PHYSICAL REVIEW D VOLUME 19, NUMBER 6 15 MARCH 1979

Two-point function of nonlinear spinor theory

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(Received 14 September 1978)

The influence of interaction on the nature of the quantum field is investigated for a self-interacting spinor theory. Nonlinear equations for the two-point function are formulated. The solutions of these equations demonstrate that fields in interaction exhibit novel features. In particular, the modified short-distance behavior indicates the absence of pathological features, implying self-regulation of nonlinear systems with concomitant noncanonical quantization.

I. INTRODUCTION

The quantization of a physical system proceeds in two stages. In the first step the particle-wave duality finds expression in the replacement of the classical variables by operators of specified commutation relations. The occurrence of particle creation and of virtual transitions is accommodated by regarding the classical fields as operator-valued fields. In the canonical formalism of field theory the quantum fields satisfy commutation relations which are independent of the interaction. Attempts to verify the validity of the free-field commutation relations in the interacting situation encountered obstacles. Thus the formalism of field theory is permeated by notions obtained by means of perturbation theory. In particular, interacting field theory is disfigured by the occurrence of radiative corrections which may become arbitrarily large. This feature divides field theory into that which can be realized and that which cannot. After the renormalization procedure has been carried out the canonical commutation relations are destroyed and the pathological contributions are eliminated at the expense of introducing arbitrary mass and coupling parameters into the theory, which preclude the dynamical explanation of important attributes of the theory itself. It would thus seem that a more complete understanding of the nature of interacting field theory is of importance.

In this work a nonlinear spinor theory is investigated as a prototype of features intrinsic to nonlinear quantum field theory. The theory is allowed to develop outside the chartered course of perturbation theory by adopting an approach related to that pioneered[1] by Heisenberg and by Mitter. The novel features exhibited by classical nonlinear systems (intrinsic nonlinear features such as self-modulation, self-focusing, the emergence of nonlinear excitations such as vortices, and the existence of nonlinear waves that exhibit particlelike characteristics) in a natural way lead to the question of aspects intrinsic to nonlinear quantum field theory.[2] On general grounds nonlinear quantum phenomena may be expected to be of relevance in the high-energy regime where nonlinearity due to feedback becomes important.[3] Throughout the analysis a special premium is placed on nonlinearity as engendered by the two-point function.

In Sec. II the model is specified, a functional method is introduced to generate the ingredients required for the analysis, and the equations satisfied by the multipoint functions are formulated. In Sec. III the equation for the two-point function is cast into a form amenable to solution. In Sec. IV the solutions of the nonlinear equations are investigated, and questions related to the breakdown of scale invariance and spontaneous mass generation are considered. In Sec. V the solutions are subjected to the boundary conditions imposed by quantum field theory and, finally, in Sec. VI the implications of the results for nonlinear field theory are discussed.

II. EQUATION OF MOTION AND GENERATING FUNCTIONAL

This work investigates the features intrinsic to a nonlinear field theory based on an equation of motion of the form

$$i\gamma\cdot\partial\psi(x)-m\psi(x)\pm l^2[\bar\psi(x)O^\mu\psi(x)]O_\mu\psi(x)=0\,, \quad (2.1)$$

which may formally be deduced from the Lagrangian density

$$\mathcal{L}=\frac{i}{2}(\bar\psi\gamma^\mu\partial_\mu\psi-\partial_\mu\bar\psi\gamma^\mu\psi)-m\bar\psi\psi \pm\tfrac{1}{2}l^2(\bar\psi O^\mu\psi)(\bar\psi O_\mu\psi)\,. \quad (2.2)$$

For the present purpose this model may be regarded as a kind of incubator for studying nonlinear field theory. According to the canonical formalism of quantum field theory, the classical field variables are promoted to dynamical variables satisfying the equal-time anticommutation relations

$$\delta(x_0 - y_0)\{\pi(x), \psi(y)\} = i\delta(x-y)\,, \tag{2.3}$$

where the canonical momentum conjugate to ψ is given by

$$\pi = \frac{\delta \mathcal{L}}{\delta(\partial_0 \psi)} = i\overline{\psi}\gamma_0 = i\psi^\dagger\,. \tag{2.4}$$

The canonical quantization procedure implies that the second quantization proceeds independently of the self-interaction present in the theory and attributes a length dimension of $-\frac{3}{2}$ to the canonical field. A traditional way of arriving at the canonical commutation relations consists in establishing Eq. (2.3) in the free-field case and transforming from the fields defined at a fixed time to the interacting Heisenberg fields at arbitrary times by unitary transformations. However, for fields possessing an infinite number of degrees of freedom the interaction representation does not, strictly speaking, exist for relativistic invariant theories. As implied by Haag's theorem,[4] the interaction picture on which perturbation theory is based does not exist, at least in a state space in which the free-field commutation relations are also represented.

The massless equation (2.1) is of particular interest for it remains invariant under scale transformations.[5] Consider the infinitesimal scale transformation defined on Minkowski space by

$$x'_\mu = (1+\epsilon)x_\mu\,, \tag{2.5a}$$

$$\partial'_\mu = (1-\epsilon)\partial_\mu \tag{2.5b}$$

and the spinor field cotransforms according to

$$\psi'(x') = (1+d\epsilon)\psi(x)\,, \tag{2.6}$$

where d denotes the scale dimension of the spinor field. The form invariance of the massless Eq. (2.1) is maintained provided that the scale dimension of the field is subcanonical, $d = -\frac{1}{2}$. Scale invariance of the theory in turn may serve as a rationale for the absence of (bare) mass terms in Eq. (2.1). The scale invariance hints at the possibility that results different from the canonical quantization may obtain. In view of the presence of self-interaction in the theory, however, it may be anticipated that scale invariance may break down through spontaneous mass generation so that the scale-invariant situation may be reflected only by very special solutions.

