Dual transformation in Abelian gauge theories*

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The dual transformation discovered in the two-dimensional Ising and planar Heisenberg models is applied to gauge theories in four dimensions. It is shown that after the dual transformation the Abelian Higgs model gives the same partition function as the relativistic hydrodynamics of Kalb and Ramond and of Nambu coupled to the Higgs scalar, and that these two theories have various dual relations. In our hydrodynamics a quantized vortex is created by the phase component of the Higgs scalar in a similar manner as the magnetic string of Nielsen and Olesen is created in the Abelian Higgs model.

I. INTRODUCTION

In solid-state physics, dual transformation plays an important role.[1-3] For example, the critical temperature of the two-dimensional Ising model was obtained explicitly[1] with the help of this dual transformation without calculating the partition function explicitly. This dual transformation relates the Ising model defined on a square lattice to the same model defined on its dual lattice. Recently, José *et al.*[3] have shown by using the dual transformation that the two-dimensional planar Heisenberg model (the XY model) is transformed into a spin-wave theory[4] with a vortex excitation. In their dual transformation a *Fourier transform formula* plays a key role [see Eq. (2.7)]. So, it is not too much to say that the dual transformation is a kind of Fourier transformation performed in the integrand of the partition function. There is also an interesting work by Savit[5] studied independently of the work of José *et al.* in the U(1)-invariant lattice theories in arbitrary dimension. Since then, this dual transformation has been studied[6] in the Abelian lattice gauge theory in connection with Mandelstam-'t Hooft duality.[7]

The purpose of the present paper is to study this dual transformation in conventional gauge theories in four dimensions. Using this dual transformation, we show that the Abelian Higgs model gives the same partition function as the *relativistic hydrodynamics* of Kalb and Ramond[8] and of Nambu[9] coupled to a scalar field. In this new type of hydrodynamics, a vorticity source is created by the phase component of the Higgs scalar $\chi(x)$. The strength of this vorticity source is proportional to the integer n if χ varies by $2\pi n$ in a full turn around a specified string S.

For simplicity, we identify S with the x^3 axis and search for a simple classical solution. We find that there exists as a classical solution a static circulation flow of fluid around the x^3 axis and that the total flux of the velocity potential is quantized. This classical solution found in our hydrodynamics corresponds to the Nielsen-Olesen vortex solution[10] in the Higgs model. In our classical solution an "electric" flux, a flux of the (0,3) component of the velocity potential, is squeezed in a manner similar to that of the magnetic flux in the Nielsen-Olesen solution.

We also derive various dual relations between the Green's functions of the Higgs model and those of the hydrodynamic model. In order to derive these relations, it is necessary to prove a dual relation between the partition function of the Higgs model and that of the hydrodynamic model in the presence of an external tensor source.

In the next section, we show that the Abelian Higgs model has the same partition function as the relativistic hydrodynamics of Kalb and Ramond and of Nambu coupled to a scalar field. A generalized formula in the presence of a fermion is also given there. In Sec. III, we obtain a classical solution in the hydrodynamic model and give various dual relations between the Higgs model and the hydrodynamic model.

II. THE DUAL TRANSFORMATION

In this section, we will show that the Abelian Higgs model gives the same partition function as the relativistic hydrodynamics of Kalb and Ramond[8] and of Nambu[9] coupled to a scalar field.

The Higgs model is described by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left|(\partial_\mu + ieA_\mu)\phi\right|^2 - V(\phi)\,, \qquad (2.1)$$

where

$$V(\phi) = \mu^2\phi^\dagger\phi + \tfrac{1}{4}\lambda(\phi^\dagger\phi)^2 \quad (\mu^2 < 0)\,. \qquad (2.2)$$

As usual, ϕ and A_μ denote a complex scalar field and an Abelian gauge field, respectively, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Using the path-integral method, the partition function of the Higgs model is given by

$$Z=\int \mathfrak{D}A_\mu(x)\int \mathfrak{D}|\phi(x)|\,|\phi(x)|\int \mathfrak{D}\chi(x)\prod_x \delta(\chi(x)-n\theta(x))\exp\Big[i\int d^4x\,\mathfrak{L}(x)\Big], \qquad (2.3)$$

with $|\phi(x)|$ and $\chi(x)$ defined by

$$\phi(x)=|\phi(x)|e^{i\chi(x)}\,. \qquad (2.4)$$

In Eq. (2.3), we fix $\chi(x)$ to be $n\theta(x)$, where $\theta(x)$ is the azimuthal angle of x around the specified string S, and n is an integer. When $n=0$, this fixing corresponds to the unitary gauge condition. For $n\neq 0$, $\theta(x)$ is not well defined on the world sheet of that string S, but the measure of this string is vanishing so that $\Pi_x\delta(\chi(x)-n\theta(x))$ is useful to make the partition function (2.3) finite.

In Eq. (2.3), we perform the following transformation using antisymmetric tensor fields $W_{\mu\nu}$:

$$\exp\Big[i\int d^4x(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu})\Big]\propto\int \mathfrak{D}W_{\mu\nu}(x)\exp\Big\{i\int d^4x[-\tfrac{1}{4}(m^2W_{\mu\nu}W^{\mu\nu}+2m\tilde{W}_{\mu\nu}F^{\mu\nu})]\Big\}, \qquad (2.5)$$

with

$$\tilde{W}_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}W^{\lambda\rho}\,, \qquad (2.6)$$

where $\epsilon_{\mu\nu\lambda\rho}$ takes the value +1 or −1 according to whether $(\mu\nu\lambda\rho)$ is an even or odd permutation of (0123), and m is an arbitrary mass parameter. It is instructive to understand that Eq. (2.5) is a *Fourier transform formula* in the path-integral method. Recently José *et al.*[3] have shown that the planar Heisenberg model is transformed into a spin-wave theory[4] with a vortex excitation with the help of the Fourier transform formula

$$e^{V(\theta(\vec{r})-\theta(\vec{r}'))}=\sum_{S(\vec{r},\vec{r}')=-\infty}^{+\infty} e^{\tilde{V}(S(\vec{r},\vec{r}'))}e^{iS(\vec{r},\vec{r}')[\theta(\vec{r})-\theta(\vec{r}')]}\,, \qquad (2.7)$$

where we have used the notations in Ref. 3. Our transformation (2.5) corresponds to the transformation (2.7) in solid-state physics. Now, substituting (2.5) into (2.3), we perform an integration over $A_\mu(x)$,

$$\int \mathfrak{D}A_\mu(x)\exp\Big\{i\int d^4x[e^2|\phi|^2A_\mu(x)A^\mu(x)+A^\mu(x)(m\partial^\nu\tilde{W}_{\nu\mu}+ie\phi^\dagger\overleftrightarrow{\partial}_\mu\phi)]\Big\}$$
$$\propto\exp\Big[i\int d^4x\Big(-\frac{1}{4}\frac{m^2}{e^2|\phi|^2}[V_\mu(x)+i(e/m)\phi^\dagger\overleftrightarrow{\partial}_\mu\phi]^2+4i\delta^{(4)}(0)\ln(e|\phi|/m)\Big)\Big], \qquad (2.8)$$

where we have introduced the velocity vector $V_\mu(x)$ of relativistic hydrodynamics using the velocity potential $W_{\mu\nu}$,[9]

$$V_\mu(x)\equiv\partial^\nu\tilde{W}_{\nu\mu}(x)\,. \qquad (2.9)$$

In Eq. (2.8), the $\delta^{(4)}(0)$ term represents

$$\prod_x[1/(e^2|\phi|^2/m^2)^{1/2}]^4 \qquad (2.10)$$

and $\overleftrightarrow{\partial}_\mu=\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu$. In the hydrodynamics, the velocity vector is a physical quantity and satisfies the equation of continuity

$$\partial^\mu V_\mu=0\,, \qquad (2.11)$$

which holds automatically because of (2.9). Integration over $A_\mu(x)$ corresponds to integration over the phase component $\theta(\vec{r})$ of the classical spin in the planar Heisenberg model.[3] In our case, however, Eq. (2.8) gives no constraint because there exists the bilinear term $e^2|\phi|^2A_\mu A^\mu$ in Eq. (2.8). From Eqs. (2.3), (2.5), and (2.8), we obtain

$$Z\propto\int \mathfrak{D}W_{\mu\nu}(x)\int \mathfrak{D}|\phi(x)|\int \mathfrak{D}\chi(x)\prod_x\delta(\chi(x)-n\theta(x))$$
$$\times\exp\Big[i\int d^4x\Big(-\frac{1}{4}\frac{m^2}{e^2|\phi|^2}[V_\mu(x)+i(e/m)\phi^\dagger\overleftrightarrow{\partial}_\mu\phi]^2-\frac{m^2}{4}(W_{\mu\nu})^2$$
$$+|\partial_\mu\phi|^2-V(\phi)+3i\delta^{(4)}(0)\ln(e|\phi|/m)\Big)\Big]\,. \qquad (2.12)$$

