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Phenomenological equation of state for quark matter

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A model for quark matter, similar to the SLAC bag model, is used as a phenomenological basis in view of astrophysical illustrations. This model, although crude, is expected to be sufficient for this type of application where orders of magnitude only are required at this stage. This model consists of colored quarks interacting via scalar gluons (responsible for the confinement) and colored vector gluons. Although color-singlet states are assumed to be the states physically realized in nature, this assumption is not essential and can easily be relaxed. A covariant statistical-mechanical formalism based on the use of techniques similar to those of plasma physics is given. A Bogolubov-Born-Green-Kirkwood-Yvon hierarchy is obtained for the relevant statistical quantities (i.e., covariant Wigner functions, moments of the gluon fields, etc.). It is truncated at the third order (neglect of three-body correlations). The thermodynamical quantities are then evaluated and the resulting equation of state is applied to neutron stars.

I. INTRODUCTION

The success of the quark hypothesis in elementary-particle physics has led to the natural idea that the main features of elementary-particle interactions could be used—via a specific quark model—in a (more or less) realistic theory of relativistic dense matter. Such a theory is particularly needed in relativistic astrophysics where extremely high densities are supposed to take place in dense stars (neutron stars) or in the early universe. Several attempts have been performed in this direction either in order to get a "third family of dense stars"[1] constituted of a conventional neutron star containing a quark phase[2-10] (or a pure quark star[11]) *or* in connection with problems of the early universe in its hadronic era (or before).[12-15] Other applications in astrophysics are related to quasars[16-19] or solar neutrinos.[20]

The most recent papers on the subject generally rest heavily on the property of asymptotic freedom[21]: high-energy experiments (deep-inelastic scattering) suggest that for large momentum transfers quark partons are quasifree inside hadrons. In the framework of quantum chromodynamics (QCD) it can be shown that this property holds true for ultradense matter. Once this property is admitted, it is assumed that there exists a quark phase inside hadronic matter (and hence a first-order phase transition between these two phases) so that the quark chemical potential and the pressure are obtained (see Fig. 1) through an assumed Maxwell's construction that links a hadron (e.g. neutron) equation of state to the free Fermi gas equation of state satisfied by the free quarks. It goes without saying that such a phase transition from quarks to hadrons depends strongly (both the temperature-density regime at which it occurs *and* its very existence) on the model of interactions between quarks put at the onset of the derivation of the thermodynamical properties of the system. For instance, one can choose the MIT bag model[5] or the SLAC bag model[22] or a gauge model,[7,23] etc. On the other hand one can also adopt a different point of view in which one tries to reproduce the main features of quark interactions known from elementary-particle physics (i.e., "low"-energy data) in a relativistic and semiphenomenological way while requiring that they are still true at higher densities. Doing so one can reasonably expect to obtain plausible results at densities slightly higher than the nuclear density or than the highest densities obtained in collisions of elementary particles (10^{16}–10^{17} g/cm^3).

In an interesting paper such an approach has indeed been performed by Bowers, Gleeson, and Pedigo[8] on the basis of a model developed by these authors[24,25] and designed to describe baryonic matter. In their model the interaction between quarks is mediated by a scalar gluon field, supposed to take account of the quark confinement, and a gluon vector field: the two free parameters of their model are chosen so as to reproduce the typical binding energy of a nucleon inside a nucleus *and*

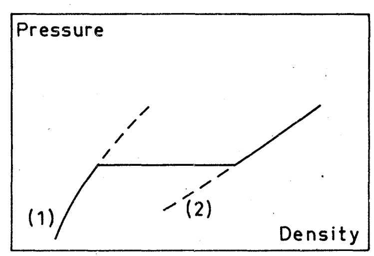

FIG. 1. Matching of the nuclear equation of state (curve 1) and of the quark equation of state (curve 2) via a Maxwell's construction (assuming the existence of a first-order phase transition between the nuclear and the quark phases).

the nuclear saturation density. It is also assumed that the same model (suitably rescaled for quarks) applies both to nuclear and quark matter. Next these authors discuss some astrophysical consequences and, in particular, the stability of dense stars involving a quark phase.

However, while reproducing nuclear matter features, their model does not reproduce the main "low"-energy elementary-particle data. Let us specify this point more precisely. First, their quarks are not colored[26] and hence the correct symmetry properties of the lowest-lying baryons cannot be obtained.[27] Next, in order to get the correct hyperfine splitting of mesons and baryons spectra (both in signs and relative sizes), the vector field must necessarily transport color[28] and belong to a SU_3 (color) octet. Furthermore, their interpretation of the first-order phase transition of their model as a quark-baryon transition is not correct. The correct interpretation can be obtained by looking at the curve (Fig. 2) showing the effective mass of the quarks as a function of the chemical potential. The Maxwell's construction induces in this diagram an analogous construction which shows that the two phases consist of (i) particles with a heavy mass, referring thereby to a *quark phase* and (ii) particles with a light mass forming a *confined phase*. That the latter phase is *not* a baryon phase can be seen from the nature of their approximation: the Hartree approximation provides a *collective* bound state while it is necessary to deal with three-body correlations to obtain a baryon phase.

In this paper we adopt a semi-phenomenological model that is an extension of the model of Bowers, Gleeson, and Pedigo.[8] The spirit in which the confinement is made in our model is analogous to the one at the origin of the SLAC bag model[22] (see also Lee[29]). The basic assumptions are given below and are taken from elementary-particle-physics data:

(i) *Quark confinement* is obtained by using a real

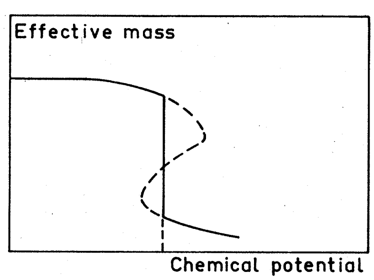

FIG. 2. Qualitative behavior of the effective quark mass as a function of the chemical potential: the first-order phase transition considered by Bowers *et al.* (Ref. 8) amounts, in this diagram, to replacing the dashed line by the vertical straight line (constancy of the chemical potential in both phases).

scalar quantum field mediating the interaction between quarks as in several models.[8,22] It should be noticed that such scalar gluons give rise to correct orders of magnitude for the nucleons magnetic moments.[30]

(ii) Quarks are *very massive* whenever free and have *low effective masses* as a result of the scalar interaction, in conformity with, e.g., deep-inelastic scattering experiments.

(iii) Correct *symmetry properties* are taken into account by *colored* quarks, the color group being SU_3 (color) $\equiv SU_{3'}$.

(iv) A *colored vector*-gluon field—belonging to the regular $SU_{3'}$ representation—takes account of the *hyperfine structure* of the spectra of baryons and mesons.

(v) The physically admissible states are *color singlets* as suggested by the presently observed elementary particles. This assumption can be relaxed without any particular problem.

The difference between this model and the SLAC bag model occurs mainly in the confinement mechanism: in the SLAC bag model there exists a complex scalar field, a component of which gives rise—as in our case—to the quark confinement while the other is used to make the vector-gluon field (a gauge field) a massive field via the Higgs mechanism; in our model the vector field is massive from the beginning.

In this paper we have adopted a strictly phenomenological point of view so that our results are probably valid at densities higher than the nuclear density, although a precise estimation of their domain of validity is difficult to assess. In principle, this model describes the baryon phase as well as the quark phase. However, owing to the nature of the statistical assumptions used in the calculations, it is probably well suited to the quark phase or—possibly—to a meson phase. Among the possible drawbacks of this model is the fact that it does not contain—at least explicitly—the asymptotic-freedom property. However it will be shown that a similar property is contained in the model. One can also think of our massive vector-gluon field as being an *approximation* for a non-Abelian gauge field and hence asymptotic freedom might show up. This is discussed in the last section of this article.

In Sec. II the basic dynamical equations of the model are given and also the basic statistical-mechanical techniques are provided. In Sec. III the equations of a Bogolubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy for our quark plasma are obtained and applied in Sec. IV to the case of thermodynamical equilibrium. In Sec. V numerical results are obtained in some particular cases of astrophysical interest and these results are discussed in Sec. VI. All the notations are standard.

II. BASIC EQUATIONS OF THE MODEL

The system described in this paper consists of a quark plasma built up from colored quarks [described by a spin-$\frac{1}{2}$ field ψ, or by $\psi_{AA'\alpha}(x)$ where α is the usual spinor index, where A is the flavor index (here SU_3), and where A' is the color index (here SU_3')] interacting via scalar gluons (described by the field ϕ, a scalar both in SU_3 and SU_3') and colored spin-1 gluons (described by the field A^μ_B). The basic dynamical equations follow from the Lagrangian

$$\begin{aligned} L = \bar{\psi}_A(x)\{i\gamma\cdot[\partial - ig_V A^B(x)\Lambda_B] - m + g_S\phi(x)\}\psi_A(x) \\ + (1/8\pi)[\partial_\mu\phi(x)\partial^\mu\phi(x) - \mu_S{}^2\phi^2(x) - \tfrac{1}{2}\lambda\phi^4(x)] \\ + (1/8\pi)[\partial_\mu A_{\nu B}(x)\partial^\mu A^{\nu B}(x) - \mu_V{}^2 A^\lambda_B(x)A^B_\lambda(x)]. \end{aligned} \quad (2.1)$$

In these equations the matrices Λ_B are the usual Gell-Mann matrices.

At this point it must be strongly emphasized that our model is only a *phenomenological model*: it is not supposed to be a complete and consistent field-theoretical model. If we have in mind the most current bag models (SLAC's or MIT's), our vector colored gluon field must be considered as an *approximation* of a gauge field (the gauge group being the color group) that amounts to a kind of linearization of the equations of motion. Let us specify this last point more precisely (details are given in Ref. 23) from the equations of motion satisfied by a gauge field A^μ_B; they read

$$\begin{aligned} \partial^\mu\partial_\nu A^\nu_A - \Box A^\mu_A - ig_V C^{BC}_A(A^\nu_C\partial_\nu A^\mu_B + A^\mu_B\partial_\nu A^\nu_C) - g_V\bar{\psi}\gamma^\mu\Lambda_A\psi \\ - ig_V(\partial^\mu A^\rho_C - \partial^\rho A^\mu_C - ig_V C^{BD}_C A^\mu_B A^\rho_D)C^C_{AE}A^E_\rho = 0 \end{aligned} \quad (2.2)$$

(plus a gauge condition). (The C's are the color group structure constants.) They can be approximated in the following way: If we are interested in two-body correlations only (thus neglecting higher-order correlations) the nonlinear terms of Eq. (2.2) can be approximated by

$$A\otimes A\otimes A \sim \sum_{\substack{\text{permutations}\\ \text{of } A}} \langle A\otimes A\rangle\otimes A\,, \quad (2.3)$$

leading thereby to a linear equation which can be shown to reduce (in the Coulomb gauge and in thermodynamical equilibrium) or to be approximated (in the Landau gauge) by a Klein-Gordon equation. It should also be noticed that, within this approximation, μ_V is connected to $\langle A\otimes A\rangle$ and thus leads to self-consistent equations.[23]

From this brief discussion it is clear that, e.g., Eq. (2.1) should not necessarily be supplemented by the usual transversality condition $\partial_\mu A^\mu_B = 0$ as it should be if A^μ_B is to represent a true massive spin-1 field. It is also clear that such a condition *cannot* be imposed since the color current transported by the quarks is not conserved. However, in the sequel, it will be shown that, within our (statistical) approximations (e.g., neglect of three-body correlations, thermodynamical equilibrium) the following two equations are satisfied:

$$\partial_\mu\langle A^\mu_B\rangle = 0\,,\quad \partial_\mu\langle A^\mu_B(x)A^\nu_C(x')\rangle = 0\,.$$

Therefore, there will be no inconsistency (and no ghosts) of our phenomenological model with a gauge model within the approximations effected (see Appendix A).

From Eq. (2.1) it is seen that the dynamics of the system are determined when the *six* constants m, $\mu_S{}^2$, $\mu_V{}^2$, g_S, λ, g_V are known. These constants have to be determined from elementary-particle data (see below). It should also be noticed that since SU_3 does not play any dynamical role in the model, this latter is valid as well for SU_4 or other groups. The masses of the various quarks have been taken to be alike and it is in fact easy to account for the heavier mass of the strange (or the charmed) quark by putting instead of m a mass matrix in Eqs. (2.1).

A. Statistical description of the quark plasma

Such a statistical description, needed to obtain, e.g., an equation of state, is based on a covariant Wigner[31] function and connected functions. A formalism using covariant Wigner functions has been developed and discussed elsewhere[32,33] so that, in this paper, we only adapt our previous definitions to the quark plasma and give the minimum necessary for the understanding of the remainder of the article.

(1) The basic tool is the one-particle Wigner function defined by

$$\begin{aligned} F_{AA'\alpha,BB'\beta}(x,p) = \frac{1}{(2\pi)^4}\int d^4R\,\exp(-ip\cdot R) \\ \times\langle\bar{\psi}_{BB'\beta}(x+R/2) \\ \otimes\psi_{AA'\alpha}(x-R/2)\rangle\,, \end{aligned} \quad (2.4)$$

where the angular brackets $\langle\,\rangle$ denote a quantum statistical-mechanical average

$$\langle\cdots\rangle \equiv \mathrm{Tr}\{\bar{\rho}\cdots\}\,, \quad (2.5)$$

$\bar{\rho}$ being the density operator which describes the statistical state of the system. One can easily check [from Eq. (2.4)] that the quark four-current J^μ and momentum-energy tensor $T^{\mu\nu}$ are given by

$$J^\mu(x) = \mathrm{Tr}\,\mathrm{Tr}\,\mathrm{Tr}'\left(\int d^4p\,\gamma^\mu F(x,p)\right)\,, \quad (2.6)$$

$$T^{\mu\nu}(x) = \mathrm{Tr}\,\mathrm{Tr}\,\mathrm{Tr}'\left(\int d^4p\,\gamma^\mu p^\nu F(x,p)\right)\,, \quad (2.7)$$

$$\langle\bar\psi(x)\psi(x)\rangle = \mathrm{Tr}\,\mathrm{Tr}\,\mathrm{Tr}'\left(\int d^4p F(x,p)\right). \qquad (2.8)$$

The symbols Tr occurring in these last equations respectively indicate the trace over spin, SU_3, and SU_3' indices. These quantities are of primordial importance in the following.

(2) Besides $F(x,p)$ an important quantity is

$$\tilde F(x,p) = \frac{1}{(2\pi)^4}\int d^4R \exp(-ip\cdot R)\bar\psi(x+R/2) \otimes \psi(x-R/2), \qquad (2.9)$$

which is nothing but the covariant Wigner operator. From Dirac's equations a straightforward calculation shows that $\tilde F(x,p)$ satisfies the equation

$$\begin{aligned}[][i\gamma\cdot\partial + 2(\gamma\cdot p - m)]\tilde F(x,p) &= \frac{-2g_S}{(2\pi)^4}\int d^4x'd^4p' \exp[-ip'\cdot(x-x')] \times \tilde F(x,p-p'/2)\phi(x') \\ &\quad - \frac{2g_V}{(2\pi)^4}\int d^4x'd^4p' \exp[-ip'\cdot(x-x')]\gamma^\mu\Lambda_B \times \tilde F(x,p-p'/2)A_\mu^B(x') \qquad (2.10)\end{aligned}$$

and also an "adjoint" equation which must be solved *simultaneously*. Once these equations are joined to the equations of motion written in the form

$$[\Box + \mu_S{}^2 + \lambda\phi^2(x)]\phi(x) = 4\pi g_S\,\mathrm{Tr}\left(\int d^4p\tilde F(x,p)\right), \qquad (2.11)$$

$$(\Box + \mu_V{}^2)A_B^\mu(x) = 4\pi g_V\,\mathrm{Tr}\left(\int d^4p\gamma^\mu\Lambda_B\tilde F(x,p)\right), \qquad (2.12)$$

they constitute the set of the *generating equations* of the relativistic quantum BBGKY hierarchy that describes fully the quark plasma. This will be illustrated in detail in the next section.

(3) Another quantity of importance is

$$\chi_B^\mu(x,x';p) \equiv \langle\tilde F(x,p)A_B^\mu(x')\rangle, \qquad (2.13)$$

to which one must add

$$\underline{\chi}_B^\mu(x',x;p) \equiv \langle A_B^\mu(x')\tilde F(x,p)\rangle. \qquad (2.14)$$

These quantities arise in a natural way when one takes the average value of both sides of Eqs. (2.10) and the adjoint equation: the left-hand sides involve $F(x,p)$ while the right-hand sides contain χ or $\underline{\chi}$. In turn χ is obtained by multiplying, e.g., Eq. (2.10) from the right by $A_B^\mu(x')$ and averaging: χ is then connected to the more complex quantity

$$\langle\tilde F(x,p)A_B^\mu(x')A_C^\nu(x'')\rangle \qquad (2.15)$$

and the process can be indefinitely repeated, leading consequently to an infinite hierarchy of equations for more and more complex quantities.

(4) The fluctuations of one-quark quantities can be obtained from

$$\mathfrak{F}(x,p;x',p') \equiv \langle\tilde F(x,p)\otimes\tilde F(x',p')\rangle. \qquad (2.16)$$

For instance, fluctuations of color current are given by [$\langle J_A^\mu(x)\rangle \equiv 0$ since the average value is taken over color singlets]

$$\langle\tilde J_A^\mu(x)\tilde J_B^\nu(x')\rangle - \langle\tilde J_A^\mu\rangle\langle\tilde J_B^\nu\rangle = \langle\tilde J_A^\mu(x)\tilde J_B^\nu(x')\rangle, \qquad (2.17)$$

with

$$\tilde J_A^\mu(x) = \mathrm{Tr}\left(\int d^4p\gamma^\mu\Lambda_A\tilde F(x,p)\right). \qquad (2.18)$$

Therefore they are given by

$$\langle J_A^\mu(x)J_B^\nu(x')\rangle = \mathrm{Tr}\,\mathrm{Tr}\left(\int d^4pd^4p'\gamma^\mu\gamma^\nu\Lambda_A\Lambda_B\mathfrak{F}(x,p;x',p')\right) \qquad (2.19)$$

where the first Tr refers to the first (implicit) indices of $\mathfrak{F}$ and the second Tr to the second (implicit) indices in this same quantity.

III. BBGKY HIERARCHY FOR THE QUARK PLASMA

Let us derive the first two sets of equations of the BBGKY hierarchy. If three-body correlations had to be considered (and they should be in the case where reactions like

baryon $\rightleftharpoons$ three quarks

have to be dealt with) we should also derive the third set of equations.

The first set of equations is easily obtained by averaging both sides of the generating equations (2.10) to (2.12) and found to be

$$\begin{aligned}[][i\gamma\cdot\partial + 2(\gamma\cdot p - m)]F(x,p) &= \frac{-2g_S}{(2\pi)^4}\int d^4x'd^4p' \exp[-ip'\cdot(x-x')]\langle\tilde F(x,p-p'/2)\phi(x')\rangle \\ &\quad - \frac{2g_V}{(2\pi)^4}\int d^4x'd^4p' \exp[-ip'\cdot(x-x')]\gamma^\mu\Lambda_B\chi_\mu^B(x,x';p-p'/2), \qquad (3.1)\end{aligned}$$

$$\left(\Box + \mu_S{}^2 + \lambda\frac{\langle\phi^3(x)\rangle}{\langle\phi(x)\rangle}\right)\langle\phi(x)\rangle = 4\pi g_S\,\mathrm{Tr}\left(\int d^4pF(x,p)\right). \qquad (3.2)$$

There is no equation for $\langle A_B^\mu\rangle$, since this quantity vanishes identically, because of the fact that we have *restricted* the physical states to be *color singlets*. Had we relaxed this assumption we would have found

$$(\Box+\mu_V^2)\langle A_B^\mu(x)\rangle = 4\pi g_V \,\mathrm{Tr}\left(\int d^4p\gamma^\mu\Lambda_B F(x,p)\right). \quad (3.3)$$

Note also that the average color four-current of the quarks [i.e., essentially the right-hand side of Eq. (3.3)] vanishes for the same reason as for $\langle A_B^\mu\rangle$. Indeed, when the physical states occurring implicitly in the averaging operation $\langle\rangle$ are taken to be color singlets, the (color) tensor $F_{A'B'}(x,p)$ is necessarily proportional to $\delta_{A'B'}$ and hence the trace on SU_3, indices reduces to a trace on Λ_B [note that $\Lambda_B \equiv \|\Lambda_{B,A'B'}\|$, with $B=1,2,\ldots,8$ and $(A',B')=1,2,3$] which trace is zero. The second set of equations of the BBGKY hierarchy is much more involved, essentially because of the fact that the generating equations (2.10) to (2.12) can be multiplied from the right or from the left by one of the operators $[\tilde F(x'',p''),\ \phi(x''),\ A_B^\mu(x'')]$ and next averaged. Among the thirty equations of the second set of equations of the hierarchy one finds

$$[i\gamma\cdot\partial+2(\gamma\cdot p-m)]\chi_C^\lambda(x,x'';p) = \frac{-2g_S}{(2\pi)^4}\int d^4x'd^4p'\exp[-ip'\cdot(x-x')]\langle\tilde F(x,p-p'/2)\phi(x')A_C^\lambda(x'')\rangle$$
$$-\frac{2g_V}{(2\pi)^4}\int d^4x'd^4p'\exp[-ip'\cdot(x-x')]\gamma^\mu\Lambda_B\langle\tilde F(x,p-p'/2)A_\mu^B(x')A_C^\lambda(x'')\rangle \quad (3.4)$$

and similar equations for $\underline{\chi}_C^\lambda$. For the scalar gluons one finds[32-34]

$$(\Box+\mu_S^2)\langle\phi(x)\phi(x')\rangle+\lambda\langle\phi^3(x)\phi(x')\rangle = 4\pi g_S\,\mathrm{Tr}\left(\int d^4p\,\langle\tilde F(x,p)\phi(x')\rangle\right), \quad (3.5)$$

while there is no equation for $\langle\phi(x)A_B^\mu(x')\rangle$ since this quantity vanishes identically when averaging over color singlets. This property implies the following condition on $\chi_B^\mu(x,x'',p)$:

$$\mathrm{Tr}\left(\int d^4p\chi_B^\mu(x,x'';p)\right)=0, \quad (3.6)$$

and a similar condition for $\underline{\chi}_B^\mu$ [it represents nothing but the vanishing of the source term of the equation for $\langle\phi(x)A_B^\mu(x')\rangle$].

For the vector gluons one finds

$$(\Box+\mu_V^2)\langle A_B^\mu(x)A_C^\lambda(x'')\rangle = 4\pi g_V\,\mathrm{Tr}\left(\int d^4p\gamma^\mu\Lambda_B\chi_C^\lambda(x,x'';p)\right). \quad (3.7)$$

It should be noticed that the equation satisfied by the average value of commutator $[A_B^\mu(x),A_C^\nu(x')]$ can easily be obtained from the generating equation (2.12); multiplying this equation from the right and next from the left, subtracting the result and averaging, one gets

$$(\Box+\mu_V^2)\langle[A_B^\mu(x),A_C^\nu(x')]\rangle = 4\pi g_V\,\mathrm{Tr}\left(\int d^4p\gamma^\mu\Lambda_B[\underline{\chi}_C^\nu(x',x;p)-\chi_C^\nu(x,x';p)]\right). \quad (3.8)$$

This equation shows that the use of the approximation

$$\langle[A_B^\mu(x),A_C^\nu(x')]\rangle\sim 0 \quad (3.9)$$

should be accompanied by

$$\underline{\chi}_C^\lambda(x',x;p)\sim\chi_C^\lambda(x,x';p), \quad (3.10)$$

and conversely. The use of such an *Ansatz* will be referred to as a *quasi-classical approximation*.

It is clear, at this step, that in order to obtain solutions of these equations the hierarchy must be truncated at some order and with some physically (more or less) plausible *Ansatz* as is the case in usual plasma physics. This is the object of the next section.

IV. EQUILIBRIUM AND THE EQUATION OF STATE

In a thermodynamical equilibrium state the BBGKY equations can be somewhat simplified owing to the invariance of the system under space-time translations, which requires

$$F(x,p)\equiv F_{\mathrm{eq}}(p), \quad (4.1)$$

$$\langle\phi(x)\rangle\equiv\phi, \quad (4.2)$$

$$\chi_A^\mu(x,x';p)\equiv\chi_A^\mu(x-x';p), \quad (4.3)$$

$$\langle A_B^\mu(x)A_C^\nu(x')\rangle\equiv\mathcal{Q}_{BC}^{\mu\nu}(x-x'), \quad (4.4)$$

$$\langle\phi(x)\phi(x')\rangle\equiv\Phi(x-x'). \quad (4.5)$$

In order to obtain the basic thermodynamical quantities such as the pressure, the energy density, etc., we must first calculate the momentum-energy tensor $T^{\mu\nu}$ of the system. From this quantity we have

$$\rho = T^{\mu\nu} u_\mu u_\nu , \tag{4.6}$$

$$P = -\tfrac{1}{3}\Delta^{\mu\nu}(u^\rho) T_{\mu\nu} \tag{4.7}$$

[with $\Delta^{\mu\nu}(u^\rho) \equiv g^{\mu\nu} - u^\mu u^\nu$] ($\rho$ is the invariant energy density and P is the pressure.) u^μ is the average four-velocity of the medium (see below). Equations (4.6) and (4.7) lead immediately to the equations of state of the system. To these equations one must also add the normalization equation (2.6). This equation relates the equilibrium particle density $n_{\rm eq}$ to the macroscopic parameters of $F_{\rm eq}(p)$. If the Wigner distributions for the gluons are defined as

$$f_{(S)}(x,k) = \frac{1}{(2\pi)^4}\int d^4R \exp(-ik\cdot R) \times \langle \phi(x+R/2)\phi(x-R/2)\rangle \tag{4.8}$$

(scalar gluons),

$$f^{\mu\nu}_{(V)BC}(x,k) = \frac{1}{(2\pi)^4}\int d^4R \exp(-ik\cdot R) \times \langle A^\mu_B(x+R/2) A^\nu_C(x-R/2)\rangle \tag{4.9}$$

(vector gluons),

then the *total* momentum-energy tensor of the quark plasma reads

$$\begin{aligned} T^{\mu\nu} = {} & \mathrm{Tr}\Big(\int d^4p\, p^\mu \gamma^\nu F(x,p)\Big) \quad \text{(quarks)} \\ & + \int d^4k f_{(S)}(k)[k^\mu k^\nu - \tfrac{1}{2} g^{\mu\nu}(k^2 - \mu_S{}^2)] \quad \text{(scalar gluons)} \\ & + \int d^4k f^{B\lambda}_{(V)\lambda B}(k)[k^\mu k^\nu - \tfrac{1}{2} g^{\mu\nu}(k^2 - \mu_V{}^2)] \quad \text{(vector gluons)}, \end{aligned} \tag{4.10}$$

where use has been made of the stationarity and of the homogeneity of the system.

The expression of the momentum-energy tensor for scalar particles in terms of a Wigner distribution has first been given by Cooper and Sharp.[35] Note that in order to get, e.g., the scalar gluon part of $T^{\mu\nu}$, it is sufficient to know $\langle \phi^2(x)\rangle$ since, for instance, one has

$$\langle \partial\phi\cdot\partial\phi\rangle = \tfrac{1}{2}\Box\langle\phi^2\rangle + \mu_S{}^2\langle\phi^2\rangle - \langle\bar\psi\psi\phi\rangle ,$$

an expression occurring in $T^{\mu\nu}$ (scalar gluons). Note also that in Eq. (4.10) the $\lambda\langle\phi^4\rangle$ term has been omitted: in fact it necessitates *a two-body Wigner distribution.*

Let us now investigate the BBGKY hierarchy at equilibrium and let us begin with the equations for the gluons.

A. Gluons quantities at thermodynamical equilibrium

For the scalar gluons the first equation of the hierarchy Eq. (3.2) reduces to

$$\mu_S{}^2\phi + \lambda\langle\phi^3(0)\rangle = 4\pi g_S \,\mathrm{Tr}\Big(\int d^4p F_{\rm eq}(p)\Big) \tag{4.11}$$

and ϕ is known when $F_{\rm eq}(p)$ (the equilibrium one-particle Wigner function) and $\langle\phi^3(0)\rangle$ are known. In order to close the hierarchy for the scalar gluons let us briefly recall their role in the model: they have been introduced only to obtain the confinement of quarks and thus, in this respect, only the *collective* behavior of the scalar field is important at this first step. Of course, correlations either of the scalar gluons or of the quarks with the scalar gluons can play a role, but in a first study they will be neglected and we shall focus our attention on this collective aspect (the dominant one as far as confinement is concerned). Therefore, in order to express these approximations analytically, the following *Ansatze* are used

$$\begin{aligned} &\langle\phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle \sim \prod_{i=1}^{i=n}\langle\phi(x_i)\rangle , \\ &\langle \tilde F(x,p)\prod_{i=1}^{i=n}\phi(x_i)\rangle \sim F(x,p)\prod_{i=1}^{i=n}\langle\phi(x_i)\rangle , \\ &n = 1, 2, \cdots . \end{aligned} \tag{4.12}$$

In other words, $\langle\phi(x)\rangle$ is a quasi-classical field (a coherent state of the scalar gluon field).[33,34] Thence, Eq. (4.11) reduces to

$$\mu_S{}^2\langle\phi\rangle + \lambda\langle\phi\rangle^3 = 4\pi g_S \mathrm{Tr}\Big(\int d^4p F_{\rm eq}(p)\Big) . \tag{4.13}$$

In fact, the only role—in this approximation—of the scalar field is to modify the mass of the quarks, a consequence of which is the existence of peculiar phase transitions[32-34] that are interpreted in the next section. It can be shown (by a numerical analysis) that the $\langle\lambda\phi^3\rangle$ term only modifies the *numerical* value of the effective mass of the quarks[34] and not the qualitative behavior[36] of the system as a function of its thermodynamical parameters. Therefore we can simply set $\lambda = 0$ and rescale the coupling constant g_S. In fact the $\lambda\phi^3$ term is necessary to renormalize the theory. However in the Hartree approximation there is no gluon-gluon collision and hence no particular need of such a counterterm.

Let us look at the vector gluons. The situation is now slightly more complicated since $\langle A^\mu_B\rangle \equiv 0$ and we are forced to go over to the second equations of the hierarchy, i.e., to Eqs. (3.7) and (3.4). Fourier transforming the latter equations we get

$$(k^2-\mu_V{}^2)\hat{\mathcal{A}}^{\mu\lambda}_{BC}(k)=-4\pi g_V \mathrm{Tr}\left(\int d^4p\gamma^\mu\Lambda_B\hat{\chi}^\lambda_C(k,p)\right). \tag{4.14}$$

It should be noticed that $\hat{\mathcal{A}}^{\mu\lambda}_{BC}\neq\hat{\mathcal{A}}^{\lambda\mu}_{CB}$. The caret designates a Fourier transform.

B. Quarks quantities at thermodynamical equilibrium

(1) Owing to the aforementioned invariance under spacetime translations, the equation for the one-particle Wigner function (3.1) (and the nonwritten adjoint equation) is

$$(\gamma\cdot p-M)F_{\mathrm{eq}}(p)=g_V\int d^4p'\gamma^\mu\Lambda_B\hat{\chi}^B_\mu(p';p-p'/2), \tag{4.15}$$

$$F_{\mathrm{eq}}(p)(\gamma\cdot p-M)=g_V\int d^4p\hat{\chi}^B_\mu(p';p+p'/2)\gamma^\mu\Lambda_B, \tag{4.16}$$

once all the possible (and elementary) integrations are performed. In Eqs. (4.15) and (4.16), M is an *effective mass* for the quarks, given by

$$M=m-g_S\langle\phi\rangle. \tag{4.17}$$

The general solutions of these equations have the form

$$F_{\mathrm{eq}}(p)=F_1(p)+g_V\frac{\gamma\cdot p+M}{p^2-M^2}\int d^4p'\gamma^\mu\Lambda_B\hat{\chi}^B_\mu(p';p-p'/2), \tag{4.18}$$

$$F_{\mathrm{eq}}(p)=F_1(p)+g_V\int d^4p'\hat{\underline{\chi}}^B_\mu(p+p'/2)\gamma^\mu\Lambda_B\frac{\gamma\cdot p+M}{p^2-M^2}, \tag{4.19}$$

where F_1 and F_2, respectively, are homogeneous solutions of Eqs. (4.15) and (4.16). One can show that $F_1\equiv F_2$ and therefore that the following identity

$$(\gamma\cdot p+M)\int d^4p'\gamma^\mu\Lambda_B\hat{\chi}^B_\mu(p';p-p'/2)=\int d^4p'\chi^B_\mu(p';p+p'/2)\gamma^\mu\Lambda_B(\gamma\cdot p+M) \tag{4.20}$$

is satisfied. Furthermore, F_1 is a solution of

$$(\gamma\cdot p-M)F_1(p)=F_1(p)(\gamma\cdot p-M)=0,$$

and hence is a "free" solution.

(2) Let us now investigate the equation (3.4) for χ^μ_B. It involves third-order quantities like

$$\langle\tilde{F}(x,p-p'/2)A^\mu_B(x')A^\lambda_C(x'')\rangle.$$

We want, at least as a first approximation, to close the BBGKY hierarchy keeping second-order quantities only. A possible ansatz to achieve this goal is a pairing approximation of the form

$$\langle\tilde{F}\otimes A\otimes A\rangle\sim F\mathcal{A}, \tag{4.21}$$

where $\langle A\rangle\equiv 0$ has been taken into account. Such an approximation scheme somewhat amounts to neglecting three-body correlations and is common in conventional plasma physics whether quantum or not.[37] Including now Eq. (4.21) into Eq. (3.4) and taking the Fourier transformation of the result, one obtains

$$[\gamma\cdot(p-k/2)-M]\hat{\chi}^\lambda_C(k,p)=-g_V\Lambda_A\gamma^\mu F_{\mathrm{eq}}(p+k/2)\hat{\mathcal{A}}^{A\lambda}_{C\mu}(k) \tag{4.22}$$

and also the similar equation

$$\hat{\chi}^\lambda_C(k,p)[\gamma\cdot(p+k/2)-M]=-g_V F_{\mathrm{eq}}(p-k/2)\hat{\mathcal{A}}^{\lambda A}_{\mu C}(k)\Lambda_A\gamma^\mu. \tag{4.23}$$

Equations (4.22) and (4.23) resemble very strongly equations obtained elsewhere for QED plasmas[38,39] (apart from the color indices and matrices) if one identifies $\mathcal{A}$ with the perturbed electromagnetic field (with respect to a zero equilibrium background field), χ with the perturbed Wigner function (with respect to the equilibrium function F_{eq}). Therefore the system (4.22) and (4.23), to which Eqs. (4.14) and its adjoint must be joined, will lead to dispersion relations (in the sense of plasma physics) very similar to those obtained for QED plasmas. To show this briefly we first solve[39] the system (4.22) and (4.23) as

$$\chi^\lambda_C(k,p)=-g_V\hat{\mathcal{A}}^{A\lambda}_{C\mu}(k)\left\{\frac{\gamma\cdot(p-k/2)+M}{(p-k/2)^2-M^2+i\epsilon}\Lambda_A\gamma^\mu F_{\mathrm{eq}}(p+k/2)\right.$$
$$\left.+F_{\mathrm{eq}}(p-k/2)\frac{\gamma\cdot(p+k/2)+M}{(p+k/2)^2-M^2-i\epsilon}\Lambda_A\gamma^\mu\right\}, \tag{4.24}$$

where the $i\epsilon$ terms have been added so as to choose the correct homogeneous solution of the system. Inserting now the solution (4.24) into Eq. (4.14), one gets the following *homogeneous* equation

$$(k^2-\mu_V{}^2)\hat{\alpha}^{\mu\lambda}_{BC}(k)=-4\pi g_V{}^2\,\mathrm{Tr}\left\{\int d^4p\gamma^\mu\Lambda_B\left[\frac{\gamma\cdot(p-k/2)+M}{(p-k/2)^2-M^2+i\epsilon}\Lambda_A\gamma^\nu F_{\rm eq}(p+k/2)\right.\right.$$
$$\left.\left.+F_{\rm eq}(p-k/2)\frac{\gamma\cdot(p+k/2)+M}{(p+k/2)^2-M^2-i\epsilon}\Lambda_A\gamma^\nu\right]\right\}\hat{\alpha}^{A\lambda}_{C\nu}(k)\,. \tag{4.25}$$

This last equation shows that the polarization operator is nothing but the coefficient of $\hat{\alpha}^{A\lambda}_{C\nu}(k)$ [in the right-hand side of Eq. (4.25)]

$$\Pi^{\mu\nu}_{AB}(k)=-4\pi g_V{}^2\,\mathrm{Tr}\left(\int d^4p\gamma^\mu\Lambda_B[\cdots]^{\nu}_{A}\right). \tag{4.26}$$

Once all the traces are performed the polarization tensor reduces to

$$\Pi^{\mu\nu}_{AB}(k)=-\frac{4\pi g_V{}^2}{M}\delta_{AB}\int d^4p[p^\mu p^\lambda-k^\mu k^\lambda/4-(p^2-k^2/4)g^{\mu\lambda}]\left[\frac{f(p+k/2)-f(p-k/2)}{k\cdot p-i\epsilon}\right], \tag{4.27}$$

where use has been made of the approximation that consists of replacing $F_{\rm eq}(p)$ by its *zeroth-order expression*[32,38] in $g_V{}^2$:

$$F_{\rm eq}(p)=\frac{\gamma\cdot p+M}{4M}f_{\rm eq}(p)\,, \tag{4.28}$$

with[38]

$$f_{\rm eq}(p)=\frac{2M}{(2\pi)^3}\sum_{\pm}\int_{p'^2=M^2}\frac{d^3p'}{|p'_0|}\frac{\delta^{(4)}(p\mp p')\theta(\pm p'^0)}{\exp[\beta(u^\mu p'_\mu\mp\epsilon_f)]+1}\,. \tag{4.29}$$

Introducing now[38,39] the conventional plasma frequency

$$\omega_P{}^2\equiv 4\pi g_V{}^2 n_{\rm eq}/M \tag{4.30}$$

(where $n_{\rm eq}$ is the invariant *numerical* density of the quarks) of the quark plasma and the same notation as in Refs. 38 and 39, i.e.,

$$I\equiv 2\int d^4p\frac{f_{\rm eq}(p+k/2)-f_{\rm eq}(p-k/2)}{k\cdot p-i\epsilon}\,, \tag{4.31}$$

$$\Omega_P{}^2\equiv\frac{8\pi g_V{}^2}{M}\int d^4p f_{\rm eq}(p)\,, \tag{4.32}$$

$$K^{\lambda\mu}\equiv\frac{2}{n_{\rm eq}}\int d^4p\,p^\lambda p^\mu\frac{f_{\rm eq}(p+k/2)-f_{\rm eq}(p-k/2)}{k\cdot p-i\epsilon}\,, \tag{4.33}$$

the polarization tensor can be rewritten as

$$\Pi^{\lambda\mu}_{AB}(k)=3g_{AB}\left\{\omega_P{}^2K^{\lambda\mu}+k^2\Delta^{\lambda\mu}(k)\frac{\omega_P{}^2I}{4n_{\rm eq}}+\Omega_P{}^2g^{\lambda\mu}+g^{\lambda\mu}\frac{4\pi g_V{}^2}{M}\int d^4p f_{\rm eq}(p)\left[\frac{(k\cdot p)^2-M^2k^2}{(k\cdot p)^2-k^4/4}\right]\right\} \tag{4.34}$$

(with $g_{AB}\equiv\frac{1}{3}\delta_{AB}$ and also $g^{AB}=3\delta^{AB}$) $\Omega_P{}^2$ being the *actual* plasma frequency of the system. One can also check that

$$k_\mu\Pi^{\lambda\mu}_{AB}(k)=0\,.$$

In the derivation of this last equation [used also in the derivation of Eqs. (4.36) and (4.37) below] the relation

$$\partial_\mu\alpha^{\mu\lambda}_{AB}=0 \tag{4.35}$$

has been used (see Appendix A).

C. The quasi-gluon excitation spectrum

The homogeneous system (4.25) to which the "Lorentz-gauge" condition (4.35) is added leads to the excitation spectrum of the vector plasmons in the same way as in the QED case,[38,39] and hence this derivation will not be given. The result is

$$\omega^2-k^2-\mu_V{}^2+\left(-\omega_P{}^2K^{00}-\frac{\omega_P{}^2I}{4n_{\rm eq}}-\tilde{\Omega}_P{}^2(k)\right)+\frac{\omega}{4\pi k}\omega_P{}^2K^{30}=0 \tag{4.36}$$

(for longitudinal plasmons) and

$$\omega^2 - k^2 - \mu_V{}^2 + \left(-\omega_P{}^2 K^{11} + \frac{\omega_P{}^2 I}{4n_{\rm eq}} + \tilde{\Omega}_P{}^2 \right) = 0 \quad (4.37)$$

(for transverse plasmons), with

$$\tilde{\Omega}_P{}^2 \equiv \Omega_P{}^2 + \Omega_P{}^2(k)\,,$$

$$\Omega_P{}^2(k) \equiv \int d^4p f_{\rm eq}(p) \frac{(k\cdot p)^2 - M^2 k^2}{(k\cdot p)^2 - k^4/4} \left(\frac{4\pi g_V{}^2}{M} \right),$$

where the *only* difference with the QED case arises from a factor 2 occurring because of color, In Eqs. (4.36) and (4.37) use has been made of a frame of reference where $u^\mu = (1,0,0,0)$. Furthermore, $\vec{k}$ has been chosen so that the waves propagate in the z direction, i.e., we have set (in these equations only) $k^\mu \equiv (\omega, 0, 0, k)$. These general dispersion relations have been thoroughly discussed in several cases of physical interest by Tsytovich many years ago in a well-known paper[40] and thus we shall not enter into the details of their discussion.

D. Determination of $\hat{\mathfrak{A}}^{\mu\nu}_{AB}$

Let us now determine $\hat{\mathfrak{A}}$ in the framework of the approximations considered in this section (the lowest orders of a perturbation expansion of $\hat{\mathfrak{A}}$ are given in Appendix B).

A glance at Eq. (4.25) for $\hat{\mathfrak{A}}^{\mu\nu}_{AB}(k)$ shows that this important quantity satisfies a homogeneous equation and therefore cannot be uniquely determined without any further information. In fact, such an information must be (and is) provided by thermodynamical considerations. Indeed the above equations are valid for arbitrary spacetime invariant systems whether in equilibrium or not. The only place where thermodynamical considerations entered in the above equations was in the derivation of Eq. (4.27) for the polarization tensor where it was assumed that $F_{\rm eq}(p)$ was given by Eqs. (4.28) and (4.29), referring thereby to the grand canonical ensemble.[32,38]

In this paragraph $\hat{\mathfrak{A}}^{\mu\nu}_{AB}(k)$ will be determined by a direct reasoning referring to such an ensemble and, for consistency, taking account of the approximation (4.21).

First, it should be remarked that $\hat{\mathfrak{A}}^{\mu\nu}_{AB}$ is nothing but the covariant Wigner function (4.9) of the vector gluons.

Next, symmetry considerations joined to the Lorentz-type condition (4.35) and the use of color-singlet states, show that

$$\hat{\mathfrak{A}}^{\mu\nu}{}_{AB}(k) = l(k)\Delta^{\mu\nu}(k)\delta_{AB}\,, \qquad (4.38)$$

where $l(k)$ is a scalar function of k^μ to be determined below by thermodynamical considerations.

Within the approximations considered in this section the vector gluons satisfy a homogeneous equation [depending on $F_{\rm eq}(p)$] and can therefore be considered as "free" quasi-particles endowed with a particular excitation spectrum [e.g., Eqs. (4.36) and (4.37)]. Since the system is in a thermodynamical state, these quasi-particles obey Bose-Einstein statistics. More precisely, the grand canonical density operator for the vector gluons in the approximation under consideration should read

$$\rho_{\rm vect.\ gluons} \sim \exp\left(-\beta \sum_{i=L,T} \sum_{\vec{k}} \omega_i(\vec{k}) a_B^{+i}(\vec{k}) a_i^B(\vec{k}) \right) \qquad (4.39)$$

in a Lorentz frame where $u^\mu = (1, \vec{0})$; L and T designate the longitudinal (it could also be shown that there exists a zero sound mode) and transverse modes, respectively. In Eq. (4.39) an infinite vacuum term has been dropped. $\omega_i(k)$ is the quasi-gluon excitation spectrum and $a^i_B(\vec{k})$ is the destruction operator of a quasi-gluon of color B, in the mode i and momentum $\vec{k}$.

Finally the free-like form of $\rho_{\rm vect.\ gluons}$ leads to the usual Bose-Einstein factor so that one can write

$$\hat{\mathfrak{A}}^{\mu\nu}_{AB}(k) = \frac{\delta_{AB}\Delta^{\mu\nu}(k)}{\exp(\beta u_\mu k^\mu) - 1}\,\delta(D_{\rm tr}(k) D_{\rm long}(k))\,, \qquad (4.40)$$

where Eqs. (4.36) and (4.37) have formally been rewritten as

$$D_{\rm tr}(k) = 0\,, \quad D_{\rm long}(k) = 0\,,$$

respectively.

E. Summary

Let us now briefly summarize the basic equations and approximations made in this section.

The one-particle Wigner function $F_{\rm eq}$ can be obtained from χ[Eq. (4.18)] and the free equilibrium solution [Eqs. (4.28) and (4.29)] as a zeroth-order term in g_V. χ is itself obtained from $F_{\rm eq}$ and through Eq. (4.24). $\hat{\mathfrak{A}}$ is determined from equilibrium statistical mechanics *and* from $F_{\rm eq}$ via the dispersion relation that one can get from Eq. (4.25) and Eq. (4.35).

Apart from the assumptions necessary to close the hierarchy [Eqs. (4.12) and (4.21)], the above scheme does not contain any approximation. However it is not quite simple—although not impossible—to solve this set of equations. Nevertheless, we approximated $F_{\rm eq}$ by its zeroth-order expression in g_V and as a result the above scheme was much simpler. As a first illustration of the formalism it is perhaps worthwhile to deal with this (more or less) simple approximation, which is considered in the next section. Also, it can be remarked that the next approximation, i.e., the one where F_{eq} is calculated at order g_V^2, is obtained easily from Eqs. (4.18) in which χ is taken at the lowest order in g_V. This lowest order is obtained from Eq. (4.24) in which (i) F_{eq} is now the free equilibrium Wigner function (4.28) and (ii) $\hat{\mathfrak{a}}$ is still given by a Bose-Einstein-type relation *but* now including the *free* dispersion law $k^2=\mu_V^2$. The next order is of course much more difficult to obtain.

V. ILLUSTRATION OF THE FORMALISM

Our model is rather crude and can presumably give orders of magnitude only. Consequently it seems suited to astrophysical applications where one generally contents oneself precisely with orders of magnitude. Therefore we shall *illustrate* our model in one particular case: the case of a (dense) quark star, i.e., the case of a neutron star with a quark core.

We have said "illustrate" and not "apply" for the following reason. The parameters of the model—as is clear from the fit performed below—are largely uncertain and no trustable conclusions can be effected as to the physical problems under study. At most, qualitative conclusions may be drawn. In our opinion, this unfortunate feature is shared by the other available models.

A. Estimation of the parameters of the model

It is clear that, because the crudeness of the model and also because of the speculative and somewhat uncertain domain of its applications, one can content oneself with orders of magnitude for the five constants of the model: m, μ_V, μ_S, g_V, g_S. Furthermore, even the elementary-particle-physics data cannot give a much more precise fit of these constants, but once more, orders of magnitude. For example, if one considers the effective mass of a typical quark, one can find a wide range of admitted numerical values (e.g., from 5 MeV to 355 MeV), the latter depending on the specific model adopted to interpret the raw experimental data. Therefore we proceed in a rough way and shall test (numerically) the sensitivity of our results to the variations of the constants of the model.

(1) The range of the forces between quarks must not exceed the dimension of a typical hadron, say a nucleon, so that

$$(\mu_S, \mu_V) \gtrsim 1\ \text{GeV}.$$

On the other hand, this range cannot be too small, otherwise the quarks would not interact enough to be bound and also to produce the baryon and meson spectra. In the following we adopt the *plausible* values

$$\mu_S=\mu_V \sim 1\ \text{GeV}. \tag{5.1}$$

(2) As to the *actual* mass m of the quarks, it is preferable to keep it as a *free parameter* ranging in the domain

$$20\ \text{GeV} \lesssim m \lesssim 10^4\ \text{GeV}, \tag{5.2}$$

as indicated in a recent review.[41]

(3) Let us now estimate the coupling constant g_S. Since the effect of the scalar gluons is only (in the spirit of the model) to endow the quarks with an effective mass, we have to look for a situation where such an effective mass can be (at least roughly) estimated. We expect that a quark *inside* a nucleon (or inside nuclear matter near close packing[42]) has—roughly—an effective mass of the order of one third of the nucleon mass, or 300–350 MeV. For quark matter at 0°K interacting through a scalar quantum field, the effective mass M is connected to m, to g_S, and to the Fermi momentum through[43]

$$\Gamma=\frac{2(1-M)}{(M/m)^3 G(\xi m/M)}, \tag{5.3}$$

where

$$\Gamma=\frac{4}{\pi} g_S \left(\frac{m}{\mu_S}\right)^2 d \tag{5.4}$$

(where d is the degeneracy factor due to internal symmetries), and ξ is the Fermi momentum corresponding to the quark numerical density at the nucleon close-packing density (expressed in units of the quark mass m); i.e., $\xi=0.79\ (\text{GeV})/m(\text{GeV})$. In Eq. (5.3) the function $G(x)$ is given by[43]

$$G(x)=x(x^2+1)^{1/2}-\sinh^{-1}x. \tag{5.5}$$

The results are shown in Table I. It should be noticed [see Eq. (5.4)] that only a particular combination of the constants g_S and μ_S appears in this evaluation, as in the work of Rafelski.[30]

TABLE I. Values of Γ for several quark masses. g_S^2 and g_S are calculated with $\mu_S=1$ GeV and fall within the range of values considered by Rafelski (Ref. 29) for m <200 GeV.

m (GeV)	Γ	g_S^2	g_S
20	0.912×10^5	179	13.4
40	0.736×10^6	361	19
60	0.249×10^7	544	23.3
80	0.592×10^7	726	26.9
100	0.116×10^8	908	30.1
200	0.927×10^8	1 820	42.7
500	0.145×10^{10}	4 550	67.5
1000	0.116×10^{11}	9 110	95.5
2000	0.928×10^{11}	18 200	135

(4) The remaining parameter to be estimated is g_V, the coupling constant of the vector gluons. Since these gluons are supposed to give rise to the spectrum of elementary particles, g_V will be fitted with one of the most precise data, i.e., with the spectrum of charmonium. At the lowest order, the exchange of vector gluons gives rise to a Yukawa-type potential of the form

$$V(r) = -\frac{g_V^2}{r}\exp(-\mu_V r) + V_0\,, \tag{5.6}$$

where V_0 is a constant supposed to take account of the remainder of the interaction. Using the generally accepted value of 1.5 GeV for the effective mass of the charmed quark, one finds

$$g_V \lesssim 0.5 \text{ (for all } V_0) \tag{5.7}$$

by fitting the first energy levels of charmonium (i.e., 3.1 GeV for the ground-state energy and 3.7 GeV for the first excited level[44]). In fact, there is a wide range of possibilities for g_V [satisfying Eq. (5.7)] depending on the value of V_0. However, a reasonable value is $g_V \sim 0.5$ (for $V_0 \sim 936$ MeV) since it is consistent with the coefficient of $1/r$ of the empirical potential

$$V(r) = -\alpha/r + \alpha' r \tag{5.8}$$

(with $\alpha \sim 0.3-0.6$) used in the phenomenology of elementary particles.

B. Quark core in neutron stars?

Let us now illustrate the above formalism on the case of cold (0 °K) quark matter expected to exist in the core of neutron stars. Although people are more and more interested in possible quark effects in nuclear matter,[45] it seems reasonable to admit that quarks manifest themselves at least when the density of matter reaches the point where the nucleons are close packed, i.e., when it reaches roughly nine times the nuclear saturation density.[46]

When $T = 0$ °K (or, equivalently, when $\beta \to \infty$) the basic equations of the system reduce to those obtained by Kalman[43]; i.e., the vector field does not play any role within the (statistical) assumptions (neglect of three-body correlations) used in this paper. This can be seen directly on Eqs. (B1)–(B3) [$\mathcal{A}^{(0)} \to 0$ when $\beta \to \infty$ in Eq. (B2), which property leaves us with the zeroth-order (in g_V) expression for F_{eq}]. From a physical point of view, this is also—in part—a consequence of our assumption that physical states are color singlets. Indeed, had this assumption been relaxed the vector gluons could have led to a collective (coherent) interaction (since $\langle A_B^\mu \rangle \neq 0$).

Thus Kalman's results[43] have to be used with the large values of Γ given in Table I. However, before discussing the resulting equations of state and their physical consequences, a few words are perhaps necessary as to the matching with the nuclear equation of state. As mentioned in Sec. I the equations of state in the two regimes (i.e., nuclear or quark regimes) are usually matched either merely at a given density[47] or by *assuming* a first-order phase transition (and a subsequent Maxwell's construction) between the two regimes. However this last procedure is possible only (i) when there is *actually* such a phase transition, therefore assuming also that three-body correlations are fully taken into account and (ii) when this phase transition is a *first-order* one, a property that might not be verified.

Let us discuss a little bit further these last points.

Although all the works on quark stars do assume the existence of such a phase transition, one can give arguments—not proofs—against this very existence. Indeed, (and except in the MIT bag model for which the considerations given below are irrelevant) the picture people have in mind is that of strongly interacting baryonic matter getting ionized into its quark constituents at high densities and/or temperatures. However, a simple model shows that—possibly—there is no phase transition: a hydrogen plasma may have two states (besides liquid and solid), at given pressure and density, a neutral one (hydrogen atoms ~ baryons), and an ionized one (electrons + protons ~ quarks); and one can pass *continuously* from one state to the other without any phase transition, of any order whatsoever; there is only a change in the degree of ionization and, in fact, there is no collective behavior of the system that could be indicated by, e.g., a large coherence length.[48] If the assumed phase transition does not exist there is no obvious way out for a correct matching of the equations of state describing the two regimes and we are left with the difficult problem of finding a *unique* quark equation of state valid in any case. In particular, not only three-body correlations must be completely taken into account (a very difficult problem, even in classical statistical mechanics[49]) but also bound states....

In order to get the equation of state for the quarks, the Fermi energy at which it is thought to be valid has to be evaluated. This can be achieved by taking the quark Fermi momentum at the close-packing baryonic density. For an *effective* quark mass of 0.35 GeV, this Fermi momentum is $f = 0.79$ GeV and the corresponding Fermi energy is $\epsilon_F = 0.86$ GeV. With these numbers, Kalman's equation of state is shown in Fig. 3 for three different masses for the quarks, i.e., 20 GeV, 60 GeV, 100 GeV. The effective mass versus the Fermi momentum is represented in Fig. 4 while

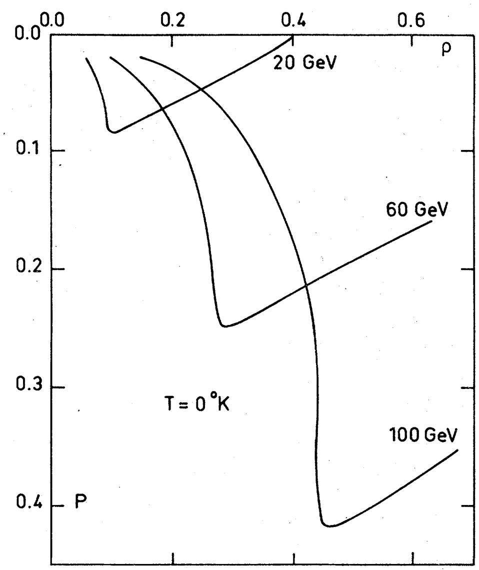

FIG. 3. The equation of state for cold quark matter for three different values of the quark mass (20 GeV, 60 GeV, 100 GeV). P and ρ are expressed in units of m^4 (m in GeV).

the energy density as a function of the particle density is depicted in Fig. 5. In Fig. 6 the energy per particle has been plotted against the particle density.

Let us now comment on these results. The equations of state shown in Fig. 3 exhibit a typical first-order phase transition; the negative pressures account for a *collective* confinement while the asymptotic branches behave like $p \sim \frac{1}{3}(\rho - \rho_B)$, i.e., as in the MIT bag equation of state (the values of ρ_B are however much higher: 1.23×10^{22} g/cm^3, 3.08×10^{24} g/cm^3, and 4.05×10^{25} g/cm^3 for $m = 20$ GeV, 60 GeV, and 100 GeV respectively[50]). The Max-

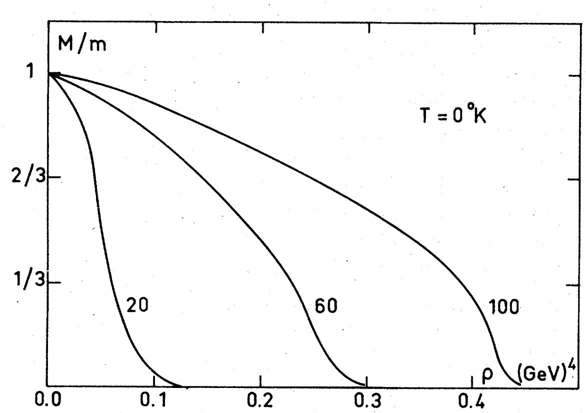

FIG. 4. The effective mass of the quarks is plotted against the Fermi momentum for three different values of the quark mass (20 GeV, 60 GeV, 100 GeV).

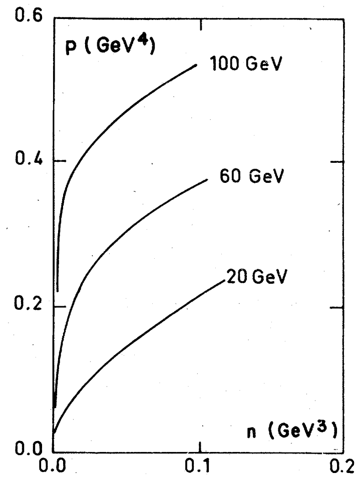

FIG. 5. The typical concavity of the $\rho(n)$ curve in the phase-transition region is shown for $m = 20$ GeV, 60 GeV, 100 GeV. ρ is expressed in units of m^4 and n in units of m^3 (m in GeV).

well's construction leaves us with the positive-pressure part of the asymptotic branches only.[51] As in Kalman's article[43] the nature of this phase transition (a collective bound state) is inferred from Fig. 6 from which it can be seen that the energy per particle is less than the quark mass (20 GeV in the case of Fig. 6) in the whole transition region; after, it grows indefinitely. Figure 5 shows the usual[52] concavity of the $\rho(n)$ curves in the transition region. An important feature is that it represents also a transition between two regimes of effective masses for the quarks (see Fig. 2 for a qualitative behavior). It is important to notice that the phase transition present in this quark equation of state has nothing to do with a quark/nuclear matter phase transition.

Assuming now the existence of *another* phase transition, i.e., between a nuclear matter phase (described, e.g., by the Chin-Walecka[53] or Bowers

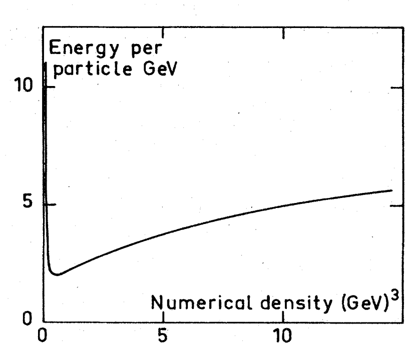

FIG. 6. The energy per particle as a function of the quark density shows that, in the phase-transition region, $E < m = 20$ GeV.

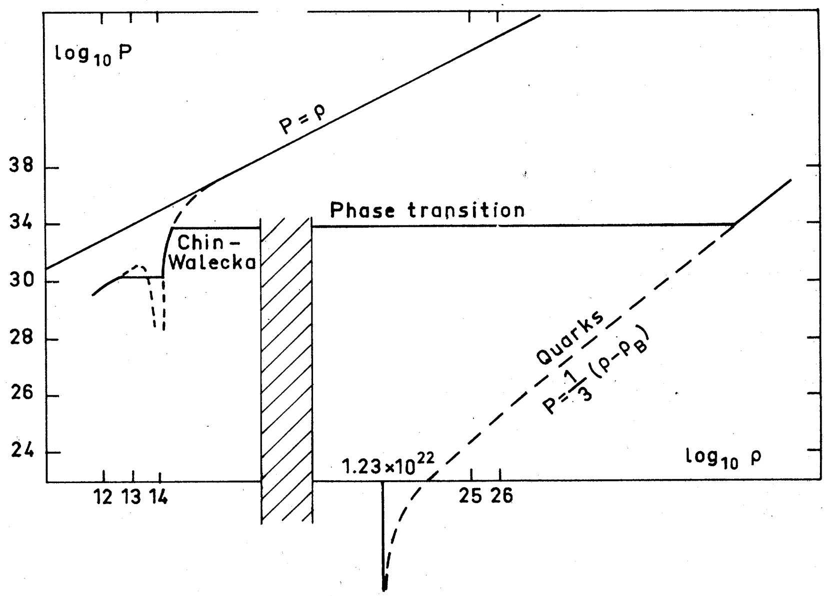

FIG. 7. Matching of the nuclear (a) and the quark (b) equation of state for $m = 20$ GeV. Notice the large plateau present in this model.

et al.[8] equation of state) and a quark phase [described here by $p = \frac{1}{3}(\rho - \rho_B)$], of first order, it is sufficient to assume a plateau of constant pressure (in the p-n^{-1} diagram) at equal chemical potential in both phases. The resulting equation of state is depicted in Fig. 7 and has a rather large plateau compared to other models. This is a typical feature of this model valid whatever the parameters; this is realized as long as the quark mass has a large value (leading to a large Γ) $\gtrsim 20$ GeV.

What are the consequences of this last feature as to neutron stars? Firstly, we observe that the higher the quark mass the larger the plateau. Secondly, unlike Bowers *et al.*[8] who claim that, in such a case, the equation of state is "soft", in our opinion, it is the opposite statement which seems to be correct. Indeed, *inside* the neutron star and starting from its center, one never passes by the transition region continuously but rather abruptly from a quark core to a neutron shell of much lower energy density: the transition region will be only a few Fermis thick. This is due to the gravitational field which *separates* the two phases in much the same way as it separates water from vapor on the earth.

Thus we are led—in this model—to the following picture of a neutron star: a very massive quark core (in a recent model[46] the quark core can contain 70% of the total mass) surrounded by a more conventional shell of neutron matter. The hardness of the equation of state in the quark core ($p \sim \frac{1}{3}\rho$) and the high energy density in this region can yield higher masses than usually considered (the maximum mass in the model alluded to above[46] is 2.92 $M_\odot$). However, the precise mass of such a neutron star is *extremely sensible* to the matching with the nuclear matter equation of state, and hence to this equation of state itself. Therefore it is probably much too early to draw definitive conclusions. Numerical estimations and the possible existence of a "third family of dense objects" will be the subject of a separate paper.

VI. DISCUSSION AND CONCLUSION

Let us now discuss the main points of this article, i.e., the model, the formalism, and its illustration.

(1) The model used thoughout this paper is a direct extrapolation of "low"-energy data. In fact, it is rather crude (although perhaps sufficient for astrophysics) and should be elaborated further in connection with elementary-particle data. This was, however, outside the scope of the paper. The result of this crudeness is reflected in the somewhat uncertain values obtained for the constants of the model, although they are within the range of other estimations.

Let us now come to the important question of asymptotic freedom. This property is not satisfied by the model, as is clear from previous studies.[54] However, the results of the preceding section show (on the basis of a numerical analysis and also after a glance at the basic equations considered at high densities and/or temperatures) that at *high energy densities* (equivalently, at high Fermi momenta or energies) the quark equation of state is of the same form as the one for the free gas, i.e., as $p \sim \frac{1}{3}\rho$. This is of course not the asymptotic-freedom property although this feature is true for asymptotically free models. Furthermore, since the effective mass of the quarks rapidly decreases as their density (or the Fermi energy) increases (see Fig. 4), for high-energy collisions, in which many quark partons can be excited, Bjorken scaling can be recovered. Hence, even though the model is not asymptotically free, it possesses features consistent with deep-inelastic experiments.

The next point to be discussed deals with the use of color singlets. This assumption, although directly suggested by the experimental data, is not necessarily true at much higher energies than those currently considered. However, this assumption can easily be relaxed as shown elsewhere where use is made of color-singlet states for equilibrium while off-equilibrium perturbations were colored: the simplest approximation was the Hartree approximation and not the next one that necessitates the use of the second set of equations of the BBGKY hierarchy. The case of equilibrium in Hartree approximation with non-color-singlet states will be studied elsewhere.

(2) In the derivation of our equations no infinity has appeared, a feature that might seem surprising. In fact, the vacuum terms, responsible for some of the usual infinities, have merely been ignored. For instance, the equation (4.29) for $f_{eq}(p)$ contains a vacuum term not written in this equation. This term introduces infinities in several places and its omission corresponds to using a *new* definition for the covariant Wigner function, namely, $:F:$, where the symbol $::$ represents the usual normal product.[55] The use of such a new definition allows the separation of the infinities from the relevant statistical quantities. Let us specify this point more precisely by looking at the generating equation satisfied by $\tilde{F}_{new}$. A straightforward calculation yields[32]

$$\frac{2^5 g_S}{(2\pi)^4}\int d^4x' d^4p' [\langle 0|\tilde{F}_{old}(x,p-p')\phi(x')|0\rangle - F_{vac}(p-p')\phi(x')]\exp[-2ip'\cdot(x-x')],$$

$$\frac{2^5 g_V}{(2\pi)}\int d^4x' d^4p' [\langle 0|\tilde{F}_{old}(x,p-p')\Lambda_B\gamma_\mu A^{\mu B}(x')|0\rangle - F_{vac}(p-p')\Lambda_B\gamma_\mu A^{\mu B}(x')]\exp[-2ip'\cdot(x-x')]$$

for the two new terms added to the generating equation. We have set $F_{vac} \equiv \langle 0|\tilde{F}|0\rangle$. They contain two vacuum terms and two other terms *linear* in the fields ϕ and A^μ_B. Furthermore, they do not depend on F_{new} (i.e. they are density independent) and therefore they refer to *vacuum effects*. Actually the term involving ϕ leads to the vacuum polarization by the scalar gluons and can be renormalized as usual.[32,56] The second term, involving A^μ_B, also leads to a vacuum polarization effect of the vector gluons. However, it cannot be renormalized since a theory with a massive vector field *is not* renormalizable when the coupling occurs through a nonconserved current. Nevertheless, the infinity carried by this term can be removed in much the same way as used by Chin[56] who dealt with a model similar to ours, although aimed at a description of nuclear rather than quark matter. His idea is to add specific counterterms in the Lagrangian at each order of approximation (see also Ref. 31). For instance, if we use a sequence of approximations that consists in truncating the successive correlations[57] of the relevant statistical functions, at each step one (or several) counterterms would be added to the Lagrangian. