRANSITION

In the preceding sections, we suggested that the QCD vacuum responds to the insertion of external color sources (quarks) by creating a region of space-time around the source in which the instanton density is substantially reduced. We further suggested that there might exist sharp boundaries (occurring presumably at some critical field strength, E_c) between a low instanton density phase and the normal dense phase. The occurrence of such a first-order phase transition would of course provide a fundamental QCD rationale for the bag picture of hadron structure. Our aim in this section is to show that the desired phase transition does occur.

In order to explore this question we will evaluate the instanton density as a function of a static external field E. There are two competing effects at work to determine the density (beyond the coupling-constant renormalization effect which determines the dilute-gas instanton density). On the one hand, the instantons have long-range dipole interactions which can be attractive and will drive the instanton gas toward collapse. (Since the instantons are not point objects, the collapse is not to a catastrophic infinite density state, but simply to a configuration where the instantons overlap significantly and are probably not too well described as instantons.) On the other hand, the presence of an external field brings a phenomenon (shortly to be explained in detail) much akin to magnetostriction into play: The external field drives the dipole density down, the reduction factor being greater for dipoles with larger moment (scale size). For strong-enough external field, the increase of density suppression with scale size dominates the ideal gas growth of density with scale size and renders the net instanton density not only finite, but small enough that the tendency toward collapse due to the dipolar interactions is defeated. Not surprisingly, as the external field is reduced, the density increases and eventually becomes large enough that the dipolar interactions take over and cause an abrupt collapse to a dense phase. Fortunately for us, the critical density is so low that the approximations we must make to explore the critical region appear quite reasonable. For the same reason, the properties of the dilute phase, which is where the quarks reside in the bag model, will be amenable to quantitative calculation. To explore these questions, we must work out the thermodynamics of instantons in an external field with special attention to the conditions which must be met at a phase boundary.

As we have noted in Sec. II, the Euclidean vacuum-to-vacuum amplitude, in the semiclassical approximation, is equivalent to the grand canonical partition function,

$$Z(\xi)=e^{PV}$$
$$=\sum_{n_+,n_-}\frac{1}{n_+!\,n_-!}V^{n_++n_-}\left[\int\frac{d\rho}{\rho}\xi(\rho)n_0(\rho)\right]^{n_++n_-}, \tag{48}$$

where P is the pressure of the gas,

$$n_0(\rho)=\frac{D(\rho)}{\rho^4}=\frac{C_0x^6e^{-x}}{\rho^4}$$

is the bare density of instantons, and $\xi(\rho)$ is the activity of an instanton or anti-instanton of scale size ρ, which we introduce to simplify some arguments. Of course, we must eventually set $\xi(\rho)$ equal to 1.

The pressure is a function of the activity. On the other hand, so is the density of instantons, or anti-instantons, via the relation

$$2n_+(\rho)=2n_-(\rho)=\frac{1}{V}\xi(\rho)\frac{\partial}{\partial\xi(\rho)}\ln Z(\xi)\bigg|_{\xi=1}$$
$$=\xi(\rho)\frac{\partial P}{\partial\xi(\rho)}\bigg|_{\xi=1}. \tag{49}$$

It is therefore possible to introduce the free energy per unit volume, $F(n)$, which is simply the Legendre transform of the pressure

$$F(n)=\int[\ln\xi(\rho)]2n(\rho)\frac{d\rho}{\rho}-P(\xi). \tag{50}$$

It then follows, by definition, that

$$\frac{\partial F(n)}{\partial n(\rho)}=2\ln\xi(\rho)=0 \quad (\text{when } \xi=1). \tag{51}$$

Thus, the density can be calculated by either evaluating the pressure as a function of the activity and using Eq. (49) or minimizing the free energy with respect to the density.

For the perfect gas of instantons and anti-instantons, the pressure is simply

$$P=2\int\frac{d\rho}{\rho}\xi(\rho)n_0(\rho)$$
$$=2\int\frac{d\rho}{\rho}n_0(\rho) \quad \text{for } \xi=1. \tag{52}$$

Therefore,

$$n(\rho)=\xi(\rho)n_0(\rho)$$
$$=n_0(\rho) \quad \text{for } \xi=1, \tag{53}$$

and

$$F_0(n)=\int\frac{d\rho}{\rho}2n(\rho)\left[\ln\frac{n(\rho)}{n_0(\rho)}-1\right]$$
$$=-\int\frac{d\rho}{\rho}2n(\rho) \tag{54}$$

when

$$\frac{\partial F_0}{\partial n(\rho)}=0.$$

To explore the issues raised in the beginning of this section, we must introduce an external color field and include the dipolar interactions between instantons. The dipole-dipole interactions enter in two ways: First, they change the electrostatic energy of the external field insofar as they influence the way in which μ (permeability) depends on density (this effect has been studied in detail in Sec. IV). Second, even in the absence of an external field, the dipoles line each other up and tend to increase the density. To obtain a simplified demonstration of the phase transition we shall provisionally neglect the second of these effects. At a later stage, we will include the neglected interactions and find that the qualitative picture is not altered significantly.

We will study the dependence of the free energy on a background Minkowski field, E, and instanton density, $n(\rho)$. Depending on whether one wants the free energy as a functional of E or D, there are two useful energy functionals[10]

$$F(n,D)=F_0(n)+\tfrac{1}{2}ED, \tag{55}$$
$$\tilde{F}(n,D)=F_0(n)-\tfrac{1}{2}ED \tag{56}$$

[$F_0(n)$ at this stage is just the ideal gas free energy] whose first variations satisfy

$$\delta F=2\int\frac{d\rho}{\rho}\ln\xi(\rho)\delta n(\rho)+E\delta D,$$
$$\delta\tilde{F}=2\int\frac{d\rho}{\rho}\ln\xi(\rho)\delta n(\rho)-D\delta E. \tag{57}$$

As explained earlier, to obtain the equilibrium configuration of the system, we must set $\xi(\rho)=1$. Therefore, the equation to be solved to determine n is either $(\partial F/\partial n)_D=0$ or $(\partial \tilde{F}/\partial n)_E=0$—either relation gives the same physics because E and D are related to each other. At phase boundaries of the type we are interested in, however, it is $\tilde{F}$ which governs the possibility of equilibrium and for that reason we will henceforth consider only $\tilde{F}$.

We also wish to explore the possibility of equilibrium between two phases. For simplicity we will take the phase boundary to be a plane with E tangential to the boundary (Fig. 3). According to Sec. V, E, since it is tangential, must be continuous across the boundary [hence our interest in $\tilde{F}(n, E)$]. In equilibrium the normal forces on the boundary must balance. When $\partial\tilde{F}/\partial n=0$ the stress tensor is

$$\theta_{ik}=E_i D_k+\delta_{ik}\tilde{F}(n, E)\,, \tag{58}$$

and, since E is tangential, the condition for normal force balance reduces to continuity of $\tilde{F}$:

$$\tilde{F}(n_1, E_c)=\tilde{F}(n_2, E_c)\,. \tag{59}$$

In what follows we will study the response of the system to "large" external fields—where "large" means large enough to put the system in the dilute phase. We will nonetheless assume that the medium responds linearly, i.e., that

$$D=\frac{1}{\mu}E=\epsilon E \tag{60}$$

with μ given as a function of local instanton density by Eqs. (42) and (43). This will be correct as long as E is small enough that the magnetization implied by Eq. (60) is less than the magnetization, M_{sat}, obtained by aligning all the dipoles in color space. Since the dipole moment of a single instanton is essentially $2\rho^2$, the saturation magnetization is

$$M_{\text{sat}}=\int\frac{d\rho}{\rho}n(\rho)2\rho^2\,. \tag{61}$$

On the other hand, the linear magnetization induced by a field E is

$$4\pi^2 M\simeq\frac{\mu-1}{\mu}E\,. \tag{62}$$

Therefore, the linear response condition is

$$4\pi^2 M_{\text{sat}}>\frac{\mu-1}{\mu}E \tag{63}$$

or

$$8\pi^2\int\frac{d\rho}{\rho}n(\rho)\rho^2>\frac{\mu-1}{\mu}E\,. \tag{64}$$

We *assume* that this is also the appropriate condition for Minkowski fields.

We can now calculate the instanton density, $n(\rho)$, in an external field by minimizing $\tilde{F}(E,n)=F_0-(1/2\mu)E^2$ with respect to $n(\rho)$, and using Eq. (42) to obtain μ as a functional of $n(\rho)$. We find that

$$0=\frac{\partial\tilde{F}}{\partial n(\rho)}=2\ln\frac{n(\rho)}{n_0(\rho)}+\frac{E^2}{2\mu^2}\frac{\partial\mu}{\partial n(\rho)}\,. \tag{65}$$

But since, according to Eqs. (42) and (43),

$$\frac{\partial\mu}{\partial n(\rho)}=\left(\frac{2\mu^2}{1+\mu^2}\right)\frac{\partial\eta}{\partial n(\rho)}=\left(\frac{2\mu^2}{1+\mu^2}\right)\frac{\pi^2}{2}x(\rho)\rho^4\,, \tag{66}$$

this reduces to

$$n(\rho)=n_0(\rho)\exp\left[-\frac{2}{1+\mu^2}\frac{\pi^2E^2}{8}x(\rho)\rho^4\right]. \tag{67}$$

This means that the density can be expressed in terms of E and a μ which itself is determined by the density:

$$\begin{aligned}\mu&=\eta+(1+\eta^2)^{1/2}\,,\\ \eta&=\frac{\pi^2}{2}\int\frac{d\rho}{\rho}x(\rho)\rho^4 n(\rho)\\ &=\frac{\pi^2}{2}\int\frac{d\rho}{\rho}\rho^4 x(\rho)n_0(\rho)\\ &\qquad\times\exp\left[-\frac{2}{1+\mu^2}\frac{\pi^2E^2}{8}x(\rho)\rho^4\right].\end{aligned} \tag{68}$$

This amounts to a system of nonlinear equations to be solved for $\mu(E)$, from which we can determine $n(\rho)$.

Before solving the equation let us discuss the physical significance of Eq. (67). We note that the density of instantons is *reduced* by the presence of the external field E, and the larger the instanton is, the larger the reduction factor is. This is simply the phenomenon of magnetostriction, as it occurs in QCD. The thermodynamic reason for this is that, while the (unperturbed) free energy, F_0, is raised by decreasing the density of instantons, the electrostatic energy $(-E^2/2\mu)$ is lowered even more since μ decreases with decreasing $n(\rho)$. Thus, in a paramagnetic medium of the type we are considering, with $\epsilon=1/\mu<1$, the application of an external field expels part of the medium from the region in which the field exists. This is the converse of electrostriction in a dielectric medium, with $\epsilon>1$, where the medium is sucked into the region with nonvanishing E.

Furthermore, we note that E acts like an infrared cutoff, suppressing the effect of large instantons, and causing the density, $n(\rho)$, to peak at small ρ and large $x(\rho)$. This means that for large-enough E we should be able to trust our semiclassical treatment of the instanton gas, as well as the approximation of replacing this gas by a dipole gas.