Further discussion is facilitated by introducing a compact notation for Eq. (2.1):

$$D_{ik}\psi_k + V_{im,nl} : \overline{\psi}_m \psi_n \psi_l : = 0\,, \tag{2.7}$$

where space-time, spinor, and other degrees of freedom are incorporated into a single index. In the local formulation of the theory the interacting term is given by

$$V_{im,nl} = \delta(x_i - x_m)\delta(x_n - x_i)\delta(x_n - x_l)V'_{im,nl}\,, \tag{2.8}$$

where for a scalar interaction $V'_{im,nl} = \delta_{in}\delta_{ml}$. The kinetic part of Eq. (2.7) is given by

$$D_{ik} = i\gamma^\mu \delta(x_i - x_k)\frac{\partial}{\partial x^k_\mu} - m\delta(x_i - x_k)\,. \tag{2.9}$$

The appearance of δ functions in the foregoing is of formal significance only and is a consequence of the compact notation employed. In view of the circumstance that the fields in quantum field theory are represented by operator-valued distributions, the definition of the interaction term requires closer attention. In order to give a definition of the products of fields at the same space-time point, knowledge of the nature of the quantized field is required, a question which is subject to the solution of the nonlinear equation itself. A heuristic approach must therefore be taken by adopting a definition motivated by linear field theory, as follows:

$$\begin{aligned} &:\overline{\psi}(x_m)\psi(x_n)\psi(x_l): \\ &\quad = T\overline{\psi}(x_m)\psi(x_n)\psi(x_l) \\ &\qquad + \theta(::)[\eta(x_n|x_m)\psi(x_l) - \eta(x_l|x_m)\psi(x_n)]\,, \end{aligned} \tag{2.10}$$

where the function $\theta(::)$ is defined to be equal to one if the Wick product is employed in Eq. (2.7) and zero otherwise, and the two-point function, η, is defined below. The definition (2.10) for the field operators at the same space-time point is adequate for the study of the two-point functions of this work, since the Wick product of the field operators suffices to regularize the two-point functions involving field-operator products at the same space-time point. The time-ordering operation, T, is defined by

$$\begin{aligned} &T\psi(x_k)\overline{\psi}(x_b) \\ &\quad = \theta(x_{0k} - x_{0b})\psi(x_k)\overline{\psi}(x_b) - \theta(x_{0b} - x_{0k})\overline{\psi}(x_b)\psi(x_k) \\ &\quad = \tfrac{1}{2}\epsilon(x_k - x_b)\{\psi(x_k), \overline{\psi}(x_b)\} + \tfrac{1}{2}[\psi(x_k), \overline{\psi}(x_b)]\,. \end{aligned} \tag{2.11}$$

Actually, the singularities of the two-point function cannot be presupposed, as is done in the canonical formalism but is to be determined from the equations of motion in conjunction with requirements imposed by the physical boundary conditions such as microscopic causality.

The equations satisfied by the multipoint functionals of interest may be derived by a functional derivative technique[6] from the generating functional

$$U=\exp[i(\bar{u}_k\psi_k+\bar{\psi}_l u_l)]\,, \tag{2.12}$$

where $\bar{u}$ and u denote independent classical spinor sources which anticommute with each other and with spinor field operators. As generator for the time-ordered field operators define furthermore

$$W=\langle 0|TU|0\rangle\,. \tag{2.13}$$

The τ and η functionals respectively, are defined by

$$\begin{aligned}\tau(a_1\cdots a_k|b_1\cdots b_l)&=\langle 0|T\psi_{a_1}\cdots\psi_{a_k}\bar{\psi}_{b_1}\cdots\bar{\psi}_{b_l}|0\rangle\\&=i^{l-k}\frac{\delta^{k+l}}{\delta\bar{u}_{a_1}\cdots\delta\bar{u}_{a_k}\delta u_{b_1}\cdots\delta u_{b_l}}W\bigg|_{\bar{u}=u=0}\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\eta(a_1\cdots a_k|b_1\cdots b_l)&=i^{l-k}\frac{\delta^{k+l}}{\delta\bar{u}_{a_1}\cdots\delta\bar{u}_{a_k}\delta u_{b_1}\cdots\delta u_{b_l}}H\bigg|_{\bar{u}=u=0}\,,\end{aligned} \tag{2.15}$$

where H is defined by

$$H=\ln W\,. \tag{2.16}$$

In particular, it follows by considering the response of H to the appropriate classical sources that

$$\eta(a|b)=\tau(a|b) \tag{2.17a}$$

and

$$\eta(kl|mn)=\tau(kl|mn)-\eta(l|m)\eta(k|n)+\eta(k|m)\eta(l|n)\,. \tag{2.17b}$$

The above equations in conjunction with the equation of motion imply that

$$D_{ik}\frac{\delta}{\delta\bar{u}_k}W=i\langle 0|TD_{ik}\psi_k U|0\rangle-iC_{ib}u_bW\,, \tag{2.18}$$

where

$$C_{ib}=\delta(x_{0i}-x_{0b})(\gamma_0)_{ik}\langle 0|\{\psi_k(x_i),\bar{\psi}_b(x_b)\}|0\rangle\,, \tag{2.19}$$

which in turn yields the following equation for the generating functional:

$$\exp(-H)D_{ik}\frac{\delta}{\delta\bar{u}_k}\exp(H)+\exp(-H)V_{im,nl}\left\{\frac{\delta^3}{\delta\bar{u}_n\delta\bar{u}_l\delta u_m}+\theta(::)\left[\eta(n|m)\frac{\delta}{\delta\bar{u}_l}-\eta(l|m)\frac{\delta}{\delta\bar{u}_n}\right]\right\}\exp(H)=-iC_{ib}u_b\,. \tag{2.20}$$

The equation conjugate to Eq. (2.7) reads

$$\bar{\psi}_i\overleftarrow{\mathrm{D}}_{ik}+V_{im,lk}\bar{\psi}_i\bar{\psi}_m\psi_l=0\,, \tag{2.21}$$

where

$$\overleftarrow{\mathrm{D}}_{ik}=-i\frac{\overleftarrow{\partial}}{\partial x_i^\nu}\gamma^\nu\,\delta(x_i-x_k)-m\,, \tag{2.22}$$

which yields

$$\exp(-H)\left(\frac{\delta}{\delta u_j}\exp(H)\right)\overleftarrow{\mathrm{D}}_{jr}-\exp(-H)V_{st,ar}\left\{\frac{\delta^3}{\delta\bar{u}_q\delta u_t\delta u_s}+\theta(::)\left[\eta(q|s)\frac{\delta}{\delta u_t}-\eta(q|t)\frac{\delta}{\delta u_s}\right]\right\}\exp(H)=i\bar{u}_aC_{ar}\,. \tag{2.23}$$

Forming derivatives of Eqs. (2.20) and (2.23) with respect to the external sources yields systems of coupled equations for the multipoint functions. In perturbation theory, the η functions represent the sum of all connected Feynman diagrams of a given number of outgoing and incoming lines. The system of equations for the η functions brings the nonlinear features of the equations to the forefront. Judging from the novel features exhibited by nonlinear classical equations and recalling the role that feedback effects due to nonlinearity might play in the quantum problem in achieving self-consistency, special attention is given in the following to nonlinearity.