Integration over $\chi(x)$ leads us to

$$Z \propto Z^* = \int \mathfrak{D}W_{\mu\nu}(x) \int \mathfrak{D}|\phi(x)| \exp\left[i \int d^4x \; \mathfrak{L}^*_{\text{eff}}(x)\right], \tag{2.13}$$

where the effective Lagrangian density $\mathfrak{L}^*_{\text{eff}}(x)$ is defined by

$$\mathfrak{L}^*_{\text{eff}}(x) = \mathfrak{L}^*(x) + 3i\delta^{(4)}(0)\ln(e|\phi|/m) \tag{2.14}$$

and

$$\mathfrak{L}^*(x) = -\frac{m^2}{2e^2|\phi|^2}\tfrac{1}{2}(V_\mu)^2 - \tfrac{1}{4}m^2(W_{\mu\nu})^2 + \frac{1}{2}\frac{2\pi m}{e}W_{\mu\nu}\omega^{\mu\nu} + (\partial_\mu|\phi|)^2 - V(|\phi|) \tag{2.15}$$

with a vorticity source $\omega^{\mu\nu}$. This vorticity source is created by the phase component $\chi(x)$ of the Higgs scalar

$$\omega^{\mu\nu} \equiv \frac{1}{4\pi}\epsilon^{\mu\nu\lambda\rho}(\partial_\lambda\partial_\rho - \partial_\rho\partial_\lambda)\chi(x). \tag{2.16}$$

Of course, $\omega^{\mu\nu}=0$ where $\chi(x)$ is regular. In our case, however, $\chi(x)=n\theta(x)$ and is singular (for $n\neq 0$) on the world sheet of the specified string S, so that $\omega^{\mu\nu}\neq 0$ on this world sheet. If we parametrize a position y_μ on this world sheet of the string by timelike and spacelike parameters τ and σ, we have[11]

$$\omega^{\mu\nu} = n\int\int d\tau\, d\sigma\, \delta^{(4)}(x-y(\tau,\sigma)) \times\left|\frac{\partial(y^\mu, y^\nu)}{\partial(\tau,\sigma)}\right|. \tag{2.17}$$

Here it is necessary to discuss a relation (2.14) between $\mathfrak{L}^*_{\text{eff}}$ and $\mathfrak{L}^*$. Let us remind the reader that the original path-integral formula is written in terms of dynamical variables $q_i(x)$ and their conjugate momenta $p_i(x)$,[12]

$$Z_N \propto \int \mathfrak{D}q_i(x)\mathfrak{D}p_i(x) \times\exp\left\{i\int d^4x\left[\sum_i p_i\dot{q}_i - \mathcal{H}_N(p,q)\right]\right\}, \tag{2.18}$$

where $\mathcal{H}_N(p,q)$ is a Hamiltonian density. In the case of a nonlinear Lagrangian density $\mathfrak{L}_N$ such as

$$\mathfrak{L}_N(x) = \tfrac{1}{2}\sum_{i,j}\mathring{q}_i f_{ij}(q)\mathring{q}_j + \sum_i g_i(q)\mathring{q}_i + h(q), \tag{2.19}$$

integration of (2.18) over $p_i(x)$ gives

$$Z_N \propto \int \mathfrak{D}q_i(x)\exp\left[i\int d^4x\; \mathfrak{L}_N(x)_{\text{eff}}\right], \tag{2.20}$$

and $\mathfrak{L}_N(x)_{\text{eff}}$ is derived from $\mathfrak{L}_N(x)$ by the following formula[12,13]:

$$\mathfrak{L}_N(x)_{\text{eff}} = \mathfrak{L}_N(x) - i\delta^{(4)}(0)\sum_i\{\ln[f(q)]^{1/2}\}_{ii}. \tag{2.21}$$