These equations can easily be solved by introducing the variable ($\bar{\mu}$ is the renormalization scale parameter)

$$\alpha \equiv \frac{2}{1+\mu^2}\,\frac{\pi^2(E\bar{\mu}^2)^2}{8} \tag{69}$$

and solving for η and E in terms of α. Equation (68) becomes

$$\eta = (\pi^2/22)C_0\int_0^\infty dx\,x^7 \exp\left[-x\left(1+\alpha e^{-4x/11}\right)\right],$$
$$E/\bar{\mu}^2 = \frac{2}{\pi}\left[\alpha(1+\mu^2)\right]^{1/2}. \tag{70}$$

To proceed, we need a numerical value for C_0 which can only be obtained by making an explicit choice of coupling-constant definition. By a simple extention of 't Hooft's results,[14] we find that $C_0 = 1.51\times10^{-3}$ for his Pauli-Villars renormalization scheme and that $C_0 = 1.06\times10^2$ for his dimensional renormalization scheme (these numbers are for SU_3 and incorporate a correction factor of 64 needed to compensate for a normalization error in Ref. 14). Dimensionless physical quantities should not of course depend on the choice of coupling definition; we will in fact show that the relative factor of 10^5 in density between Pauli-Villars and dimensional renormalization does not change the physics. For a first look at the consequences of Eq. (70), we shall in fact adopt an "intermediate" renormalization scheme which leads to the value $C_0 = 0.097$. (This is largely a matter of convenience: The normalization error mentioned above was brought to our attention after the completion of this paper—we had thought that $C_0 = 0.097$ corresponded to Pauli-Villars.) Once the outlines of the solution have been established, we will present the corresponding results for the true Pauli-Villars and dimensional schemes.

The solution to these equations, presented as a D versus E curve, is shown in Fig. 5. In Table I we have listed the values, at representative points along that curve, of permeability (μ), field (E), actual magnetization (M), saturation magnetization ($M_{\rm sat}$), and the value of x at which the integrand of the integral for η is peaked (x_p). Note that at all points $M \ll M_{\rm sat}$ which verifies that the linear response approximation is adequate.

The outstanding feature of Fig. 5 is the existence of a point (point 3) where the slope of the D versus E curve changes sign. According to Eq. (57), if we set $\xi = 1$ (thereby eliminating n as an independent variable), the second variation of F with respect to D is just $(\partial E/\partial D)$. Therefore, the

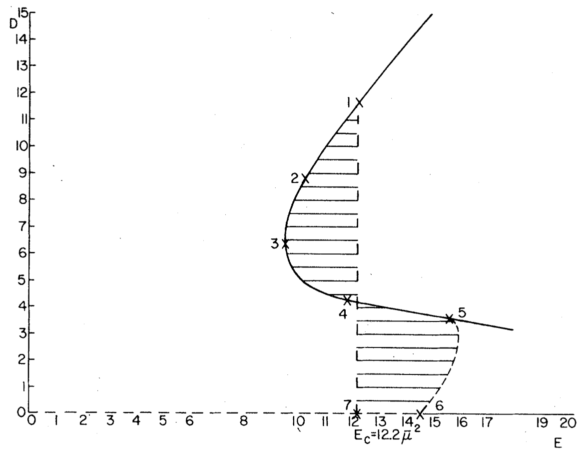

FIG. 5. The equation of state, $D(E)$, of the QCD vacuum in the presence of an external field E, in units of $\bar{\mu}^2$ ($\bar{\mu}$ = renormalization scale parameter), without instanton interactions taken into account.

condition for thermodynamic stability is

$$\left(\frac{\partial E}{\partial D}\right)_{\xi=1} > 0\,, \tag{71}$$

which condition is obviously violated below point 3. An important point is that the passage from the stable to the unstable regime occurs at such a small coupling and density that we should be able to trust the calculation which establishes the existence of the instability.

If the theory is not catastrophically unstable, the D versus E curve must eventually turn around, signalling the passage to a new, stable and presumably much denser phase. Our notions of how and where the turnaround occurs are rather conjectural, but our guess as to what happens is indicated by the dashed line joining points 5 and 6 in Fig. 5. We will not go into detail here, but the rationale for this is as follows: As one moves along the unstable portion of the curve the density of instantons increases and the effective coupling decreases. At point 5 x_p is about equal

TABLE I. The values of the permeability (μ), the electric field ($E\mu^{-2}$), the magnetization ($4\pi^2 M$), the maximum saturated magnetization ($4\pi^2 M_{\rm sat}$), and the peak value of x in the η integral (x_p) for representative points of the $D(E)$ curve (Fig. 5).

	μ	$E\mu^{-2}$	$4\pi^2 M$	$4\pi^2 M_{\rm sat}$	x_p
1	1.05	12.25	0.58	2.28	21
2	1.17	10.3	1.49	4.92	19.1
3	1.47	9.47	3.04	10.39	17.7
4	2.76	11.82	7.54	25.56	16
5	4.37	15.65	12.06	39.92	15
6	∞	14.6			?

to the value at which instantons ionize into merons.[1] We therefore expect μ to rise suddenly and D to decrease correspondingly suddenly to a very small value.

If the passage to the condensed phase is roughly as described above, one will determine the critical field, E_c, at which the two phases are in equilibrium by the Maxwell construction. According to Eq. (59) we must find an E_c such that the points in the two phases have the same free energy:

$$\bar{F}(n, E_c) = \bar{F}(n', E_c)\,. \tag{72}$$

But, by Eq. (57), $d\bar{F}|_{\xi=1} = -D(E)dE$ and Eq. (72) is equivalent to

$$\int_{E_c}^{E_c} D(E)dE = 0\,. \tag{73}$$

Applying this condition to our conjectured extension of Fig. 5 to the condensed phase, we find the critical field to be $E_c = 12.3\bar{\mu}^2$. At this field strength a (very) dilute phase of instantons with $\mu = 1.05$ is in equilibrium with a dense phase with $\mu \gg 1$.

Before we discuss the phenomenological implications of this phase structure of the QCD vacuum, we should verify that it is not significantly altered when we include the instanton interaction effects which were neglected in the foregoing discussion. What we neglected was the shift in energy of each instanton due to the fact that it is embedded in a medium which can respond to its presence and react back on it. To compute this energy shift we make use of the sort of model explored in Sec. IV: The medium is replaced by a continuum with the same dielectric constant and the dipole under consideration is placed in a vacuum cavity of radius R within the continuous medium. Since we are mainly interested in the long-range dipole tail of the interaction between instantons, the replacement of the instanton gas by a continuous medium should be accurate. The placement of the instanton whose chemical potential is being computed within a spherical cavity requires some discussion since the end result will depend on the cavity radius. For the moment it is probably simplest to regard the cavity radius as providing a short-distance cutoff on instanton interactions. In Appendix II we shall analyze in some detail the instanton-anti-instanton interaction as a function of separation. We find that the dipole form is accurate as long as the separation is larger than 2 to 3 times the instanton scale size. At smaller separations the instantons actually turn into merons, and the action tends smoothly to some finite limit at zero separation. Therefore, if we wish to use the dipole interaction to characterize the interaction between a given instanton and the instantons of the surrounding medium, we must prevent them from coming closer than 2 or 3 times the instanton scale size. This function will be served by the cavity as long as its radius is taken to be 2 to 3 times the scale of the instanton inside. This scheme neglects the very short-range interactions between instantons, a neglect which should not be too significant so long as the overall density is low.

The shift in chemical potential of a dipole in a cavity in a medium relative to vacuum is just

$$\frac{1}{4g^2}\int_{|x|<R}(F_c)^2 d^4x + \frac{1}{4g^2}\int (F_m)^2 d^4x - \frac{1}{4g^2}\int (F_0)^2 d^4x\,, \tag{74}$$

where F_c is the field inside the cavity, F_m is the field (due to the dipole inside the cavity) in the medium, and F_0 is the vacuum field of the dipole. These fields are worked out in Appendix A where we show in particular that

$$(F^a_{\mu\nu})_c = (F^a_{\mu\nu})_0 + \frac{\mu-1}{\mu+1}\frac{4\rho^2\bar{\eta}_{a\mu\nu}}{R^4} \tag{75}$$

if the dipole within the cavity is an instanton of scale size ρ and where μ is the permeability of the surrounding medium. The difference between F_c and F_0 is a uniform field (which we call the reaction field) generated by the response of the medium to the dipole within the cavity. The shift in chemical potential is basically due to the interaction of the cavity dipole with the reaction field. A direct application of the expression [Eq. (25)] given earlier for the interaction energy of a dipole with a uniform field would give an energy shift

$$-12x(\rho)\frac{\mu-1}{\mu+1}\left(\frac{\rho}{R}\right)^4. \tag{76}$$

A careful evaluation of Eq. (74) (supplied in Appendix C) shows that this is wrong by a factor of two and that the shift in energy of the dipole due to the presence of the medium is actually

$$-6x(\rho)\frac{\mu-1}{\mu+1}\left(\frac{\rho}{R}\right)^4. \tag{77}$$

As expected, the chemical potential is *reduced* by the presence of the medium and the instanton density is increased.

Since the chemical potential is just the variation of free energy with respect to density, the above result shows that the interacting free energy, F, satisfies

$$\frac{\partial F}{\partial n(\rho)} = \frac{\partial F_0}{\partial n(\rho)} - 12x(\rho)\frac{\mu-1}{\mu+1}\left(\frac{\rho}{R}\right)^4, \tag{78}$$

where F_0 is the perfect gas free energy [Eq. (54)] (a factor of two has appeared because the variation is actually taken with respect to the common density of instantons and anti-instantons).

An interesting check on this calculation is obtained by comparing it with the standard virial expansion. In the absence of an external field, the condition for equilibrium is $\partial F/\partial n = 0$. If we apply this to Eq. (78) we obtain an expression for $n(\rho)$ which reduces, on expansion in powers of density to

$$n(\rho) = n_0(\rho)\left[1 + 3x(\rho)(\mu_{DG} - 1)\frac{\rho^4}{R^4} + O(n_0^2)\right], \qquad (79)$$

where μ_{DG} is given by Eq. (37) [note that $\mu_{DG} - 1$ is $O(n_0)$]. We may compare this with the virial expansion for the density, taking the interaction between instanton pairs to be dipolar down to separation R and excluding configurations with smaller separation. The result is that the $O(n_0)$ and $O(n_0^2)$ terms in Eq. (79) agree with the virial expansion, thus reassuring us that Eq. (78) is correct at least in the low-density limit.

To find the instanton density in an external field we must minimize $\tilde{F}(n, E) = F(n) - E^2/2\mu$. With the help of Eqs. (78) and (66) we find that

$$\left.\frac{\partial \tilde{F}}{\partial n(\rho)}\right|_E = 2\ln\left(\frac{n(\rho)}{n_0(\rho)}\right) - 12x(\rho)\frac{\mu - 1}{\mu + 1}\left(\frac{\rho}{R}\right)^4 + \frac{E^2}{1 + \mu^2}\frac{\pi^2}{2}x(\rho)\rho^4. \qquad (80)$$

Therefore, the equilibrium density is

$$n(\rho) = n_0(\rho)\exp[-\alpha x(\rho)(\rho\overline{\mu})^4], \qquad (81)$$

where

$$\alpha = \frac{2}{1 + \mu^2}\frac{\pi^2 E^2}{8\overline{\mu}^4} - \frac{6}{(R\overline{\mu})^4}\frac{\mu - 1}{\mu + 1}. \qquad (82)$$

Note that α, which governs the net decrease in instanton density, is not of definite sign: The effect of the background Minkowski field, E, is positive, while the effect of the interaction with other instantons is negative, corresponding to an *increase* in density. In our approximation, the interaction of a particular instanton with the medium is through an induced uniform cavity field and one might wonder why its effect is different from that of the background Minkowski field. The reason is that in the Euclidean picture, the fields due to instantons are all real while the electric fields due to physical charges (call them Minkowski fields) are *imaginary* (this is because they are real in the Minkowski picture and acquire an i on continuation to the Euclidean picture). Consequently, the square of Euclidean and Minkowski electric fields have different sign and, as we see in Eq. (82), different physical effects.