III. NONLINEAR EQUATIONS FOR THE TWO-POINT FUNCTION

In this section nonlinear equations satisfied by the two-point function are derived. First of all, variation of Eq. (2.20) with respect to the source u_j, and consideration of the limit where all sources go to zero, result in the equation

$$D_{ik}\eta(k|j)+V_{im,nl}\eta(nl|mj)+(1-\theta)V_{im,nl}[\eta(n|j)\eta(l|m)-\eta(l|j)\eta(n|m)]=iC_{ij}\,. \tag{3.1}$$

The equation conjugate to Eq. (3.1) may be derived from Eq. (2.23) by employing the same procedure as above:

$$\eta(a|j)\overleftarrow{\mathrm{D}}_{jr} - V_{st,qr}\eta(aq|ts) - (1-\theta)V_{st,qr}[\eta(q|t)\eta(a|s) - \eta(a|t)\eta(q|s)] = iC_{ar}\,. \tag{3.2}$$

The desired equation for the two-point function is obtained by operating onto Eq. (3.1) by $\overleftarrow{\mathrm{D}}_{jr}$ from the right and employing Eq. (3.2) in conjunction with the equation satisfied by $\eta(nl|mj)\overleftarrow{\mathrm{D}}_{jr}$. The equation for the latter quantity is obtained by operating with $\delta^3/\delta\bar{u}_n\,\delta\bar{u}_m\delta u_l$ onto Eq. (2.23), with the result

$$\eta(nl|mj)\overleftarrow{\mathrm{D}}_{jr} + V_{st,qr}\theta[\eta(q|t)\eta(nl|ms) - \eta(q|s)\eta(nl|tm)] = V_{st,qr}A_{nlq,mts}\,, \tag{3.3}$$

where A is given by

$$A_{nlq,mts} = \frac{\delta^3}{\delta\bar{u}_n\,\delta\bar{u}_l\,\delta u_m}\left[\exp(-H)\frac{\delta^3}{\delta\bar{u}_q\,\delta u_t\delta u_s}\exp(H)\right] \tag{3.4}$$

$$= -\eta(q|m)\eta(l|t)\eta(n|s) + \eta(q|m)\eta(n|t)\eta(l|s) - \eta(q|m)\eta(nl|ts) - \eta(lq|mt)\eta(n|s) + \eta(nq|mt)\eta(l|s)$$
$$+\eta(q|t)\eta(nl|ms) - \eta(nl|mt)\eta(q|s) - \eta(l|t)\eta(nq|ms) + \eta(n|t)\eta(lq|ms) - \eta(nlq|mts)\,. \tag{3.5}$$

Substituting Eqs. (3.2) to (3.5) into Eq. (3.1) and neglecting the higher-point functions, the equation for the two-point function reads

$$\begin{aligned} D_{ik}\eta(k|j)\overleftarrow{\mathrm{D}}_{jr} = &- V_{im,nl}V_{st,qr}[\eta(q|m)\eta(n|t)\eta(l|s) - \eta(q|m)\eta(l|t)\eta(n|s)]\\ &-(1-\theta)V_{im,nl}V_{st,qr}\eta(n|m)[\eta(l|t)\eta(q|s) - \eta(q|t)\eta(l|s) + iC_{lr}]\\ &-(1-\theta)V_{im,nl}V_{st,qr}\eta(l|m)[\eta(q|t)\eta(n|s) - \eta(n|t)\eta(q|s) - iC_{nr}] + iC_{ij}\overleftarrow{\mathrm{D}}_{jr}\,. \end{aligned} \tag{3.6}$$

Specializing to the scalar-interaction Eq. (2.8) and employing Eqs. (2.9) and (2.22), it follows that

$$\begin{aligned} -\left(i\gamma^\mu\frac{\partial}{\partial x_\mu} - m\right)\eta(x|y)\left(i\gamma^\nu\frac{\overleftarrow{\partial}}{\partial y_\nu} + m\right) = &\, l^4\{\eta(x|y)\eta(y|x)\eta(x|y) - \eta(x|y)\mathrm{Tr}[\eta(y|x)\eta(x|y)]\}\\ &+(1-\theta)l^4[\eta(x|x)\eta(x|y)\eta(y|y) - \eta(x|x)\eta(x|y)\mathrm{Tr}\eta(y|y) + i\eta(x|x)C]\\ &+(1-\theta)l^4\mathrm{Tr}\eta(x|x)[\eta(x|y)\mathrm{Tr}\eta(y|y) - \eta(x|y)\eta(y|y) - iC]\\ &-iC\left(i\gamma^\nu\frac{\overleftarrow{\partial}}{\partial y_\nu} + m\right). \end{aligned} \tag{3.7}$$

Utilizing Lorentz invariance, the two-point function may be expressed in the form

$$\eta(x|y) = i\gamma\cdot(x-y)f(s) + g(s)\,, \tag{3.8}$$

where s denotes the Minkowski length squared

$$s = (x-y)^\mu(x-y)_\mu = (x_0-y_0)^2 - (\vec{x}-\vec{y})^2\,. \tag{3.9}$$

Substituting Eq. (3.8) into Eq. (3.7), using the results that

$$\gamma^\mu\gamma_\mu = 4,\quad \gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \tag{3.10}$$

and transforming to the variable s, the following coupled equations result:

$$s\frac{d^2f}{ds^2} + 3\frac{df}{ds} + m\frac{dg}{ds} - \tfrac{1}{4}m^2 f = \lambda(sf^3 + fg^2) - 3\lambda(1-\theta)fg(0)^2 \tag{3.11}$$

and

$$s\frac{d^2g}{ds^2} + 2\frac{dg}{ds} - m\left(s\frac{df}{ds} + 2f\right) - \tfrac{1}{4}m^2 g = \lambda(g^3 + sf^2g) - 3\lambda(1-\theta)gg(0)^2\,, \tag{3.12}$$

where $\lambda = 3l^4/4$. In view of the locality of the theory, the equations for the two-point function contain terms involving the two-point function at the same space-time point. In general, the singular nature of the two-point function on the light cone would render such terms meaningless. As indicated by the above equations, these terms are accompanied by compensating contributions, as a consequence of interpreting the self-interaction of the theory as the Wick product of fields. The inhomogeneous term of the coupled set of equations has according to Eq. (2.19), a bearing on the question of the quantization of the interacting theory, the answer to which is not known *a priori*. In fact, the question of quantization is conditional on the solution of the homogeneous equations subjected to appropriate boundary conditions.