Let us show that the relation (2.14) between $\mathfrak{L}^*_{\text{eff}}$ and $\mathfrak{L}^*$ is nothing but an example of (2.21). If we introduce $\vec{e}(x)$ and $\vec{b}(x)$ as[9]

$$e^i \equiv W^{0i}, \tag{2.22a}$$

$$b^i \equiv \tfrac{1}{2}\epsilon_{0ijk}W^{jk}, \tag{2.22b}$$

we have

$$V_0 = V^0 = \vec{\nabla}\cdot\vec{b}, \tag{2.23a}$$

$$V_i = -V^i = [\dot{\vec{b}} + (\vec{\nabla}\times\vec{e})]^i. \tag{2.23b}$$

So, there are four dynamical variables $\vec{b}(x)$ and $|\phi(x)|$, and three nondynamical variables $\vec{e}(x)$ in our model described by $\mathfrak{L}^*$. The coefficients f_{ij} of Eq. (2.19) are in our case

$$f_{ij} = \frac{m^2}{2e^2|\phi|^2}\delta_{ij} \quad (i,j=1,2,3). \tag{2.24}$$

Then we obtain Eq. (2.14) from Eq. (2.21), and we have proved that Z^* [Eq. (2.13)] is a partition function of the nonlinear Lagrangian density $\mathfrak{L}^*$ [Eq. (2.15)].

In the above discussion, we have used terminology in hydrodynamics such as velocity field, velocity potential, and vorticity source since $\mathfrak{L}^*$ represents the massive relativistic hydrodynamics of Kalb and Ramond[8] and of Nambu[9] coupled to a scalar field $|\phi(x)|$. The relativistic hydrodynamics discussed by Kalb and Ramond and by Nambu is described by the following Lagrangian density:

$$\mathfrak{L}_1 = -\tfrac{1}{2}(V_\mu)^2 - kW_{\mu\nu}\omega^{\mu\nu} \tag{2.25}$$

or

$$\mathfrak{L}_2 = -\tfrac{1}{2}(V_\mu)^2 - \tfrac{1}{4}m^2(W_{\mu\nu})^2 - kW_{\mu\nu}\omega^{\mu\nu}. \tag{2.26}$$

It should be stressed that the Lagrangian density

$\mathcal{L}^*(x)$ obtained above [Eq. (2.15)] is the relativistic hydrodynamics of a massive version (2.26) coupled to the Higgs scalar $|\phi(x)|$, and that the partition function Z^* [Eq. (2.13)] of $\mathcal{L}^*(x)$ is proved to be proportional to the partition function Z of the Higgs Lagrangian $\mathcal{L}(x)$ [Eq. (2.1)]. Following solid-state physics, we may call the transformation of $\mathcal{L}(x)$ to $\mathcal{L}^*(x)$ a *dual transformation*, and these two theories described by $\mathcal{L}(x)$ and $\mathcal{L}^*(x)$ may be called dually related to each other.

In concluding this section, we register a formula in the presence of a fermion. If we add a term

$$\Delta\mathcal{L}=\bar{\psi}(i\not{\partial}-M)\psi+e\bar{\psi}\gamma_\mu\psi A^\mu \qquad (2.27)$$

to $\mathcal{L}$ [Eq. (2.1)], we obtain an additional term to $\mathcal{L}^*$ [Eq. (2.15)], namely

$$\Delta\mathcal{L}^*=\bar{\psi}(i\not{\partial}-M)\psi-n\bar{\psi}\gamma_\mu\psi\partial^\mu\theta+\frac{1}{2}\frac{2\pi m}{e}W^{\mu\nu}\omega^F_{\mu\nu}-\frac{1}{4}\frac{1}{|\phi|^2}(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi), \qquad (2.28)$$

where

$$\omega^F_{\mu\nu}=\frac{1}{4\pi}\epsilon_{\mu\nu\lambda\rho}\partial^\lambda\left(\frac{1}{|\phi|^2}\bar{\psi}\gamma^\rho\psi\right). \qquad (2.29)$$

III. THE VORTEX SOLUTION AND THE DUAL RELATIONS

In this section, we will search for a most simple classical solution of hydrodynamics described by $\mathcal{L}^*(x)$ [Eq. (2.15)], namely, a static circulation flow of fluid around the x^3 axis.