If we choose R to be fixed (independent of the scale size of the instanton inside the cavity), then the equations for η and E are as easy to solve as in the case where the back reaction of the medium on the instanton density is neglected. We simply compute $\eta = (\mu^2 - 1)/2\mu$ and E as functions of α,

$$\eta = (\pi^2/22)C_0\int_0^\infty dx\, x^7\exp[-x(1 + \alpha e^{-4x/11})], \qquad (83)$$
$$\frac{E}{\overline{\mu}^2} = \frac{2}{\pi}\left[(1 + \mu^2)\left(\alpha + \frac{6}{(R\overline{\mu})^4}\frac{\mu - 1}{\mu + 1}\right)\right]^{1/2}$$

and eliminate α to get η (and therefore μ and D) as functions of E. A safer procedure, given the significance of R, would be to choose it proportional to the scale size of the instanton inside the cavity. The problem of solving for $D(E)$ becomes much more difficult, but the results are not much different than those which follow from Eq. (83). The basic reason for this is that in the regions of interest the instanton density is quite sharply peaked in scale size and it does not much matter whether we take R fixed or proportional to ρ. We take the easy course and solve Eq. (83).

Before presenting the solution we will review the considerations which affect our choice of R. Our basic picture of the significance of the cavity leads us to set R equal to the distance at which the instanton–anti-instanton interaction begins to deviate from the simple dipole formula. According to Appendix B, this means

$$R \simeq 2.2\rho, \qquad (84)$$

where ρ is the scale size of the instanton in the cavity. On the other hand, the cavity must be small enough that the probability of finding a second instanton inside it is small. This will be true if

$$\pi^2\int\frac{d\rho}{\rho}n(\rho)R^4 < 1. \qquad (85)$$

This condition can usefully be expressed in terms of $f = \pi^2\int(d\rho/\rho)n(\rho)\rho^4$, the fraction of space-time filled by instantons. If we substitute Eq. (84) into Eq. (85) we have

$$f < \frac{1}{(2.2)^4} = 0.043. \qquad (86)$$

This condition is surely too strong, and we will not be too concerned if it is violated by as much as a factor of two. We must also require that the linear response approximation be valid. This requires two things. First, the external field must not be so strong as to align all the dipoles of the medium. This condition, which was discussed previously, does not depend on R and requires that

$$4\pi^2|M| = \frac{\mu-1}{\mu}E < 4\pi^2|M_{\text{sat}}|$$
$$= 8\pi^2 \int \frac{d\rho}{\rho} n(\rho)\rho^2 . \qquad (87)$$

(As in our previous treatment it turns out that this condition is well satisfied in the domain where our other approximations are valid.) Second, it must be the case that the instantons themselves do not completely align neighboring anti-instantons. If, for example, the density were too high and R very small, then the field produced by an instanton might be so strong as to produce a magnetization (just outside the cavity) greater than the saturated magnetization of the medium. Since the maximum field produced by the instanton is at the edge of the cavity, and equal to $|B| = 4\sqrt{12}\,\rho^2/R^4$, this condition is

$$4\pi^2|M^-|_{|x|=R} = \frac{\mu-1}{\mu}\,\frac{4\sqrt{12}\,\rho^2}{R^4}$$
$$< 4\pi^2|M^I_{\text{sat}}| , \qquad (88)$$

where M^I_{sat} is the maximum magnetization attainable by aligning all the dipoles of a given self-duality:

$$M^I_{\text{sat}} = \sqrt{12} \int \frac{d\rho}{\rho} 2n(\rho)\rho^2 . \qquad (89)$$

This condition turns out to be extremely well-satisfied in the domain where we trust our other approximations.

To extract useful information from Eq. (83) we make a specific numerical choice of $R\bar{\mu}$, construct the corresponding D versus E curve, determine the scale size ρ_p at which the instanton density peaks, and vary R until R/ρ_p is near the value 2.2. In Fig. 6 we plot D versus E for the values $R_1\bar{\mu} = 0.34$, $R_2\bar{\mu} = 0.42$, and $R_3\bar{\mu} = 0.55$. The corresponding values of R/ρ_p, determined at the points where $\partial E/\partial D = 0$, are 1.9, 2.3, and 2.9 and therefore in the range of interest.

The general features of this curve are much the same as in the case of no interaction. In particular, we still find an instability in the region of very low density and small μ. Indeed, the

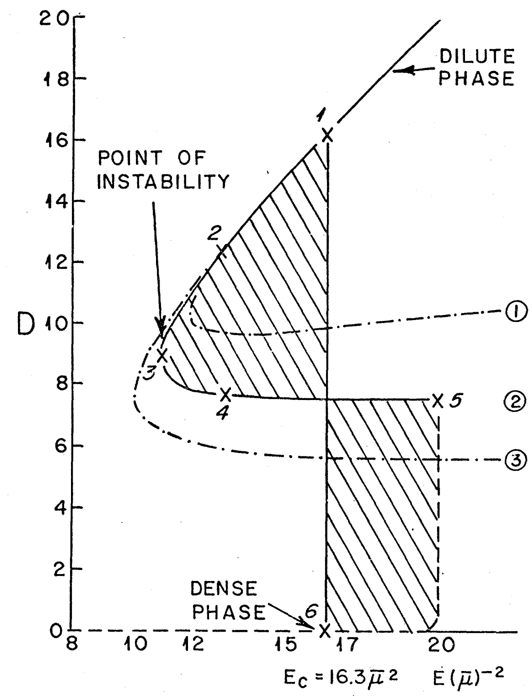

FIG. 6. The equation of state, $D(E)$, of the QCD vacuum in the presence of an external field E, in units of $\bar{\mu}^2$ ($\bar{\mu}$ = renormalization scale parameter), with instanton interactions taken into account. The solid curve holds for a cavity size $R\bar{\mu} = 0.42$, whereas the dash-dotted curves (labeled 1 and 3) refer to $R\bar{\mu} = 0.34$ and 0.55, respectively.

effect of instanton interactions is to *reduce* the value of μ at which the instability occurs. For the values of $R_i\mu$ cited above, we find that at the instability points:

$$E_1 = 11.87\bar{\mu}^2 , \quad \mu_1 = 1.17 , \quad x_1 = 19.1 ,$$
$$E_2 = 10.85\bar{\mu}^2 , \quad \mu_2 = 1.23 , \quad x_2 = 18.7 ,$$
$$E_3 = 10.1\bar{\mu}^2 , \quad \mu_3 = 1.30 , \quad x_3 = 18.3 ;$$

while in the case of no interaction we had

$$E = 9.47\bar{\mu}^2 , \quad \mu = 1.47 , \quad x = 17.7 .$$

In Table II, we give the values of μ, $E\bar{\mu}^2$, $4\pi^2|M|$,

TABLE II. The values of the permeability (μ), the electric field ($E\bar{\mu}^{-2}$), the magnetization ($4\pi^2 M$), the maximum saturated magnetization ($4\pi^2 M_{\text{sat}}$), the peak value of x in the η integral (x_p), and the fraction of space-time occupied by instantons (f) for representative points on the $D(E)$ curve for $R\bar{\mu} = 2.3$ with instanton interactions taken into account (Fig. 6).

	μ	$E/\bar{\mu}^2$	$4\pi^2\|M\|$	$4\pi^2\|M_{\text{sat}}\|$	x_p	f
1	1.016	16.3	0.26	0.7	22.6	0.0014
2	1.051	12.9	0.63	2.3	20.7	0.005
3	1.17	10.85	1.58	6.3	18.7	0.02
4	1.67	12.94	5.2	13.5	17.7	0.06
5	2.7	20	12.6	25.6	16.8	0.15
6	∞?	$E_c = 16.3$	?	?	?	?

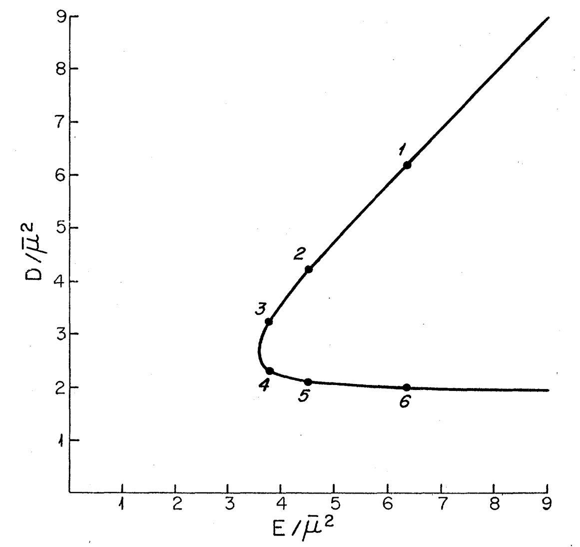

FIG. 7. The D vs E curve for the Pauli-Villars renormalization scheme. Values of μ, etc., at the labeled points are tabulated in Table III.

$4\pi^2|M_{sat}|$, x_p (the value of x at which the η integral is peaked) and f (the fraction of space occupied by instantons) at representative points along the curve for $R=R_2$. All of these results are not sensitive to the fact that we have taken R to be independent of ρ. If we set $R=2.2\rho$ in Eq. (83), we find slight changes in these numbers [μ is slightly larger (smaller) in the stable (unstable) phase].

We should also determine the effect on the instability of changing the coupling-constant definition.

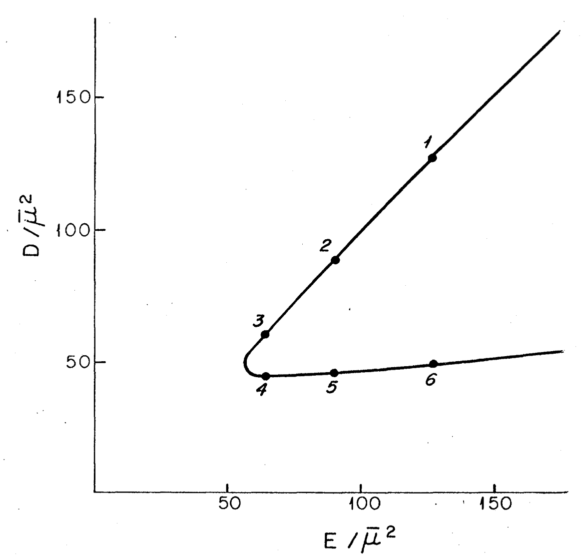

FIG. 8. The D vs E curve for the dimensional renormalization scheme. Values of μ, etc., at the labeled points are tabulated in Table IV.

TABLE III. The values of selected quantities (refer to Table II for definitions) at points 1–6 on the D vs E curve for the Pauli-Villars scheme plotted in Fig. 7.

	μ	$E/\bar{\mu}^2$	$\|M\|/\|M_{sat}\|$	x_p	f
1	1.01	6.4	0.29	17	0.002
2	1.06	4.5	0.24	13	0.008
3	1.16	3.8	0.23	13	0.02
4	1.62	3.8	0.20	12	0.08
5	2.12	4.5	0.20	12	0.13
6	3.16	6.4	0.20	13	0.22

As explained earlier, the above numerical work was done with a coupling-constant definition intermediate between 't Hooft's version of Pauli-Villars and dimensional renormalization. We have therefore determined the D vs E curve for the more conventional Pauli-Villars and dimensional renormalization schemes (with the variable cavity radius $R=2.2\rho$). The results are presented in Figs. 7 and 8 and Tables III and IV. The D vs E curves are quite similar in shape to those of Fig. 6 and the values of the dimensionless quantities μ and f are about the same at similar points on the curves. On the other hand, the values of x suffer an overall shift (for intermediate renormalization, the instability occurs at $x\sim 19$, for Pauli-Villars it occurs at $x\sim 13$ and for dimensional renormalization it occurs at $x\sim 29$) while the values of E (and $\bar{\mu}$) suffer an overall rescaling (for intermediate renormalization the value of E at the instability point is $\sim 11\bar{\mu}^2$, in Pauli-Villars it is $\sim 3.5\bar{\mu}^2$ and in dimensional renormalization it is $\sim 56\bar{\mu}^2$). The physics should of course be exactly invariant to changes in coupling-constant definition and we would presumably find it to be so if we could include the effect of quantum corrections to instanton effects. The present exercise shows that it is not unreasonable, for the conventional coupling-constant definitions, to believe that the physical effect of quantum corrections on the properties of the instability is small.