Motivated by considerations based on scaling arguments, the following functions are defined:

$$f(s)=\lambda^{-1/2}s^{-1}A(s) \tag{3.13}$$

and

$$g(s)=\lambda^{-1/2}s^{-1/2}B(s), \tag{3.14}$$

so that Eqs. (3.11) and (3.12) read

$$\left(s^2\frac{d^2A}{ds^2}+s\frac{dA}{ds}-A\right)+ms^{1/2}\left(s\frac{dB}{ds}-\tfrac{1}{2}B\right)-\tfrac{1}{4}m^2sA=\lambda(A^3+AB^2) \tag{3.15}$$

and

$$\left(s^2\frac{d^2B}{ds^2}+s\frac{dB}{ds}-\tfrac{1}{4}B\right)-ms^{1/2}\left(s\frac{dA}{ds}-A\right)-\tfrac{1}{4}m^2sB=\lambda(B^3+A^2B). \tag{3.16}$$

Furthermore, if we introduce the variable

$$\rho=\ln\frac{s}{s_0} \tag{3.17}$$

(where s_0 denotes, for the moment, an arbitrary scale of length) it follows that

$$\frac{d^2A}{d\rho^2}+ms_0{}^{1/2}e^{\rho/2}\left(\frac{dB}{d\rho}-\tfrac{1}{2}B\right)-\tfrac{1}{4}m^2s_0e^{\rho}A=A+A^3+AB^2 \tag{3.18}$$

and

$$\frac{d^2B}{d\rho^2}-ms_0{}^{1/2}e^{\rho/2}\left(\frac{dA}{d\rho}-A\right)-\tfrac{1}{4}m^2s_0e^{\rho}B=\tfrac{1}{4}B+B^3+BA^2. \tag{3.19}$$

In the next section the solutions of the nonlinear equations are investigated, subject to the following physical requirements imposed by the boundary conditions of field theory.[7] One of the most important requirements of relativistic local quantum field theory is the axiom of microcausality. The physical significance of this requirement is the independence of field components, the arguments of which are separated by spacelike intervals,

$$\{\psi(x),\bar{\psi}(y)\}=0,\quad \text{for } s<0. \tag{3.20}$$

This requirement may also be expressed by the conditions that $\text{Im}f(s)$ and $\text{Im}g(s)$ vanish for spacelike separations. Furthermore, it has been shown in axiomatic field theory that the requirement of the positivity of the energy spectrum of the total energy operator implies the analyticity of the two-point function regarded as a function of the complex variable s, except for the origin and the timelike real axis in the s plane. Since the proof of this result is based on the assumption of the existence of a complete set of positive-energy states, the analytic properties may not be realized in a theory with indefinite metric. In the nonlinear problem under discussion, it will turn out that an indefinite metric is implied by the less singular nature of the two-point function on the light cone. In addition, it is required that the theory respect the discrete symmetries of partiy, charge conjugation and time-reversal invariance.

The antiunitary nature of the time-reversal operation, leads to the requirement that $f(s)$ and $g(s)$ be real for s spacelike and $f(s,x_0)=f(s,-x_0)$ and $g(s,x_0)=g(s,-x_0)$.

IV. SOLUTION OF THE AUTONOMOUS NONLINEAR EQUATIONS

This section is concerned with the solution of the coupled set of nonlinear equations,

$$\frac{d^2A}{d\rho^2}=A+A(A^2+B^2) \tag{4.1}$$

and

$$\frac{d^2B}{d\rho^2}=\tfrac{1}{4}B+B(B^2+A^2). \tag{4.2}$$

In the massive theory the equations (3.18) and (3.19) are nonautonomous equations—explicit appearance of the independent variable ρ. As was stated in Sec. II, the mass term was introduced merely as a mnemonic device to label the spontaneous mass generation through self-interaction. The spontaneous mass generation is analyzed in the concluding part of this section. In the meantime the mass terms may be disregarded.

In Sec. II it was shown that the massless nonlinear equation remains invariant under scale transformations provided that the nonlinear field possesses a subcanonical scale dimension. In view of the possibility of mass generation due to self-interaction, it may be anticipated that the scale symmetry may break down spontaneously. It is therefore of interest to determine at what level the scale symmetry manifests itself dynamically. According to Eqs. (4.1) and (4.2), this is realized provided A and B are constants, otherwise the scale s_0 (which is related to the spontaneously generated mass) may enter into the solutions by means of the variable ρ. The constant solutions are given by $A=\pm i$, $B=0$ and $A=0$, $B=\pm i/2$ which yield

$$f=\pm\frac{i}{\lambda^{1/2}}\frac{1}{s} \tag{4.3}$$

and

$$g=\pm\frac{i}{2\lambda^{1/2}}\frac{1}{s^{1/2}}$$

$$=\pm\frac{1}{2\lambda^{1/2}}\left[i\frac{\theta(t^2-r^2)}{(t^2-r^2)^{1/2}}+\frac{\theta(r^2-t^2)}{(r^2-t^2)^{1/2}}\right]. \qquad (4.4)$$

The interpretation of the singularities in the s plane is to be entered into in the next section in connection with an analysis of the compatibilty of the various solutions with the physical boundary conditions.