The field equations obtained from Eq. (2.15) are

$$-\epsilon_{\mu\nu\lambda\rho}\partial^\lambda\left(\frac{m^2}{2e^2|\phi|^2}V^\rho\right)+m^2W_{\mu\nu}-\frac{2\pi m}{e}\omega_{\mu\nu}=0 \qquad (3.1a)$$

and

$$2\Box|\phi|+\frac{\delta V(|\phi|)}{\delta|\phi|}-\frac{m^2}{2e^2|\phi|^3}(V_\mu)^2=0. \qquad (3.1b)$$

If we rewrite Eqs. (3.1a) and (3.1b) using $\vec{e}(x)$ and $\vec{b}(x)$ analogous to electric and magnetic fields[9] [Eqs. (2.22a) and (2.22b)] we have the following equations:

$$\vec{\nabla}\times\left[\frac{m^2}{2e^2|\phi|^2}(\dot{\vec{b}}+\vec{\nabla}\times\vec{e})\right]+m^2\vec{e}-\frac{2\pi m}{e}\vec{\omega}_e=0, \qquad (3.2a)$$

$$-\vec{\nabla}\cdot\left[\frac{m^2}{2e^2|\phi|^2}(\vec{\nabla}\cdot\vec{b})\right]+\partial_0\left[\frac{m^2}{2e^2|\phi|^2}(\dot{\vec{b}}+\vec{\nabla}\times\vec{e})\right]+m^2\vec{b}-\frac{2\pi m}{e}\vec{\omega}_b=0, \qquad (3.2b)$$

and

$$2\Box|\phi|+\frac{\delta V(|\phi|)}{\delta|\phi|}-\frac{m^2}{2e^2|\phi|^3}[(\vec{\nabla}\cdot\vec{b})^2-(\dot{\vec{b}}+\vec{\nabla}\times\vec{e})^2]=0, \qquad (3.2c)$$

where $\vec{\omega}_e(x)$ and $\vec{\omega}_b(x)$ are introduced by

$$\omega_e^i\equiv\omega^{0i} \qquad (3.3a)$$

and

$$\omega_b^i\equiv\tfrac{1}{2}\epsilon_{0ijk}\omega^{jk}. \qquad (3.3b)$$

For simplicity, we identify the specified string S with the x^3 axis and study a static solution of (3.2). In this case, we have

$$\omega_e{}^3=n\delta(x^1)\delta(x^2), \qquad (3.4a)$$

$$\omega_e^i=0 \text{ for } i=1,2 \qquad (3.4b)$$

and

$$\vec{\omega}_b=\vec{0}. \qquad (3.4c)$$

Therefore, we can search a static and axially symmetric solution with the following assumptions:

$$\partial_0 e^3=\partial_0|\phi|=0,\quad e^3=e^3(r),\quad |\phi|=|\phi|(r), \qquad (3.5a)$$

$$e^i\equiv0 \text{ for } i=1,2 \qquad (3.5b)$$

and

$$\vec{b}\equiv\vec{0}, \qquad (3.5c)$$

where $r\equiv[(x^1)^2+(x^2)^2]^{1/2}$.

Under these assumptions, the field equations (3.2a)–(3.2c) are reduced to

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{m^2}{2e^2|\phi|^2}\frac{d}{dr}e^3\right)+m^2e^3-\frac{2\pi m}{e}\omega_e{}^3=0 \qquad (3.6a)$$

and

$$-2\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}|\phi|\right)+\frac{\delta V(|\phi|)}{\delta|\phi|}+\frac{m^2}{2e^2|\phi|^3}\left(\frac{d}{dr}e^3\right)^2=0. \qquad (3.6b)$$

It is easy to solve the above equations for very large r where $e^3\simeq0$ and $|\phi|\simeq v$ hold, with v a minimum point of the Higgs potential $V(|\phi|)$, namely,

$$\left.\frac{\delta V(|\phi|)}{\delta|\phi|}\right|_{|\phi|=v}=0 \text{ for } v=(-2\mu^2/\lambda)^{1/2}. \qquad (3.7)$$

Then we obtain

$$e^3\underset{r\to\infty}{\sim}aK_0(\sqrt{2}evr) \qquad (3.8a)$$

$$\underset{r\to\infty}{\sim}a\left(\frac{\pi}{2\sqrt{2}evr}\right)^{1/2}e^{-\sqrt{2}evr} \qquad (3.8b)$$

and

$$|\phi| \underset{r\to\infty}{\sim} v + bK_0((-2\mu^2)^{1/2}r) \qquad (3.9a)$$

$$\underset{r\to\infty}{\sim} v + b\left[\frac{\pi}{2(-2\mu^2)^{1/2}r}\right]^{1/2} e^{-(-2\mu^2)^{1/2}r}, \qquad (3.9b)$$

where $K_0(\kappa r)$ is the modified Bessel function satisfying

$$r\frac{d}{dr}\left[r\frac{d}{dr}K_0(\kappa r)\right] - \kappa^2 K_0(\kappa r) = 0 \qquad (3.10a)$$

and

$$K_0(\kappa r) \underset{r\to\infty}{\to} 0. \qquad (3.10b)$$

As for the velocity fields $\vec{V}(r)$, they have only nonvanishing component $V(r)_\theta$, where θ indicates the angle direction of the polar coordinates on the (x^1, x^2) plane. So we have a circulation flow around the x^3 axis. The behavior of $V(r)_\theta$ for large r is

$$V(r)_\theta = \frac{\partial}{\partial r}e^3(r)$$

$$\underset{r\to\infty}{\sim} a\sqrt{2}\,evK_0'(\sqrt{2}\,evr) \qquad (3.11a)$$

$$= -a\sqrt{2}\,evK_1(\sqrt{2}\,evr)$$

$$\underset{r\to\infty}{\sim} -a\sqrt{2}\,ev\left(\frac{\pi}{2\sqrt{2}\,evr}\right)^{1/2} e^{-\sqrt{2}evr}. \qquad (3.11b)$$

Integration of (3.2a) in the domain D of the (x^1, x^2) plane leads us to

$$-\oint_C \frac{m^2}{2e^2|\phi|^2}\vec{V}\cdot d\vec{s} + m^2\int_D e^3 dS = \begin{cases} 0 \text{ for } (0,0)\notin D, \\ \dfrac{2\pi m}{e} n \text{ for } (0,0)\in D, \end{cases} \qquad (3.12)$$

where C is the boundary of the domain D. If we choose D to be a very large domain D_∞ including the origin, the first term in Eq. (3.12) does not contribute due to Eq. (3.11b) so that we obtain a *flux quantization rule*:

$$m\int_{D_\infty} e^3 dS = \frac{2\pi}{e} n \ (n \text{ is an integer}). \qquad (3.13)$$

Now we have found a static and axially symmetric classical solution in the hydrodynamic model described by $\mathcal{L}^*(x)$ [Eq. (2.15)], the dually related model with the original Higgs model described by $\mathcal{L}(x)$ [Eq. (2.1)]. In this solution there appears a circulation flow of fluid around the x^3 axis owing to a vorticity source $\omega_e{}^3$ created by the phase component $\chi(x)$ of the Higgs scalar. The Nielsen-Olesen vortex solution[10] found in the original Higgs model is a static and axially symmetric solution with boundary conditions $|\vec{A}|\simeq 0$ and $|\phi|\simeq v$ for large r. These conditions are similar to our conditions imposed to find the classical solution in the dually transformed Lagrangian density $\mathcal{L}^*(x)$. Therefore, it is interesting to compare these two solutions. For an example, we discuss simple relations between Green's functions of the Higgs model and those of hydrodynamics.

We introduce an external source $J_{\mu\nu}$ in Eq. (2.1) through the replacement[14]

$$F_{\mu\nu}(x) \to F_{\mu\nu}(x) - J_{\mu\nu}(x). \qquad (3.14)$$

Starting from the Lagrangian $\mathcal{L}(x, J_{\mu\nu})$ obtained from $\mathcal{L}$ [Eq. (2.1)] by the replacement (3.14), and following the method given in Sec. II, we easily obtain the formula

$$Z[J_{\mu\nu}] \propto Z^*[J_{\mu\nu}]. \qquad (3.15)$$

In this formula, $Z[J_{\mu\nu}]$ and $Z^*[J_{\mu\nu}]$ are defined by replacing $\mathcal{L}(x)$ and $\mathcal{L}^*(x)$ with $\mathcal{L}(x, J_{\mu\nu})$ and $\mathcal{L}^*(x, J_{\mu\nu})$ in Eqs. (2.3) and (2.13), respectively, where

$$\mathcal{L}(x, J_{\mu\nu}) = \mathcal{L}(x) + \tfrac{1}{2}F^{\mu\nu}J_{\mu\nu} - \tfrac{1}{4}(J^{\mu\nu})^2, \qquad (3.16)$$

$$\mathcal{L}^*(x, J_{\mu\nu}) = \mathcal{L}^*(x) + \tfrac{1}{2}m\tilde{W}^{\mu\nu}J_{\mu\nu}. \qquad (3.17)$$

From the formula (3.15), we have a simple relation

$$\langle F^{\mu\nu}(x)\rangle = \langle m\tilde{W}^{\mu\nu}(x)\rangle_* \qquad (3.18)$$

between a vacuum expectation value of the Higgs model and that of hydrodynamics. This relation is easily checked with the help of Eqs. (3.15)–(3.17) and the following formula:

$$\langle F^{\mu\nu}(x)\rangle = \frac{\delta}{\delta J_{\mu\nu}(x)}\ln Z[J]\bigg|_{J=0} \qquad (3.19a)$$

and

$$\langle m\tilde{W}^{\mu\nu}(x)\rangle_* = \frac{\delta}{\delta J_{\mu\nu}(x)}\ln Z^*[J]\bigg|_{J=0}. \qquad (3.19b)$$

Decomposition of Eq. (3.18) into "electric" and "magnetic" fields reads

$$\langle \vec{E}(x)\rangle = -\langle m\vec{b}(x)\rangle_*, \qquad (3.20a)$$

$$\langle \vec{H}(x)\rangle = \langle m\vec{e}(x)\rangle_*, \qquad (3.20b)$$

where

$$E^i \equiv F^{0i}, \qquad (3.21a)$$

$$H^i \equiv \tfrac{1}{2}\epsilon_{0ijk}F^{jk}. \qquad (3.21b)$$

Equation (3.20b) shows that $\vec{H}(x)$ in the Higgs model corresponds to $m\vec{e}(x)$ in the hydrodynamic model. So it is easy to understand that the behavior of $e^3(r)$ for large r corresponds to that of $H^3(r)$ in the Nielsen-Olesen vortex solution for large r.

From Eqs. (2.9) and (3.18) we have also

$$\langle \partial_\mu F^{\mu\nu}(x)\rangle = \langle m V^\nu(x)\rangle_* . \quad (3.22)$$

Introducing an electric current $j^\nu(x)$ as

$$j^\nu(x) \equiv \partial_\mu F^{\mu\nu}(x) = 2e^2|\phi|^2\left(A^\nu(x) + \frac{1}{e}\partial^\nu\chi(x)\right), \quad (3.23)$$

where we have used the field equation of the Higgs model, we obtain from Eqs. (3.22) and (3.23) the following correspondence. Conservation of $j^\nu(x)$ in the Higgs model corresponds to that of $V^\nu(x)$ in the hydrodynamic model, and

$$\vec{A}(x) - (1/e)\vec{\nabla}\chi(x) \underset{r\to\infty}{\longrightarrow} 0$$

in the Nielsen-Olesen solution corresponds to $\vec{V}(x) \underset{r\to\infty}{\longrightarrow} 0$ in our vortex solution. As for the flux quantization, the quantization of the "electric" flux [Eq. (3.13)] in our hydrodynamic model corresponds to that of the magnetic flux in the Higgs model through Eq. (3.20b). We may call these relations (3.18), (3.20), and (3.22) as *dual relations*. Of course, it is possible to study further the dual correspondence between the Green's functions of the Higgs model and those of the hydrodynamic model by evaluating the functional derivative of Eq. (3.15) successively. For example, we have a dual correspondence

$$\langle T^* F_{\mu\nu}(x) F_{\lambda\rho}(y)\rangle + i(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda})\delta^{(4)}(x-y) = m^2\langle T^*\tilde{W}_{\mu\nu}(x)\tilde{W}_{\lambda\rho}(y)\rangle_* . \quad (3.24)$$

Therefore, there is a possibility of finding the renormalizability of the hydrodynamic model described by Eq. (2.15) by using the dual correspondence between this hydrodynamic model and the renormalizable Higgs model.

Note added. After completion of this work we learned of a work dealing with a problem similar to ours in the lattice gauge theory by M. B. Einhorn and R. Savit [Phys. Rev. D 17, 2583(1978); 19, 1198(1979)]. Their treatment is, however, considerably different from ours.

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