Returning to Fig. 6, we see no sign of the D

TABLE IV. The values of selected quantities (refer to Table II for definitions) at points 1–6 on the D vs E curve for dimensional renormalization plotted in Fig. 8.

	μ	$E/\bar{\mu}^2$	$\|M\|/\|M_{sat}\|$	x_p	f
1	1.00	127	0	35	0
2	1.01	90	0.43	33	0.0004
3	1.06	64	0.45	31	0.003
4	1.44	64	0.39	31	0.025
5	1.98	90	0.40	30	0.048
6	2.59	127	0.41	31	0.070

vs E curve turning around a second time [actually Eq. (83) leads to such a turnaround, but at a value of E and μ so large as to be ridiculous]. However, it is clear that at some point along the curve, our approximations are no longer valid.

Where do our approximations break down? If we examine the various conditions enumerated above, we find that the first problem that arises is that the density of instantons becomes too large. (There is no problem at all of saturation of the magnetization.) For example, in the case of $R=R_2$, it is roughly at point 5 ($E\approx 20$, $\mu\simeq 3$) where Eq. (85) fails indicating that instanton density has grown too large. At this point, the gas is sufficiently dense that, by virtue of the arguments of Appendix B, the instantons must be replaced by merons. In practice this means that Abelian techniques can no longer be trusted and truly non-Abelian physics takes over. We assume, without any very detailed justification, that the new "phase" is one in which μ is very large and make the guess that at point 5 the $D(E)$ curve drops rapidly to zero as illustrated in Fig. 6. If this is the case, then (by Maxwell's construction) at $E_c \sim 16.3\bar{\mu}^2$, we have two phases in equilibrium, one extremely dilute ($x_p \sim 23$, $\mu\sim 1.02$) and the other, a very dense phase with large, if not infinite, μ. The instantons density profile in the dilute phase is pictured in Fig. 9. We can then calculate the zero-field difference in free energy density between the dense and dilute phases, which we shall see below is equal to the "bag constant" B. The equilibrium condition is

$$\begin{aligned}\tilde{F}(n_1, E_c) &= F(n_1, E=0) - \frac{1}{2\mu_1}E_c^2\\ &= \tilde{F}(n_2, E_c)\\ &= F(n_2, E=0) - \frac{1}{2\mu_2}E_c^2,\end{aligned}$$

where n_1 (n_2) is the density in the dilute (dense) phase. Since $\mu_2\approx\infty$, $\mu_1\approx 1$, we have that

$$B = F(n_1, E=0) - F(n_2, E=0) \simeq \tfrac{1}{2}E_c^2, \qquad (90)$$

which yields the following estimate for the bag constant:

$$B\sim \frac{1}{2\mu_1}E_c^2 \sim (3.4\bar{\mu})^4. \qquad (91)$$

We will have more to say in the following sections concerning the phenomenological implications of a first-order phase transition between a very dilute phase (where ordinary perturbation theory notions are valid and the effective coupling is small) and a very dense phase (where the coupling is large and the behavior of the system is essentially non-Abelian). It is obvious, however, that this qualitative picture of the phase structure of the QCD vacuum must lead directly to something very closely related to the bag model. Although numerical details may be off by factors of a few, it seems to us that the methods of this section give a convincing demonstration that a dilute phase exists in a sufficiently strong external field and that the dilute phase must condense when we lower the external field beyond a critical value. This qualitative insight will surely be the key to a quantitative QCD treatment of hadron physics.

VII. A MODEL OF HADRONS

We have seen that the QCD vacuum can exist in two distinct phases. The normal vacuum in absence of colored fields or quarks, is highly paramagnetic due to densely packed instantons and meron pairs (and presumably free merons) with a very large, if not infinite, permeability. In the presence of colored gauge fields, produced say by static quarks, there can exist a dilute instanton gas phase which is in equilibrium with the normal vacuum phase. In the dilute phase the susceptibility and the effective coupling are very small. In this section we shall show how this leads to the confinement of quarks, and to a bag model of hadrons.

Let us first discuss very heavy quarks. We consider a color singlet state consisting of a quark-antiquark pair, separated by distance R. We wish to calculate the energy of this state. This should be done, of course, by evaluating the vacuum expectation value of the Wilson loop operator, $P\exp(i\oint_L A^\mu dx_\mu)$, for a loop ($L$) of spatial extent L and time extent T, as $T\to\infty$. How-

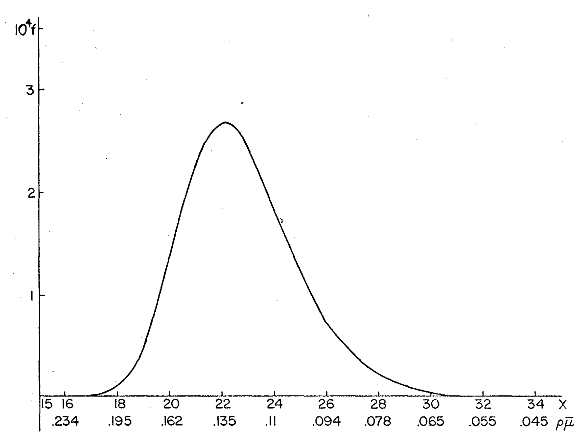

FIG. 9. The density of instantons, $10^4 f(x)$, in the dilute QCD phase. $f(x)dx$ represents the fraction of space-time occupied by instantons of scale size between ρ and $\rho+d\rho$, where $\rho\bar{\mu}=\exp(-x/11)$, and $f(x)=(0.1\pi^2/11)x^6 \times \exp[-x(1+\alpha e^{-4x/11})]$ ($\alpha\simeq 320$).

ever, since the coupling, $g^2/8\pi^2$, is so small inside the bag, we can treat the quarks as static Abelian sources of charge Qg, where

$$Q^2=\tfrac{4}{3}=-\sum_{\alpha=1}^{\infty}(\tfrac{1}{2}\lambda_\alpha^{(1)})(\tfrac{1}{2}\lambda_\alpha^{(2)})_{\text{singlet}}\,.$$

In the normal vacuum state $\mu=\infty$ (or is at least very large). Thus the color fields produced by the quarks cannot penetrate this phase and a flux tube must be formed surrounding the quarks in which the dilute phase exists and which is in equilibrium at the boundary with the dense vacuum phase [see Fig. 4(b)]. Thus we must solve the linearized Yang-Mills equations for $\vec{E}$

$$\vec{\nabla}\cdot\vec{D}=gQ[\delta(\vec{x})-\delta(\vec{x}-\vec{R})],\qquad\qquad (92)$$
$$\vec{\nabla}\times\vec{E}=0\,,\quad \vec{D}=\frac{1}{\mu_1}\vec{E}\,,$$

where $\mu_1\approx 1$ is the permeability inside the tube. The boundary conditions are clearly that the normal component of E vanish at the boundary, and that the transverse component be equal to the critical value E_c.

$$E_n=0\,,\quad E_t=E_c=\sqrt{2B}=16.43\overline{\mu}^{\,2}\,.\qquad (93)$$

The solution of this problem will determine the actual size and shape of the flux tube as well as the magnitude of E inside the tube. This will determine the density of instantons and the precise value of μ_1.

A particularly simple case arises when R becomes very large. In that limit the tube must become a cylinder of diameter d, in which E has a constant magnitude E_c (since $\vec{\nabla}\times\vec{E}=0$). (See Fig. 10.) The diameter of the tube is fixed by demanding that the flux, gQ, be equal to $\frac{1}{4}\pi d^2 D=\pi d^2E_c/4\mu_1$. Since the permeability in the dilute phase, μ_1, is essentially one, the diameter of the tube is given by

$$d=\left(\frac{4}{\pi}\frac{Q}{E_c}g\right)^{1/2}=3.04(Bx)^{-1/4}\simeq 0.4\overline{\mu}^{\,-1}\,.\qquad (94)$$

Such a flux tube leads of course, for large R, to a linearly growing interaction energy between the quarks. The energy density of the flux tube is simply the difference in free energy of the dilute and dense phase, $B=E_c^{\,2}/2\mu_1$, plus the electrostatic energy density which also equals $E_c^{\,2}/2\mu_1$. Thus the energy per unit length of flux tube ϵ is

$$\epsilon=2\frac{E_c^{\,2}}{2\mu_1}\left(\frac{\pi d^2}{4}\right)\simeq E_c gQ=14.5\left(\frac{B}{x}\right)^{1/2}\simeq 33.7\overline{\mu}^{\,2}\,.\qquad (95)$$

This picture bears many similarities to the string model. We might hope to use it to estimate the slope of the Regge trajectories, which should only depend on the value of ϵ and not on the character of the quarks at the ends of the tube. This we can do by approximating the action of a moving tube by ϵLT, where L is the spatial length of the tube and T the time of propagation. Comparison with the Nambu action for a relativistic string[12] leads us to identify ϵ with the proper tension of the string.[13] The Regge slope, α', is then determined by

$$\frac{1}{2\pi\alpha'}=\epsilon\,,\qquad\qquad (96)$$
$$\alpha'=0.011\left(\frac{x}{B}\right)^{1/2}\simeq 0.0047\overline{\mu}^{\,-2}\,.$$

This is surely a crude approximation given our rather fat flux tube. If we take $\alpha'\approx 0.9\ \text{GeV}^{-2}$ then $\overline{\mu}\simeq 73$ MeV, a value, as we shall see below, reasonably consistent with the bag radius of the nucleon.

In order to actually calculate a heavy-quark-antiquark potential that could be used in a Schrödinger equation to predict the spectrum of heavy-quark bound states, one would have to improve the above calculation. The effects of instantons inside the flux tube, the finite thickness of the surface of the tube, the resulting surface energy, and the effects of massless quarks and chiral-symmetry breaking would all have to be taken into account. These issues will be discussed below after we develop the bag model for light-quark bound states.

Turning now to light quarks we must confront the question of quark propagation in the instanton gas. Until this point we have been able to characterize the net effect of the instantons as producing a (possibly space-time-dependent) paramagnetic susceptibility. However, the existence of a finite density of instantons can have other effects. In particular, as we have shown in Ref. 1, they can induce dynamical chiral-symmetry breaking and generate a dynamical quark mass.[1] This will determine the boundary conditions for the quark field at the surface of the bag.

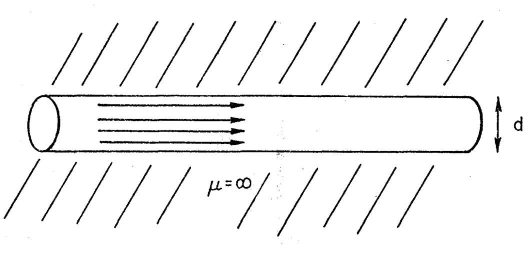

FIG. 10. A flux tube (of diameter d) for very-well-separated heavy quarks.