The structure of Eqs. (4.1) and (4.2) implies a further simplifying feature. Introduce the auxiliary quantities $a=A+iB$ and $b=A-iB$ which satisfy

$$\frac{d^2a}{d\rho^2}=\tfrac{5}{8}a+\tfrac{3}{8}b+a^2b \qquad (4.5)$$

and

$$\frac{d^2b}{d\rho^2}=\tfrac{5}{8}b+\tfrac{3}{8}a+b^2a. \qquad (4.6)$$

Thus a and b satisfy the same differential equation, which implies that $a=b$ provided that the same boundary conditions are imposed. This implies that either $A=0$ or $B=0$ so that

$$\frac{d^2A}{d\rho^2}=A+A^3, \qquad (4.7)$$

or the analogous equation satisfied by B. Thus the problem of determining the two-point function for the nonlinear system has been reduced to the solution of nonlinear differential equations which, expressed in terms of the variable ρ are analogous to equations encountered in classical systems for a single space dimension. The equations are, however, hyperbolic,[7] with concomitant singular solutions.

Equation (4.7) can be solved by quadrature. Multiplying the equation by $(dA/d\rho)$ and integrating the resulting equation, it follows that

$$\pm\frac{\rho}{\sqrt{2}}=\int_{A(0)}^{A(\rho)}dA(A^4+2A^2+c)^{-1/2} \qquad (4.8)$$

$$=\int_{A(0)}^{A(\rho)}dA[(A^2+a_1)(A^2+a_2)]^{-1/2}, \qquad (4.9)$$

where c denotes a constant of integration, $a_1=1+(1-c)^{1/2}$ and $a_2=1-(1-c)^{1/2}$. The last equation may be inverted by means of the Jacobi elliptic functions.[8] A striking feature of an Abelian integral is the fact that it depends not so much on the functional form of the integrand as on the nature of the singularities thereof. It is therefore appropriate to consider the inversion of (4.9) for the following intervals:

Case 1. This case considers $c<0$, thus both a_1 and a_2 are real, but a_2 negative. With the choice $A(0)=\infty$, it follows that[8]

$$A=\sqrt{2}\,(1-c)^{1/4}\mathrm{ds}(\pm(1-c)^{1/4}\rho\,|k^2), \qquad (4.10)$$

where the modulus k is given by

$$k^2=\frac{a_1}{a_1-a_2}=\frac{1+(1-c)^{1/2}}{2(1-c)^{1/2}}. \qquad (4.11)$$

The parameter k $(0<k<1)$ serves as a measure of the nature of the nonlinearity of the wave. In the limit where c tends to zero, the modulus k tends to one so that the expression (4.10) simplifies to

$$A=\pm\sqrt{2}\,\mathrm{csch}(\rho). \qquad (4.12)$$

In this limiting case Eq. (4.8) may be integrated directly. Expressed in terms of the variable s, the Minkowski space solution reads

$$A=\pm\frac{(2\sqrt{2}\,s_0)s}{(s^2-s_0^{\,2})} \qquad (4.13)$$

and

$$f(s)=\pm\left(\frac{2\sqrt{2}\,s_0}{\lambda^{1/2}}\right)\frac{1}{(s^2-s_0^{\,2})}. \qquad (4.14)$$

It may be mentioned in passing that the choice $A(0)=(-a_2)^{1/2}$ yields the solution

$$A=[(1-c)^{1/2}-1]^{1/2}\mathrm{nc}((1-c)^{1/4}\rho\,|k^2), \qquad (4.15)$$

which for c small is proportional to $|c|^{1/4}\cosh\rho$, thus the solution of the linearized Eq. (4.7) is

$$f(s)=\mathrm{const}\left(\frac{1}{s^2}+\mathrm{const}\right), \qquad (4.16)$$

For large values of s the solutions (4.14) and (4.16) coincide, but the behavior on the light cone is significantly modified by the presence of self-interaction.

Case 2. In this case the interval $0<c<1$ is considered. For $A(0)=0$, Eq. (4.9) yields the result

$$A=[1-(1-c)^{1/2}]^{1/2}\mathrm{sc}\left([1+(1-c)^{1/2}]\frac{1}{\sqrt{2}}\rho\,\middle|k^2\right), \qquad (4.17)$$

where the modulus is given by

$$k^2=\frac{a_1-a_2}{a_1}=\frac{2(1-c)^{1/2}}{[1+(1-c)^{1/2}]}. \qquad (4.18)$$

In the limit when c tends to one, the modulus tends to zero and Eq. (4.17) yields

$$A=\pm\tan\left(\frac{1}{\sqrt{2}}\rho\right), \qquad (4.19)$$

so that

$$f=\pm\frac{1}{\lambda^{1/2}}\frac{1}{s}\tan\left(\frac{1}{\sqrt{2}}\ln\frac{s}{s_0}\right). \qquad (4.20)$$

This solution is of the form of the scale solution (4.3) modulated by rapid oscillations in the proximity of the light cone.

Case 3. In the case where $c>1$, both a_1 and a_2 are imaginary. Thus the condition of microscopic causality requiring A real for real ρ cannot be fulfilled.

The solutions to Eq. (4.2) may be obtained from the above solution with the replacement of a_1 and a_2 by

$$b_1=\tfrac{1}{4}[1+(1-D)^{1/2}]$$

and

$$b_2=\tfrac{1}{4}[1-(1-D)^{1/2}],$$

respectively, where D denotes a constant of integration. In the first case, where $D<0$, the solution reads

$$B=\frac{1}{\sqrt{2}}(1-D)^{1/4}\mathrm{ds}(\pm\tfrac{1}{2}(1-D)^{1/4}\rho\,|k^2)\,. \tag{4.21}$$

In the limit $D=0$, the solution assumes the form

$$B=\pm\frac{1}{\sqrt{2}}\mathrm{csch}(\tfrac{1}{2}\rho)\,, \tag{4.22}$$

so that

$$g=\pm\frac{1}{\lambda^{1/2}}\frac{(2s_0)^{1/2}}{(s-s_0)}\,. \tag{4.23}$$

In the second case where $0<D<1$, the solution reads

$$B=b_2^{\ 1/2}\mathrm{sc}\left(\frac{1}{\sqrt{2}}b_1^{\ 1/2}\middle|k^2\right), \tag{4.24}$$

so that, in the limiting case $D=1$, the solution may be expressed in the form

$$g=\pm\frac{1}{2}\frac{1}{\lambda^{1/2}}\frac{1}{s^{1/2}}\tan\left(\frac{1}{2}\frac{1}{\sqrt{2}}\ln\frac{s}{s_0}\right). \tag{4.25}$$

The elliptic functions[8] are doubly periodic meromorphic functions of real and imaginary quarter periods given by the complete elliptic integrals K and iK', defined by

$$K(k^2)=K=\int_0^{\pi/2}d\theta(1-k^2\sin^2\theta)^{-1/2} \tag{4.26}$$

and

$$K'(k^2)=K'=\int_0^{\pi/2}d\theta(1-k_1^{\ 2}\sin^2\theta)^{-1/2}\,, \tag{4.27}$$

where the complementary modulus, k_1, is defined by $k_1=(1-k^2)^{1/2}$. The modulus $(0<k<1)$ serves as a measure of the nonlinearity of the propagator. In the limit where k tends to zero, the complete elliptic integral assumes the form

$$K(k^2)=\frac{\pi}{2}\left[1+(\tfrac{1}{2})^2+\left(\frac{1\times 3}{2\times 4}\right)^2k^4+\dots\right], \tag{4.28}$$

whereas for k close to one it reads

$$K(k^2)=\tfrac{1}{2}\ln\left(\frac{16}{1-k^2}\right). \tag{4.29}$$

The modulus k characterizes the nature of the nonlinearity of the propagator as may be exemplified by the expansions valid for small k,

$$\mathrm{ds}(u|k)=\frac{\pi}{2K}\csc\frac{\pi u}{2K}\,,$$

$$-\frac{2\pi}{K}\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1+q^{2n+1}}\sin(2n+1)\left(\frac{\pi u}{2K}\right) \tag{4.30}$$

and

$$\mathrm{sc}\,(u|k)=\frac{\pi}{2k_1K}\tan\left(\frac{\pi u}{2K}\right)$$

$$+\left(\frac{2\pi}{k_1K}\right)\sum_{n=1}^{\infty}\frac{q^{2n}}{1+q^{2n}}(-1)^n\sin 2n\left(\frac{\pi u}{2K}\right), \tag{4.31}$$

where the nome, q, is given by

$$q=\exp\left(-\frac{\pi K'}{K}\right)=\frac{k^2}{16}+8\left(\frac{k^2}{16}\right)^2+\dots\,. \tag{4.32}$$

Thus, for small k the nonlinear propagator is characterized by a family of poles which satisfy an equal-spacing rule in terms of the variable ρ. This is the analog of the wave-train solution of nonlinear wave equations. In the other extreme the singularities are separated by a distance in ρ space proportional to $\ln(1-k^2)$. When the modulus equals its upper limit the propagator may be termed a "solitary-wave" propagator. Finally, in the special limiting cases where the amplitude of the nonlinear solutions becomes small, the nonlinear propagators approach the propagators of the linear theory.

This section is concluded with a discussion of the solution of the coupled system of nonlinear equations (4.1) and (4.2) by a method of successive approximations. Expanding the amplitude A in a power series in terms of a parameter ϵ as follows

$$A=\epsilon a_1+\epsilon^3a_3+\epsilon^5a_5+\cdots\,, \tag{4.33}$$

effecting a similar expansion of B, and collecting terms with the same powers of ϵ, the following set of telescopic equations results:

$$\begin{aligned}
&(D-1)a_1=0\,,\\
&(D-1)a_3=a_1(a_1^2+b_1^2)\,,\\
&(D-1)a_5=a_3(a_1^2+b_1^2)+a_1(2a_1a_3+2b_1b_3)\,,\\
&(D-1)a_7=a_5(a_1^2+b_1^2)+a_3(2a_1a_3+2b_1b_3)\\
&\qquad\qquad+a_1(2a_1a_5+a_3^2+2b_1b_5+b_3^2)
\end{aligned}\tag{4.34}$$

and

$$\begin{aligned}
&(D-\tfrac{1}{4})b_1=0\,,\\
&(D-\tfrac{1}{4})b_3=b_1(b_1^2+a_1^2)\,,\\
&(D-\tfrac{1}{4})b_5=b_3(b_1^2+a_1^2)+b_1(2b_1b_3+2a_1a_3)\,,\\
&(D-\tfrac{1}{4})b_7=b_5(b_1^2+a_1^2)+b_3(2b_1b_3+2a_1a_3)\\
&\qquad\qquad+b_1(2b_1b_5+b_3^2+2a_1a_5+a_3^2)\,,
\end{aligned}\tag{4.35}$$

where

$$D=d^2/d\rho^2\,.$$

This reduces the problem to the solution of an infinite system of coupled linear inhomogeneous equations. Proceeding from the solutions $a_1=\exp(\rho)$ and $b_1=\exp(\frac{1}{2}\rho)$ of the homogeneous equations and disregarding the constants of integration encountered, the following expansions for the nonlinear amplitudes are obtained:

$$\begin{aligned}
A=&\,\epsilon e^{\rho}+\epsilon^3(\tfrac{1}{8}e^{3\rho}+\tfrac{1}{3}e^{2\rho})\\
&+\epsilon^5[(\tfrac{1}{8})^2e^{5\rho}+\tfrac{7}{72}e^{4\rho}+\tfrac{1}{6}e^{3\rho}]\\
&+\epsilon^7[(\tfrac{1}{8})^3\epsilon^{7\rho}+\tfrac{1}{12}e^{4\rho}+\tfrac{11}{576}e^{6\rho}+\tfrac{29}{432}e^{5\rho}]+\cdots
\end{aligned}\tag{4.36}$$

and

$$\begin{aligned}
B=&\,\epsilon e^{\rho/2}+\epsilon^3(\tfrac{1}{2}e^{3\rho/2}+\tfrac{1}{6}e^{5\rho/2})\\
&+\epsilon^5\left(\frac{1}{2^2}e^{5\rho/2}+\tfrac{5}{36}e^{7\rho/2}+\tfrac{1}{48}e^{9\rho/2}\right)+\cdots\,.
\end{aligned}\tag{4.37}$$

It is to be noticed that the first terms of a given power of ϵ in the above series represent the terms of a geometric series. The geometric subseries of Eqs. (4.36) and (4.37) may be summed to yield the solution (4.19) (for $\epsilon=2\sqrt{2}$) and the solution (4.22) (for $\epsilon=\sqrt{2}$), respectively. In view of the inapplicability of the principle of superposition for nonlinear systems, the partial summation of the series should not be viewed as a procedure to gain insight into the nature of more general solutions. Rather, it provides a method to gauge the mutual influence of A and B on each other around the subclass of solutions of vanishing amplitudes A or B, solved in the first part of this section. In particular, it is implied by the series (4.36) that effects due to B could significantly influence the solution (4.19) whereas the effect of A on B, as represented by Eq. (4.22), is, according to the result (4.37), not of consequence. A more complete knowledge of the general solution is still outstanding.

In summary, it is found that the solutions for the two-point function for the interacting system group into different classes with respect to their behavior under dilatation transformations. In particular, in the case of the solution (4.3) the underlying spinor field possesses a subcanonical scale dimension. Although this solution is less singular on the light cone than the free-field canonical one and is suggestive of a renormalizable theory, it represents only a very special solution which is deficient in some important aspects. In the process of transforming the autonomous equations for the two-point function to a tractable form, the introduction of a scale of length was necessary. Actually, the breakdown of scale invariance is expected to be an essential ingredient of self-interacting theories through spontaneous mass generation. The derived solutions for the two-point function, in terms of a scale of length serve as a stepping stone in this direction. The connection may be indicated in a preliminary fashion by noticing that Eq. (4.23) gives a finite contribution on the light cone so that the Wick product of the self-interacting term may be disregarded. Equating the mass term of Eq. (3.12) to the "Hartree-Fock term" given by the last term of Eq. (3.12), it follows that $m^2s_0=24$. The isolation of the mechanisms of spontaneous mass generation, which from the present point of view is locked up in radiative corrections to the underlying fields, is one of the central issues in understanding the physical origin of the mass spectrum.

V. INVARIANT FUNCTIONS IN THE PRESENCE OF INTERACTION AND BOUNDARY CONDITIONS

The solutions for the two-point function of the interacting system enumerated in the previous section define Lorentz-invariant functions which, in this section, must be subjected to the appropriate boundary conditions.

As a preliminary to the discussion, it is of interest to consider the corresponding functions of the noninteracting system

$$\begin{aligned}
S(x,y)&=\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle\\
&=i\gamma\cdot(x-y)f_0(s)+g_0(s)\,,
\end{aligned}\tag{5.1}$$

which, according to Eq. (2.1) satisfy

$$2s\frac{df_0}{ds}+4f_0-2i\gamma\cdot(x-y)\frac{dg_0}{ds}=-i\gamma_0\delta(x_0-y_0)\langle 0|\{\psi(x),\bar{\psi}(y)\}|0\rangle\,.\tag{5.2}$$

The latter equation in solved by

$$S_c(x,y)=\left(\frac{-i}{2\pi^2}\right)\frac{i\gamma\cdot(x-y)}{(s-i\epsilon)^2}\,, \tag{5.3}$$

where the normalization is adopted to yield the response due to a source of unit strength, and the singularity for the time-ordered product is to be interpreted according to the Feynman prescription. The invariant functions of physical interest prescribed by the boundary conditions may be obtained by appropriately interpreting the singularities of Eq. (5.3) and the path of integration. Adding a small imaginary part to the variables, the integration may be taken along the real s axis.

In particular, the commutator function which satisfies the homogeneous equation and microscopic causality is defined by

$$S(x,y)=S^+(x,y)+S^-(x,y)\,, \tag{5.4}$$

where

$$S^+=\left(\frac{-i}{2\pi^2}\right)\frac{i\gamma\cdot(x-y)}{[s+i\epsilon(x_0-y_0)\epsilon]^2} \tag{5.5}$$

and

$$S^-=\left(\frac{-i}{2\pi^2}\right)\frac{[-i\gamma\cdot(x-y)]}{[s-i\epsilon(x_0-y_0)\epsilon]^2}\,. \tag{5.6}$$

Employing the symbolic identities[9]

$$\frac{1}{(s\pm i\epsilon)^n}=\frac{P}{s^n}\mp\frac{i\pi(-1)^{n-1}}{(n-1)!}\,\delta^{(n-1)}(s)\,, \tag{5.7}$$

it follows that

$$\begin{aligned}S(x,y)&=\frac{1}{2\pi}\,i\gamma\cdot(x-y)\epsilon(x_0-y_0)z\,\frac{d}{ds}\,\delta(s)\\&=i\gamma\cdot\partial\Delta(x,y)\,,\end{aligned} \tag{5.8}$$

where

$$\Delta(x,y)=\frac{1}{2\pi}\,\epsilon(x_0-y_0)\delta(s) \tag{5.9}$$

denotes the commutator function of the massless free scalar field. The equal-time canonical commutator reads

$$\begin{aligned}\delta(x_0-y_0)\{\psi(x),\bar\psi(y)\}&=\delta(x_0-y_0)\gamma\cdot\partial\Delta(x,y)\\&=\gamma_0\delta(x-y)\,.\end{aligned} \tag{5.10}$$

A further invariant function of great physical interest is the causal function defined by

$$S_c(x,y)=i[\theta(x_0-y_0)S^- -\theta(y_0-x_0)S^+] \tag{5.11}$$

in which the future and the past are treated in a symmetrical manner. Utilizing Eqs. (5.5) and (5.6), Eq. (5.11) yields Eq. (5.3). The propagators usually studied are those of the massive free field theory. However, the light cone singularities of the free massive theory coincide with that of the massless one. The treatment of higher-order corrections in terms of free-field propagators involves the products of distributions that result in the appearance of pathological features which are removed by a renormalization procedure. Although this procedure has proved to be remarkably successful in a number of situations, a shadow is cast on the method due to the appearance of arbitrary parameters which preclude a dynamical explanation of attributes such as masses and coupling constants of particles—issues central to the understanding of the particle spectrum and the hierarchy of interactions. The results of the previous section demonstrate that the light-cone singularities are arrested by self-interaction and that a mechanism of self-regulation is thus operative.

The free-field exercise shows that the anticommutation relation of the free field is determined by the solution of the classical Eq. (5.2), a result which coincides with the canonical commutation rule, since (5.8) is the only solution of (5.2) which fulfills the microscopic causality condition. This strongly suggests that the commutation relations of field theory should not be looked upon as an independent postulate, but rather, that it is determined by the interacting theory itself.

The solutions of the previous section are now in turn subjected to an investigation of their compatibility with microscopic causality and the requirements of positivity. It is to be noticed first of all that the scaling solution (4.3) is ruled out by the requirement of time-reversal invariance in view of the antiunitary nature of the time-reversal operation. The general solutions to the nonlinear equations are the Jacobi elliptic functions, thus doubly-periodic meromorphic functions of real and imaginary periods and poles determined by K and iK'. The locus of the poles is determined when the argument of the elliptic functions equals $2nK+2miK'$ with n and m integers. The discussion of the analytic properties is simplified if the variable ρ is regarded as a function of $u=-s$ which is cut along the negative u axis. The positivity condition requires the absence of singularities in the u plane for $-\pi<\mathrm{Im}\rho<u$ except for the origin and the negative u axis. The singularities of Eq. (4.10), in particular, are located by

$$\mathrm{Im}\rho=(1-c)^{-1/4}\,2mK'\,, \tag{5.12}$$

which, in view of the inequality

$$K' \leq \frac{\pi}{2} - \tfrac{1}{2}\ln k^2 = \frac{\pi}{2} - \tfrac{1}{2}\ln\left[\frac{1+(1-c)^{1/2}}{2(1-c)^{1/2}}\right] \leq \frac{\pi}{2} + \tfrac{1}{2}\ln 2 < \pi \qquad (5.13)$$

may enter the physical region for arbitrary values of c. In particular, when $c=0$ the singularity approaches the real axis in agreement with the solution (4.4) which possesses both a physical and a tachyon component. The corresponding analysis for the solution (4.12) shows that for D sufficiently small the singularities remain on the unphysical sheet, in agreement with the particular solution (4.23). In the case of the solution (4.17) the singularities are located at

$$\rho = [1+(1-c)^{1/2}]^{-1/2}[(2n+1)K + i2nK']. \qquad (5.14)$$

In the limit where c appraoches 1, K tends to $\pi/2$, whereas K' tends logarithmically to infinity. Thus the complex singularities are moved off the physical sheet; the physical singularities occur for $\mathrm{Re}\rho = (2n+1)(\pi/2)$. This result corresponds to the limiting solution Eq. (4.20). In summary, the requirement of positivity selects particular solutions of the previous section for which the integration parameters of Eq. (4.7) is either zero or constrained to complete the square in Eq. (4.8), for example.

We now turn to the requirement imposed by microscopic causality. The commutator function, denoted by $C(x,y)$ corresponding to the solution (4.20), may be constructed as follows: Define

$$C(x,y) = C^{+} + C^{-}, \qquad (5.15)$$

where

$$C^{\pm} = \pm\frac{1}{\lambda^{1/2}}\frac{i\gamma\cdot(x-y)}{[s \pm i\epsilon(x_0-y_0)\epsilon]} \times \tan\left(\frac{1}{\sqrt{2}}\ln[u \pm i\epsilon(x_0-y_0)\epsilon]\right) \qquad (5.16)$$

and $u = -s/u_0$, so that, after some simplification, it follows that

$$C = \frac{\gamma\cdot(x-y)}{d\lambda^{1/2}}\left((2\pi)^2\Delta(x,y)\sin\sqrt{2}\ln\left|\frac{s}{s_0}\right| - 2\frac{P}{s}\sinh(\sqrt{2}\,\pi\epsilon(x_0-y_0)\theta(s)]\right), \qquad (5.17)$$

where

$$d = \cos\sqrt{2}\,\ln\left|\frac{s}{s_0}\right| + \cosh 2\pi\theta(s) \qquad (5.18)$$

and $\Delta(x)$ denotes the commutator function of the massless scalar theory. Thus, although the interacting theory exhibits novel features near the light cone, microscopic causality may nevertheless be fulfilled. In the case of the solution (4.14), the light-cone modification derives from the presence of a tachyon component with the accompanying violation of microscopic causality. Furthermore, even though Eq. (3.1) becomes homogeneous, the Eq. (3.7) satisfied by the two-point function is itself a second-order equation for which the inhomogeneous term survives.

In summary, the presence of interaction in conjunction with physical boundary conditions leads to a picture of fields in interaction in sharp contrast to the conventional approach, and the solutions which are acceptable on general grounds result in a physical theory which exhibits self-regulation.

VI. CONCLUSIONS

In this work, features intrinsic to nonlinear quantum field theory have been investigated with particular reference to the nature of the two-point function of the self-interacting system. Although the results may be of relevance in the wider context of nonlinear field theory, such as in the Yang-Mills system,[10] the most immediate implications are for nonlinear spinor theories of particles.

Nonlinear spinor theories have been a useful testing ground for the study of composite particle models, spontaneous symmetry breakdown, and the investigation of collective phenomena in quantum field theories. Fermions appear as excitations of the primary field, whereas boson states are described as composites of the underlying fermion fields. The dynamical treatment of nonlinear spinor theory followed two main trends. The approach based on conventional techniques of field theory has received a great deal of attention[11] and is widely known. By investigating the theory from a conventional angle, the results are more easily recognized in the familiar setting but, unfortunately, employment of objects such as free-field propagators introduces properties foreign to the nonlinear theory. In the original approach to the theory introduced by Heisenberg[1,5] it has been attempted to exploit directly the features inherent to the nonlinear system. In particular, the modified short-distance behavior of the two-point function, with regard to the absence of δ function (and derivatives thereof) on the light cone, has been incorporated into the theory by assuming that spontaneous mass generation is operative, and by imposing constraints on the spectral functions of the Lehmann-Källén spectral representation for the two-point function. This conjectured form of the two-point function had done proxy in dynamical calculations in which attempts were made to determine the introduced masses self-consistently, providing evidence for hadronic excitations[12] and electromagnetically generated fermions[13] (leptons). The physical implications of the modified short-distance behavior of the two-point function are reflected in the suppression of the role of the large-momentum components with enhanced attendant effects due to the particular infrared features.

A deeper understanding of the attendent effects of the two-point function which plays a central role in all dynamical processes may serve to place the theory on a sound footing. Although the features intrinsic to nonlinear field theory have important implications for circumventing difficulties encountered in field theory, they also come equipped with unconventional properties which raise questions about the interpretative aspects of the theory.

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