Let us recall the mechanism proposed in Ref. 1 for dynamical chiral-symmetry breaking. (We shall assume two massless quarks.) Instantons generate an effective, nonlocal four-fermion interaction of the schematic form[1,14]

$$V = \det_{i,j}[\bar{\psi}_i(1+\gamma_5)\psi_j + (\gamma_5 \leftrightarrow -\gamma_5)], \qquad (97)$$

where the determinant runs over the quark flavors. This effective interaction breaks the axial U(1) symmetry [thus solving the "U(1) problem"] and renders the SU(2) × SU(2) chirally symmetric vacuum unstable.[1] This was discussed in detail in Ref. 1, where we showed that even a moderately dilute instanton gas generates an instability of the chirally symmetric vacuum (the manifestation of this instability is the appearance of a tachyon in the $\sigma = \bar{\psi}\psi$ channel).

Thus the true vacuum state will be a Goldstone vacuum in which $\langle \bar{\psi}\psi \rangle \neq 0$. Within the dilute-gas approximation one can actually construct this vacuum by means of a Hartree-Fock equation (as illustrated in Fig. 11). Our study of this equation, as well as the much simpler case of one massless flavor, leads us to conclude that once the density of instantons is moderate, a large dynamical quark mass will be generated. By the dynamical mass we mean, of course, the term $m(p)$ in the quark propagator $S(p)^{-1} = Z(p)[\not{p} + m(p)]$. This mass is strongly momentum dependent, vanishing rapidly for large p. For small p, however, $m(p)$ is very large. The inclusion of meron pairs will increase $m(p)$ even more. Although we are unable to calculate the quark propagator in the true vacuum phase, due to the large density of instantons and ionized merons, it is reasonable to conjecture that in this phase $m(p)$ will be *very large* for moderate values of p. Thus if a bag exists, as discussed above, the inside of the bag will be chirally symmetric, since the density of instantons is so low, and the quark masses will be given by their small bare values. Outside the bag, however, the density is so high that the vacuum will exhibit spontaneous chiral-symmetry breaking, generating a *very large* quark mass (for small momentum). Thus our bag acts also as a mass bag, confining the quarks to a region where they are light.

Let us now consider light-quark bound states. Here again we expect that the quarks will create a bubble in the dense vacuum phase in which the dilute phase exists. However, now the quarks have kinetic energy which will contribute to the pressure on the surface of the bag helping to balance the vacuum pressure of the dense phase outside the bag. To treat adequately light-quark bound states one would have to go beyond our previous discussion, which was restricted to a time-independent bag surface. We shall not attempt such a derivation in this paper. Rather we shall assume that the phase boundary (the bag surface) responds slowly to the fluctuating fields created by the quarks inside the bag, and that we can therefore treat it as a static surface. This approximation must, of course, be justified or superseded by an improved treatment.

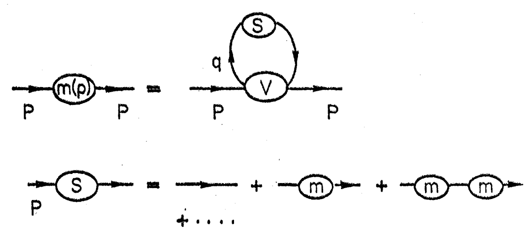

FIG. 11. The Hartree-Fock equations (dilute-gas approximation) to the quark propagator. The instanton-induced determinantal interaction, represented by V, generates, for moderate instanton densities, a dynamical quark mass $m(p)$.

We can now determine the conditions for phase equilibrium. Given a collection of quarks in a singlet state we again must solve the coupled Yang-Mills and Dirac equations (with $g^2/8\pi^2 \simeq \frac{1}{23}$) for the quarks and gluons inside the bag with the boundary condition $\hat{n}^\mu F_{\mu\nu} = 0$ on the surface (where $\hat{n}^\mu$ is the spacelike normal to the surface). Strictly speaking, there are also instanton-induced interactions inside the bag, but since the density of instantons in the dilute phase is so small we neglect these for the time being. The boundary condition on the quark field is determined by the discontinuity of the quark mass at the boundary. According to the above discussion a reasonable approximation is to take $M_{\text{quark}} \simeq \infty$ outside the bag. This leads to $i\not{n}\psi = \psi$ on the surface, which coincides with the MIT boundary condition for the quark field,[11] and appears to introduce explicit chiral-symmetry breaking. However, we must now recall that the QCD vacuum phase outside the bag is a Goldstone vacuum with zero-mass (if the bare quark masses are zero) Goldstone bosons. These will necessarily couple to the quarks at the bag surface and will restore chiral symmetry.

In lieu of a full treatment of chiral-symmetry breaking in the dense phase, the construction of the pion state and the evaluation of f_π, we shall adopt a phenomenological approach. There must be a chiral order parameter in the vacuum state and for a certain class of phenomena (soft-pion theorems, for example) we can rigorously represent its dynamics by a phenomenological nonlinear σ model, adjusted to yield an infinite quark mass outside the bag. Namely, outside the bag we consider an effective Lagrangian

$$\mathcal{L}=\bar{\psi}[i\partial\!\!\!/ - g(\sigma+i\vec{\pi}\cdot\vec{\tau}\gamma_5)]\psi + \lambda(\sigma^2+\vec{\pi}^2-\sigma_0^{\,2})^2+\tfrac{1}{2}(\partial_\mu\vec{\pi})^2+\tfrac{1}{2}(\partial_\mu\sigma)^2 , \qquad (98)$$

and adjust λ, g, and σ_0 so as to obtain in the Goldstone vacuum ($\langle\sigma\rangle=\sigma_0$) an infinitely massive quark and σ and a finite f_π. We then make the standard canonical transformation to the nonlinear σ model.

Now we have, in addition to the quark and gluon fields that live inside the bag, a $\vec{\pi}$ field that lives outside the bag. In the limit $M_q\to\infty$ it decouples from the quarks outside the bag and thus satisfies $D^\pi_\mu D^\pi_\mu\vec{\pi}=0$, where D^π_μ is the chirally covariant derivative, $D^\pi_\mu=(1+\vec{\pi}^2/f_\pi^2)^{-1}\partial_\mu$. However, the $\vec{\pi}$ field couples to the quarks on the surface of the bag. Not surprisingly, this coupling is given by

$$n^\mu\bar{\psi}\gamma_5(\vec{\tau}/2)\psi=\bar{\psi}\gamma_5(\vec{\tau}/2)\psi=f_\pi n^\mu D^\pi_\mu\vec{\pi} \qquad (99)$$

on the surface. This is exactly what is required to ensure that the total axial-vector current

$$\vec{A}_\mu=\bar{\psi}\gamma_\mu\gamma_5(\vec{\tau}/2)\psi\theta(x\in R)+f_\pi D^\pi_\mu\vec{\pi}\theta(x\notin R) \qquad (100)$$

be conserved (R is the spatial region of the bag).

Finally, we have an equation which ensures energy momentum conservation, balancing the kinetic pressure of the quarks and gluons inside the bag with the vacuum pressure and the pressure of the $\vec{\pi}$ field outside the bag. This results in

$$B=\tfrac{1}{2}E_c^{\,2}=\tfrac{1}{2}n^\mu D_\mu(\bar{\psi}\psi)-\tfrac{1}{2}(D^\pi_\mu\vec{\pi})^2-\tfrac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \qquad (101)$$

at the surface.

To summarize, our semiphenomenological equations are

$$\begin{aligned}
&i\not{D}\psi=(i\partial\!\!\!/-g\not{A})\psi=0\,, \quad \vec{x}\in R\\
&D^\mu F^{\alpha}_{\mu\nu}=\bar{\psi}\gamma_\nu\tfrac{1}{2}\lambda^a\psi\,,\\
&(D^\pi_\mu)^2\vec{\pi}=0\,, \quad \vec{x}\notin R\\
&i\not{n}\psi=\psi\,, \qquad\qquad\qquad (102)\\
&n^\mu F^{\alpha}_{\mu\nu}=0\,, \quad \vec{x}\text{ on the surface of }R\,.\\
&\bar{\psi}\gamma_5(\vec{\tau}/2)\psi=f_\pi n\cdot D^\pi\vec{\pi}\,,\\
&B=\tfrac{1}{2}n\cdot D(\bar{\psi}\psi)-\tfrac{1}{2}(D^\pi_\mu\vec{\pi})^2-\tfrac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}\,.
\end{aligned}$$

Solving these equations will produce a bag model which automatically satisfies all the soft-pion theorems of chiral symmetry. Other ("nonsoft") effects of the pion field should not be taken too seriously.

These equations are similar to those of the MIT bag model,[11] differing only in the inclusion of the Goldstone mode present in the vacuum outside the bag. The philosophy behind them is totally different.[15] Our confining paramagnetic bag arises dynamically due to the properties of the QCD vacuum[16] and is not introduced in an *ad hoc* fashion. Furthermore, the boundary conditions on the quark field are also a consequence of the dynamics of the vacuum and once the pion, present as a Goldstone excitation of the dense vacuum phase, is accounted for, chiral symmetry is preserved. Finally, the above equations are just the first-order approximation to the QCD bag for light quarks. An improved treatment will take into account the finite surface thickness of the bag (see below) and the fluctuations of the bag surface. The finite density of instantons inside the bag and on its surface will induce new interactions between the quarks. Finally, the energy shift due to zero-point oscillations in the bag, which arises from the instanton determinants with interactions taken into account, is finite and calculable.

Even at this level of approximation, there will be differences between our model and the MIT bag due to the coupling of "bag states" with the Goldstone pion. This will have an effect on the hadronic spectrum and will allow for a simple calculation of pion couplings.

The pion itself is not, strictly speaking, a bag state but rather a Goldstone excitation of the dense vacuum. Its properties (f_π, for example) are determined by this phase. For consistency, however, one should be able to see how a bag containing a quark-antiquark pair in an $I^G(J^P)=1^-(0^-)$ state would collapse to a zero-mass bound state. Following Horn and Yankielowicz[16] we include in the energy of the state the attractive interaction (in the pion channel) due to the determinantal interaction induced by instantons. For a bag of radius R this has the form CR^{-3}, as long as R is greater than the instanton size. A calculation similar to that performed in Ref. 16 shows that, with the density of instantons that exists in our dilute phase, this interaction causes the bag to contract, and the mass to decrease, to a point where the mass vanishes and the bag no longer exists. This is hardly surprising since this is simply a crude way of generating a bound-state Goldstone boson by the mechanism proposed in Ref. 1. An interesting open question is whether there exists another metastable and heavy bag-like pion.

Let us estimate the size of a hadronic bag. For a crude estimate, we simply balance the vacuum pressure, $(\partial/\partial R)(4\pi/3BR^3)$ of a spherical bag with the kinetic pressure of the quarks. For massless quarks this is simply $2.04NR^{-2}$,[11] where N is the number of quarks. For the nucleon therefore

$$R=\left(\frac{6.12}{4\pi B}\right)^{1/4}\simeq 0.25\bar{\mu}^{-1}. \tag{103}$$

This should be compared with the peaked instanton size inside the bag, which is (see Table II)

$$\rho_p=\exp(-\tfrac{1}{11}x_p)\bar{\mu}^{-1}=0.12\bar{\mu}^{-1}. \tag{104}$$

Thus the bag is roughly twice the size of the instantons inside it. One might worry that such a bag could hardly be treated as a sharp bag. However, one must remember that the fraction of space-time occupied by instantons inside the bag is very small ($f\simeq 0.0014$). Thus the probability that an instanton of size ρ_p is actually inside the bag, per unit time of order ρ_p, is $fR/\rho_p\simeq 0.01$.

A final point which must be discussed is the question of the thickness of the bag wall. Like any other phase boundary, the bag wall has a finite thickness, Δ, and we must convince ourselves that this thickness is small compared to the radius of the bag. Although many mechanisms are simultaneously at work to determine Δ, we shall assume it to be entirely due to one effect we understand rather well. This effect is an increase in instanton density near the surface due to the attraction exerted on the dilute-gas instantons inside the bag by their images in the bag surface. This image attraction causes the dilute-gas instantons to pile up near the boundary with a scale distance for density increase which we will show in the next few paragraphs to be equal to $0.2\rho\simeq 0.025\bar{\mu}^{-1}$ or one-tenth the bag radius. This is certainly small enough for us to expect surface effects to cause only small corrections to the naive zero-thickness bag picture.

To compute the wall thickness, we first assume that there is a well-defined wall (i.e., a plane boundary on one side of which $\mu\gg 1$ and on the other side of which $\mu\simeq 1$) and then study the effect of that sharp boundary on the spatial distribution of instantons in the dilute region. The effect in question arises from the interaction of individual instantons with their "image dipoles" on the other side of the surface: A simple magnetostatics calculation shows that if an instanton of scale size ρ is placed at a distance $d=\rho(1+y)$ from a sharp plane boundary between media with $\mu=1$ and $\mu=\infty$, then its action is reduced by

$$S_I=-\frac{3}{8}\frac{x(\rho)}{(1+y)^4}. \tag{105}$$

This expression is accurate only if the instanton is far enough from its image for the dipole approximation to be valid. According to the results of Appendix B, this requires that $y\gtrsim 0.1$. This causes the density of instantons of scale size ρ to vary as a function of the dimensionless distance y according to

$$n_\rho(y)=\exp\left[\frac{3}{8}\frac{x(\rho)}{(1+y)^4}\right]n_\rho(\infty), \tag{106}$$

where $n_\rho(\infty)$, of course, is the density far from the boundary.

Our study of the first-order phase transition has shown that in the neighborhood of the phase boundary the dominant value of x is of order 23. The exponential enhancement factor in Eq. (106) is therefore extremely rapidly varying: at $y=1$, 0.1, and 0 it takes on the values 1.7, 362, and 5570, respectively. Although the instantons which stand off from the surface by one scale size ($y=1$) are still in the dilute phase, those which touch the surface ($y=0$) obviously are not.

To get a rough estimate of how far from the surface the transition from dilute gas to condensed phase lies, we introduce a surface layer density of instantons, $\hat{n}_\rho(y_0)$. This is the density, per unit surface area per unit time, of surface layer instantons of scale size ρ and with $y>y_0$. It is essentially a measure of the density of extra instantons, over and above the background density, $n_\rho(\infty)$, induced by the surface. It is computed by integrating $n_\rho(y)-n_\rho(\infty)$ over the distance normal to the surface, starting from a lower cutoff $y_0\rho$. If y_0 is reasonably small, one finds that,

$$\hat{n}_\rho(y_0)=\frac{2(1+y_0)^5}{3x(\rho)}\exp\left[\frac{3x(\rho)}{8(1+y_0)^4}\right]\rho n_\rho(\infty). \tag{107}$$

A convenient dimensionless measure of how dense the surface instantons actually are is $\hat{f}$, the fraction of three-dimensional space covered by surface instantons. A straightforward calculation gives

$$\begin{aligned}\hat{f}(y_0)&=\frac{8\pi}{3}\int\rho^3\hat{n}(\rho)\frac{d\rho}{\rho}\\&\simeq\frac{16}{9\pi}\frac{(1+y_0)^5}{x_p}\exp\left[\frac{3x_p}{8(1+y_0)^4}\right]f,\end{aligned} \tag{108}$$

where x_p is the x value at which the distribution peaks (about equal to 23) and f is the fraction of four-dimensional space covered by the dilute-gas instantons inside the bag (about equal to 0.0014 according to Sec. VI).

In Sec. VI we found that the dividing line between the dilute and strongly interacting four-dimensional instanton gas was marked by f equal to a few hundredths. We shall assume that the dividing line for surface instantons comes at similar values of $\hat{f}$. If we choose $y_0=0.1$ and 0.2, we find from Eq. (108) that $\hat{f}(y_0)=0.025$ and 0.006, respectively. The former value appears to lie on the dividing line between dense and dilute phases, while the latter value appears to lie unambiguously in the dilute phase. This means that the distance over which the surface instanton density passes from dense-to-dilute values is on the order of one-tenth the peak instanton scale size. If we take this distance as a measure of the bag wall thickness, we obtain the estimate mentioned at the beginning of this discussion.

In real QCD with light (essentially massless) quarks there is another surface effect. In a simple model where the dynamical quark mass is zero in the bag and large compared to ρ^{-1} outside, the fermion determinant will contain N_f (N_f is the number of light flavors, 2 or 3) eigenvalues which approach zero like $2^{-1}(1+y)^{-3}$ for large y. Unless one or more of these eigenvalues are absorbed by an interaction with light quarks in the bag, n will contain an additional factor $2^{-N_f}(1+y)^{-3N_f}$ and $\hat{n}$ and $\hat{f}$ will contain the same factor with y replaced by y_0. We note, however, that even when massless fermions are present the main surface effect is attraction by the image dipole.

Clearly much work remains to be done in developing the physics of our bag. The most important issue from a fundamental point of view is the precise nature of the dense meron-dominated vacuum phase. However, it would appear that the dynamics of the quarks that live in dilute bubbles are relatively independent of the exact nature of this phase, at least insofar as we consider the static properties of low-lying hadronic states. Thus we can already contemplate developing a phenomenological theory of hadrons which proceeds directly from QCD with perhaps only a few parameters $(B, f_\pi, \ldots)$ to represent the, as yet, quantitatively unknown features of the dense phase. For heavy-quark bags this should be a straightforward task. Of particular interest will be the surface effects discussed above. The instantons near the surface will be of moderate density and will induce nonperturbative interactions between quarks. Previous experience[1,4,5] suggests that these will dominate perturbative gluon interactions and will have important effects on the masses, couplings, and interactions of hadrons.

VIII. SHIELDING FROM LARGE-SCALE FLUCTUATIONS

Finally, we must show that fluctuations do not destroy the bag picture. In studying this problem, we will show that the QCD bag has the important property that it shields the quarks inside it from large color field fluctuations in the vacuum. In four dimensions the bag is a long cylinder of $\mu \simeq 1$ inside a medium with $\mu = \mu_{\text{vac}} \simeq \infty$. We will show below that fluctuations are attenuated by a factor $\mu_{\text{vac}}^{-1} \simeq 0$ at the bag surface, thus shielding quarks (and anything else inside the bag) from the ill-understood and potentially dangerous large-scale vacuum fluctuations. This is just the QCD analog of the Meissner effect. The penetration of a field into a QCD bag is governed by the ratio $\mu_{\text{in}}/\mu_{\text{out}} \simeq 1/\mu_{\text{vac}} \simeq 0$ which is the same as the ratio $\mu_{\text{in}}/\mu_{\text{out}} = 0/1 = 0$ that occurs at the surface of a superconductor. From the point of view of the vacuum, the QCD bag looks like a cylinder of color superconductor.

For the deep infrared fluctuations, a linearized calculation is possible. Consider a bag of size a and an infrared vacuum fluctuation of scale size $L \gg a$. On its own scale we expect such a fluctuation to be nonlinear, with $\partial_\mu \simeq L^{-1} \simeq A^\mu_{\text{IR}}$. However, near the bag boundary, typical derivatives of the color field are of order a^{-1} and large compared to A_{IR} which is of order L^{-1}. Thus the typical infrared fluctuation is a small perturbation on the bag fields, and the problem of its propagation through the bag is essentially linear. Since the linear response of the medium is determined by μ, we can compute the correlation function $\langle A^\mu_{\text{IR}}(x) A^\nu_{\text{IR}}(y)\rangle$ (and consequently the field fluctuations) by solving the problem of linear propagation in a medium in which $\mu \simeq 1$ inside a cylinder and $\mu \gg 1$ everywhere else. We will actually solve this problem in spherical geometry in order to simplify the mathematics, without, we trust, changing the qualitative nature of the result. Consider, then, a spherical bubble of radius a with $\mu = 1$ inside and $\mu = \mu_{\text{vac}} \simeq \infty$ outside. The propagator $\langle A^\mu(x) A^\nu(y)\rangle$ is most easily computed by decomposing A_μ in partial waves. We choose

$$A^\mu(x) = \sum_L Y^\mu_L(\hat{x})\phi_L(t),$$

where $t = \ln(|x|/a)$ and the normalization of the Y's is chosen so that (in Feynman gauge) the linearized action is

$$S = \frac{8\pi^2}{g^2} \sum_L \int_{-\infty}^{\infty} dt\, \frac{\dot{\phi}_L{}^2 + \nu_L{}^2 \phi_L{}^2}{2\mu(t)}, \qquad (109)$$

where $\dot{\phi} = d\phi/dt$, $\nu_L{}^2 = 4L^2 + 1$, g is some fixed coupling constant $\simeq g(\rho_c)$, and $\mu = 1$ for $t < 0$, while $\mu = \mu_{\text{vac}} \simeq \infty$ for $t > 0$. Concentrating on a single partial wave and dropping the subscript L, it is straightforward to compute the partial-wave propagator, $\langle \phi(t)\phi(t')\rangle = G(t,t')$ which satisfies

$$-\mu(t)\frac{d}{dt}\frac{1}{\mu(t)}\frac{d}{dt}G(t,t') + \nu^2 G(t,t') = \mu(t)\frac{g^2}{8\pi^2}\delta(t-t').$$

Since $G(t,t')=G(t',t)$ it is sufficient to compute G for $t<t'$. In the limit of large $\mu_{\rm vac}$ one finds

$$G(t,t')=\frac{g^2}{8\pi^2}\frac{e^{\nu t}\cosh(\nu t')}{\nu}, \quad t<t'<0$$
$$G(t,t')=\frac{g^2}{8\pi^2}\frac{e^{\nu(t-t')}}{\nu}, \quad t<0, \quad t'>0 \qquad (110)$$
$$G(t,t')=\frac{g^2}{8\pi^2}(\mu_{\rm vac})\frac{\sinh(\nu t)e^{-\nu t'}}{\nu}, \quad t'>t>0\,.$$

One sees that the correlation between fluctuations at two points inside the bubble ($t<t'<0$) or between one point inside and one point outside ($t<0$, $t>0$) is of order $g^2/8\pi^2$ and small. In contrast, the fluctuations outside ($t'>t>0$) are of order $(g^2/8\pi^2)(\mu_{\rm vac})$ and are at least very large if not infinite. Note that (in the limit where $\mu_{\rm vac}$ is large) $(\partial/\partial t')G(t,t')$ vanishes as $t'\to 0-$ with t fixed and less than zero, while $G(t,t')$ vanishes as $t\to 0+$ with t' fixed and greater than zero. This corresponds to boundary conditions $\dot{\phi}(t)|_{t=0-}=0$ and $\phi(t)|_{t=0+}=0$ on the partial-wave amplitudes and can be shown to be equivalent to $n^\mu F^{\mu\nu}|_{t=0-}=0$ and $n^\mu \tilde{F}^{\mu\nu}|_{t=0+}=0$ for the full A^μ. Thus the fluctuations inside the bag see the bag wall as the edge of a perfect paramagnet, while the fluctuations outside see the wall as the edge of a perfect diamagnet, leading to the QCD Meissner effect described earlier.

This decoupling between the bag (including, of course, the quarks) and the large-scale vacuum fluctuations is of great importance. If it were not for this phenomenon, we could not understand hadrons without understanding the detailed infrared structure of the vacuum.

The reader can also convince himself that the correlation between fluctuations in two different bags is of order $(\mu_{\rm vac})^{-1}\simeq 0$. Thus, whatever the infrared fluctuations are, they will not lead to significant long-range forces between hadrons.

The above calculation would, of course, break down if the infrared fluctuations on large spatial scales were to become so large in amplitude that the problem could not be locally linearized. To conclude the discussion, we would like to argue that this is not likely to happen. A model of the gauge field propagator consistent with asymptotic freedom above a scale $k=\rho_c^{-1}$ and linear confinement at lower momentum is

$$D^{\mu\nu}(k)=\frac{g^{\mu\nu}}{(2\pi)^4}\begin{cases}\dfrac{g^2_{\rm AF}(k)}{k^2}, & k>\rho_c^{-1}\\[2ex] \dfrac{g_{\rm AF}{}^2(\rho_c^{-1})}{k^4\rho_c^2}, & k<\rho_c^{-1}\end{cases}$$

where $g_{\rm AF}{}^2$ is the one-loop running coupling constant. Since we expect the hadron scale size to be roughly ρ_c, a reasonable definition of what we mean by infrared is $k<\rho_c^{-1}$. From the assumed form of the propagator one can easily find the rms of the infrared part of A_μ (i.e., the part constructed out of the Fourier components with $k<\rho_c^{-1}$). The result is $\langle A_{\rm IR}\rangle\simeq\rho_c^{-1}[g^2(\rho_c)/8\pi^2]^{1/2}$. The criterion for linearization is that (bag size $\simeq\rho_c)\times\langle A_{\rm IR}\rangle\sim[g^2(\rho_c)/(8\pi^2)]^{1/2}$ should be small. But this condition is met since, as we have repeatedly had occasion to point out, $g^2/8\pi^2$ is in fact quite small at the hadron scale size. One should not take this model of the propagator too seriously, but rather as showing that the shielding property of the bag does not contradict general qualitative information about the infrared properties of QCD.

IX. CONCLUSIONS

To summarize the main points, we have seen that instantons and meron pairs are four-dimensional (color) permanent magnetic dipoles, and their presence makes the vacuum unstable against collapse into a new (strong-coupling) phase which most likely exhibits perfect paramagnetism. In the presence of a critical external color field (due to quarks, say), this phase can be in contact with a dilute-gas phase where instanton effects are relatively small. This provides a derivation (not rigorous of course but, we feel, convincing) from QCD of a picture which in first approximation is just the MIT bag, suitably modified to take chiral symmetry into account. The bag both confines (or *very* strongly binds if μ is actually finite in the vacuum) quarks and at the same time shields them from the large color field fluctuations in the vacuum.

The key element in this picture is the existence of a first-order phase transition in the presence of an external field. Instantons alone are sufficient to produce this transition. In this paper the role of merons has only been to provide a mechanism which makes it reasonable (although not certain) that μ diverges in the vacuum phase. To a certain extent it should be possible to compute many properties of hadrons without understanding the detailed character of the vacuum phase. Certainly more work (on surface effects, etc.) needs to be done, but we feel that it will soon be possible to do MIT bag-type calculations *which follow directly from QCD* and are reasonably accurate (≃10%?), a situation without precedent in strong-interaction physics.

The theoretical situation will not be completely satisfactory, however, until the vacuum phase is under control. The most basic problem is to prove or disprove that μ actually diverges (we have faith that it does).

In conclusion, we have presented an internally consistent picture of QCD which at a qualitative level agrees with what one expects phenomenologically. We have not, of course, proven in a mathematical sense that this is what happens. However, the phenomena described here occur at such a small coupling ($g^2/8\pi^2 \simeq \frac{1}{24}$) that we would find it very hard to believe that something else happens first.

Note added in proof. After completing this paper we received a report by E. V. Shuryak [Phys. Lett. 79B, 135 (1978)] in which our basic notion that bag formation is due to instanton expulsion by background color fields is also proposed. We refer the reader to his paper for a slightly different perspective on these matters. Also, we inadvertently failed to refer to the paper of Chodos and Jaffe [Phys. Rev. D 12, 2733 (1975)] on pions in the bag model.

ACKNOWLEDGMENTS

The research of C. G. C. was sponsored by the National Science Foundation under Grant No. PHY 78-01221. The research of R. F. D. was sponsored by the Department of Energy under Grant No. EY-76-S-02-2220. The research of D. J. G. was sponsored by the National Science Foundation under Grant No. PHY 77-20612.

APPENDIX A: SOME PROBLEMS IN MAGNETOSTATICS

In this appendix we shall work out two simple problems in four-dimensional magnetostatics. First we shall determine the value of the local magnetic induction $B^L_{\mu\nu}$ inside a spherical cavity which lies in a permeable medium (of permeability μ) when an external constant magnetic field, $H^{\rm ext}_{\mu\nu}$, is applied. (See Fig. 1.) This we will need to know in Sec. V in order to calculate the magnetic susceptibility. Second, we shall calculate the reaction field $R_{\mu\nu}$ acting on a dipole inside a spherical cavity due to the fact that it polarizes the medium outside the cavity. To solve these problems we must solve the four-dimensional Laplace equation

$$\Box A_\mu = 0 \tag{A1}$$

inside the cavity and outside the cavity with the following boundary conditions:

Case. 1: $A_\mu(x)$ is regular everywhere,

$$H_{\mu\nu}(x) \underset{|x|\to\infty}{\longrightarrow} H^{\rm ext}_{\mu\nu}\,; \tag{A2}$$

$$\text{Case 2: } A_\mu(x) \underset{x\simeq 0}{\simeq} -\frac{2D_{\mu\nu}x^\nu}{x^4}, \qquad H_{\mu\nu}(x) \underset{|x|\to\infty}{\longrightarrow} 0\,, \tag{A3}$$

and in both cases

$$\tilde{B}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{cases} \tilde{H}_{\mu\nu}, & |x|<R \\ \mu\tilde{H}_{\mu\nu}, & |x|>R \end{cases} \tag{A4}$$

and $\hat{x}^\mu \tilde{H}_{\mu\nu}$, $\hat{x}^\mu B_{\mu\nu}$ are continuous at $|x| = R$.

We can treat both cases at once by taking the vector potential to be

$$A_\mu(x) = \begin{cases} \dfrac{-2D_{\mu\nu}x^\nu}{x^4} - \frac{1}{2}\tilde{F}_{\mu\nu}x^\nu, & |x|<R \\ \dfrac{-2C_{\mu\nu}x^\nu}{x^4} - \frac{1}{2}\tilde{B}^{\rm ext}_{\mu\nu}x^\nu, & |x|>R \end{cases} \tag{A5}$$

where $C_{\mu\nu}$ and $F_{\mu\nu}$ are constant antisymmetric tensors to be determined.

The resulting induction field is then

$$B_{\mu\nu} = \begin{cases} H_{\mu\nu} = F_{\mu\nu} + \dfrac{4T_{\mu\alpha}(\hat{x})D_{\alpha\beta}T_{\beta\nu}(\hat{x})}{x^4}, & |x|<R \\ \mu H_{\mu\nu} = B^{\rm ext}_{\mu\nu} + \dfrac{4T_{\mu\alpha}(\hat{x})\tilde{C}_{\alpha\beta}T_{\beta\nu}(\hat{x})}{x^4}, & |x|>R\,. \end{cases} \tag{A6}$$

Matching $x^\mu B_{\mu\nu}$ at $|x| = R$ yields [where we use the fact that $\hat{x}^\mu T_{\mu\alpha}(\hat{x}) = -\hat{x}_\alpha$ and $\hat{x}^\alpha C_{\alpha\beta}T_{\beta\nu}(\hat{x}) = \hat{x}^\alpha C_{\alpha\beta}$]

$$F_{\mu\nu} - \frac{4\tilde{D}_{\mu\nu}}{R^4} = B^{\rm ext}_{\mu\nu} - \frac{4\tilde{C}_{\mu\nu}}{R^4}. \tag{A7}$$

Matching $\hat{x}^\mu \tilde{H}_{\mu\nu}$ at $|x| = R$ yields

$$F_{\mu\nu} + \frac{4\tilde{D}_{\mu\nu}}{R^4} = \frac{1}{\mu}\left(B^{\rm ext}_{\mu\nu} + \frac{4\tilde{C}_{\mu\nu}}{R^4}\right). \tag{A8}$$

Therefore, for case 1, where $D_{\mu\nu} = 0$, we have that

$$C^{(1)}_{\mu\nu} = \frac{R^4}{4}\frac{\mu-1}{\mu+1}\tilde{B}^{\rm ext}_{\mu\nu}, \qquad F^{(1)}_{\mu\nu} = \frac{2}{\mu+1}B^{\rm ext}_{\mu\nu}, \tag{A9}$$

and thus

$$B^{(1)}_{\mu\nu} = \begin{cases} \dfrac{2\mu}{1+\mu}H^{\rm ext}_{\mu\nu}, & |x|<R \\ B^{\rm ext}_{\mu\nu} + \dfrac{R^4}{x^4}\dfrac{\mu-1}{\mu+1}T_{\mu\alpha}(\hat{x})B^{\rm ext}_{\alpha\beta}T_{\beta\nu}(\hat{x}), & |x|>R\,. \end{cases} \tag{A10}$$

The local magnetic field inside the cavity is then equal to

$$B^{\rm local}_{\mu\nu} = \frac{2\mu}{1+\mu}H^{\rm ext}_{\mu\nu}.$$

In case 2, where the external field $B^{\rm ext}_{\mu\nu}$ vanishes, we have

$$C^{(2)}_{\mu\nu}=\frac{2\mu}{1+\mu}\tilde{D}_{\mu\nu},$$
$$F^{(2)}_{\mu\nu}=-\left(\frac{\mu-1}{\mu+1}\right)\frac{4\tilde{D}_{\mu\nu}}{R^4}, \tag{A11}$$

and thus

$$B^{(2)}_{\mu\nu}=\begin{cases}\dfrac{4T_{\mu\alpha}(\hat{x})\tilde{D}_{\alpha\beta}T_{\beta\nu}(\hat{x})}{x^4}-\left(\dfrac{\mu-1}{\mu+1}\right)\dfrac{4D_{\mu\nu}}{R^4}, & |x|<R\\ \left(\dfrac{2\mu}{1+\mu}\right)\dfrac{4T_{\mu\alpha}(\hat{x})\tilde{D}_{\alpha\beta}T_{\beta\nu}(\hat{x})}{x^4}, & |x|>R\ .\end{cases} \tag{A12}$$

Thus the reaction field inside the cavity, due to the medium outside the cavity which has been polarized by the dipole inside the cavity, is

$$R_{\mu\nu}=-\left(\frac{\mu-1}{\mu+1}\right)\frac{4\tilde{D}_{\mu\nu}}{R^4}. \tag{A13}$$

APPENDIX B

In this appendix we study the instanton–anti-instanton interaction, with the goal of determining the range of validity of the dipole approximation. We will also discuss the relation between close instanton–anti-instanton pairs and close meron-antimeron pairs. The first calculation involves a small instanton inside a large anti-instanton. This yields a check on our computation of the interaction of an instanton with a (nearly) constant external field (provided by the anti-instanton). A conformal transformation then produces a separated instanton–anti-instanton pair.

The spherical symmetric ansatz, $A^\mu_a=[1+\phi(t)]\eta^{\mu\nu}_a x^\nu/x^2$, with $t=\ln|x|$ is consistent with the equations of motion and produces an action

$$S=\frac{3\pi^2}{g^2}\int[\dot{\phi}^2+(\phi^2-1)^2]dt\,, \tag{B1}$$

where $\dot{\phi}=d\phi/dt$. The instanton (in the regular gauge) is the solution to $\ddot{\phi}=2\phi(\phi^2-1)$ that goes from $\phi=-1$ at $t=-\infty$ to $\phi=+1$ at $t=+\infty$ and the anti-instanton (in the singular gauge) is the solution that goes from $\phi=+1$ at $t=-\infty$ to $\phi=-1$ at $t=+\infty$. In either case the scale size ρ is e^{t_0}, where $\phi(t_0)=0$. Following Forster[17] and Polyakov[18] we will obtain constrained instanton–anti-instanton solutions by requiring that ϕ vanish at two points t_1 and t_2 with $t_2>t_1$. Since we are constraining only scale size and not orientation, this will correspond to an instanton of scale size $\rho_1=e^{t_1}$ inside an anti-instanton of scale size $\rho_2=e^{t_2}$ with the relative orientation chosen so that the interaction $-S_I$ is maximal. Let us estimate this interaction in the dipole approximation. First we make a gauge transformation so that the small instanton is in the singular gauge and Eq. (25) applies. The large anti-instanton is now in the regular gauge and the nearly uniform field $-4\bar{\eta}_{a\mu\nu}/\rho_2^2$ at its center appears as a constant external field. The maximal interaction occurs when the rotation R_{ab} specifying relative orientation is δ_{ab} and the dipole approximation to $-S_I$ is $12(\rho_1^2/\rho_2^2)(8\pi^2/g^2)$. Below we will see how this compares to the exact $-S_I$ for a constrained solution.

Thinking of the action in Eq. (B1) as that of a particle in a potential $V=-(\phi^2-1)^2$ it is easy to see the qualitative character of the constrained solution. At $t=-\infty$ the particle starts out at $\phi=1$ and rolls downhill until it reaches $\phi=0$ at $t=t_1$, at which time the constraint acts. The constraint reduces the particle's velocity and after moving uphill towards $\phi=-1$ it comes to rest at some negative ϕ greater than -1. It then falls downhill, crossing $\phi=0$ at t_2 at which point the second constraint acts and increases its velocity by just the amount needed to arrive at $\phi=+1$ at $t=+\infty$. The interaction S_I is the difference between the action for this constrained trajectory and twice the action for an instanton. Forster has computed S_I in the limit $t_2\gg t_1$ and finds precisely the dipole form $-S_I=12(\rho_1^2/\rho_2^2)(8\pi^2/g^2)$.[17]

The period for small oscillations around the bottom of the well at $\phi=0$ is $\sqrt{2}\,\pi$ and for $t_2-t_1<\pi/\sqrt{2}$ the particle can no longer go uphill between t_1 and t_2. Instead it remains at $\phi=0$ and the constrained solution corresponds to a meron-antimeron pair. The action for this configuration is $S_I=S-16\pi^2/g^2=-8\pi^2/g^2[1-\frac{3}{8}\ln(\rho_2/\rho_1)]$. We will comment on this conversion of instanton–anti-instanton pair to meron-antimeron pair later. Our goal here is to compare $f=-(g^2/8\pi^2)S_I$ as a function of $y=\rho_2/\rho_1$ computed in three different ways: (a) the dipole approximation $f=12/y^2$, (b) the meron-antimeron interaction $f=1-\frac{3}{8}\ln y$, and (c) an exact calculation which uses elliptic functions for $(t_2-t_1)>\pi/\sqrt{2}$ and the meron solution for $(t_2-t_1)<\pi/\sqrt{2}$. The results are shown in Table V. Note that the exact calculation is continuous and well-behaved at the transition $y=e^{\pi/\sqrt{2}}=9.22$ between instanton–anti-instanton pair and meron-antimeron pair. The meron result $1-\frac{3}{8}\ln y$ is of course exact for $y<9.22$ and is not a bad approximation for y out to ~12. What is remarkable, however, is that the dipole approximation $12/y^2$ works very well all the way down to $y=5$, well into the region where the configuration has become a meron-antimeron pair. Below $y\sim4$, on the other hand, the dipole approximation fails catastrophically and there is a rather sharp limiting value of $y \sim 4$-5 where it breaks down. Physically, this means that an instanton of scale size ρ_1 in an external field $-4\bar{\eta}_a^{\mu\nu}/\rho_2^2$ responds linearly as long as $\rho_2/\rho_1 \gtrsim 4-5$ but is literally torn apart into merons by larger fields.

As pointed out by Forster[17] and Polyakov,[18] a conformal transformation turns the spherical instanton inside an anti-instanton configuration into a configuration containing a separated instanton–anti-instanton pair. The kinematics are the same as those used to construct meron pairs in Ref. 1 and will not be repeated here.

If the scale sizes of the instanton and anti-instanton are ρ and ρ' and their separation is R, then the variable y corresponds to $R^2/\rho\rho'$ and we see that the dipole-dipole interaction is valid for $R^2/\rho\rho' > 4-5$, while for smaller R they must be considered as a meron-antimeron pair.

This conversion of close instanton–anti-instanton pairs to meron-antimeron pairs is just the inverse of the instability in the constrained meron solution. A meron-antimeron pair with core sizes ρ and ρ', separated by R, is unstable against breakup into an instanton–anti-instanton when $R^2/\rho\rho' > 9.22$. In the vacuum phase of QCD we expect densities such that on the average $R^2/\rho\rho' < 9.22$ and the system is presumably a soup of merons. In any case, the S_I's for a meron-antimeron pair and an instanton–anti-instanton pair are virtually identical for $12 > R^2/\rho\rho' > 5$, and one can also show that the net dipole moments of the pairs are very nearly equal.

APPENDIX C

In this appendix we evaluate the action of an instanton in a cavity of the sort discussed in Sec. VI. The strategy is to divide space into two regions $R_{\rm I}$ and $R_{\rm II}$, to be chosen such that in $R_{\rm I}$ the Yang-Mills equations can be linearized around the solution $A=0$, while in $R_{\rm II}$ they can be linearized around the instanton solution. For the specific problem stated in Sec. VI we can take $R_{\rm II}$ to be a sphere of radius r $(R > r \gtrsim \rho)$ around the instanton and $R_{\rm I}$ to be the rest of space.

An integration by parts and use of the equations of motion shows that the action contained in $R_{\rm I}$ can be expressed as a surface integral

$$\frac{1}{4g^2}\int_{R_{\rm I}} \frac{\mathrm{tr}F^2}{\mu(x)} d^4x = \frac{1}{2g^2}\int_S \frac{\mathrm{tr}F^{\mu\nu}A^\nu}{\mu(x)} dS^\mu , \tag{C1}$$

where S is the boundary (the sphere $|x| = r$) between $R_{\rm I}$ and $R_{\rm II}$ and it has been assumed (as is the case for the problem in Sec. VI) that the field A falls faster than x^{-1} at infinity. On S the field can (see Appendix A) be taken to be the field of a dipole in a constant external field; i.e.,

$$A^\mu = A^\mu_{\rm dipole} + A^\mu_{\rm ext} = -2\frac{D^{\mu\nu}x^\nu}{x^4} - \tfrac{1}{2}F^{\mu\nu}_{\rm ext}x^\nu . \tag{C2}$$

It is then a matter of algebra to show that on S the cross term between $A_{\rm dipole}$ and $A_{\rm ext}$ integrates to zero, and

$$\frac{1}{2g^2}\int_S \mathrm{tr}(F^{\mu\nu}A^\nu)dS^\mu = \frac{1}{2g^2}\int_S \mathrm{tr}(F^{\mu\nu}_{\rm dipole}A^\nu_{\rm dipole})dS^\mu + \frac{1}{2g^2}\int_S \mathrm{tr}(F^{\mu\nu}_{\rm ext}A^\nu_{\rm ext})dS^\mu . \tag{C3}$$

The method for treating region II was discussed in Ref. 1. Here we will only outline the calculation. In $R_{\rm II}$ the field A can be written as $A_0 + \delta A$, where A_0 is the field of a free instanton and δA is a small (in $R_{\rm II}$) perturbation which is regular at the origin and approaches $A_{\rm ext}$ at large x. In the action density it is sufficient to keep only linear and quadratic terms in δA, both of which can (when the equations of motion are satisfied) be expressed as total derivatives. Proceeding in this way one finds

$$\frac{1}{4g^2}\int_{R_{\rm II}} L d^4x = \frac{1}{4g^2}\int_{R_{\rm II}} L_0 d^4x - \frac{\pi^2}{4g^2}\mathrm{tr}(F^{\mu\nu}_{\rm ext}D^{\mu\nu}) - \frac{1}{2g^2}\int_S \mathrm{tr}(F^{\mu\nu}_{\rm ext}A^\nu_{\rm ext})dS^\mu , \tag{C4}$$

where L_0 is the action density of a free instanton, and we have assumed the convention that dS^μ points into (out of) $R_{\rm II}$ $(R_{\rm I})$.

The same method can be used to compute the

TABLE V. The interaction of an instanton of scale size ρ_1 inside an anti-instanton of scale size ρ_2 as a function of $y=\rho_2/\rho_1$. The exact and approximate forms of f are discussed in the text.

y	$f(y)$ Exact	$12/y^2$	$1-\frac{3}{8}\ln y$
3.50	0.530	0.980	0.530
4.00	0.480	0.750	0.480
4.50	0.436	0.593	0.436
5.00	0.396	0.480	0.396
6.00	0.328	0.333	0.328
7.00	0.270	0.245	0.270
8.00	0.220	0.188	0.220
9.10	0.172	0.145	0.172
9.22	0.167	0.141	0.167
9.38	0.161	0.137	0.161
10.75	0.119	0.104	0.109
14.37	0.063	0.058	0.001
20.60	0.030	0.028	
37.10	0.0089	0.0087	

interaction of any number of instantons and anti-instantons. Consider one instanton and one anti-instanton. The region R_{II} now consists of two small spheres, one surrounding the instanton and the other surrounding the anti-instanton. It is then straightforward to show that

$$S_I = -\frac{\pi^2}{4g^2}\mathrm{tr}[F_2^{\mu\nu}(1)D_1^{\mu\nu} + F_1^{\mu\nu}(2)D_2^{\mu\nu}], \qquad (C5)$$

where D_1 (D_2) is the moment of the instanton (anti-instanton) and $F_2^{\mu\nu}(1)$ ($F_1^{\mu\nu}(2)$) is the field of the anti-instanton (instanton) evaluated at the instanton (anti-instanton). Since $F_2^{\mu\nu}(1)D_1^{\mu\nu} = F_1^{\mu\nu}(2)D_2^{\mu\nu}$, Eq. (C5) can be written as

$$S_I = -\frac{\pi^2}{2g^2}\mathrm{tr}[F_2^{\mu\nu}(1)D_1^{\mu\nu}]$$
$$= -\frac{\pi^2}{2g^2}\mathrm{tr}(F_{\mathrm{ext}}^{\mu\nu}D_1^{\mu\nu}), \qquad (C6)$$

where $F_2^{\mu\nu}(1)$ is the external field seen by the instanton. Note that this agrees with Eq. (25), but the result used in Sec. VI is a factor of two smaller. This factor of two appears because the reaction field, unlike the field of a distant dipole or a truly constant external field, is sourceless and vanishes at infinity.

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*On leave from Princeton University.

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[7]We have often been asked what is special about one-half units of topological charge, or why do instantons not dissociate into three (or seventeen) pieces? The magnetostatic picture, of course, makes it intuitively clear that an instanton = dipole should be formed out of a pair of constituents. However, there may very well be a deeper reason for the significance of one-half units of topological charge, one that clarifies why one expects null pole = meron solutions in QCD. If one is exploring the possibility of solutions with pointlike singularities, a common mathematical strategy is to excise the singular points and fill them with some manifold. The singularities then arise only from the mapping back to ordinary Euclidean space. One can fill the singular points with nonorientable manifolds. Then the Pontryagin index is quantized with *half-integral values*. Merons are then the natural objects. The two-meron configuration can be understood in this fashion (M. Atiyah, private communication) and perhaps this method can be used to understand multiple-meron configurations.
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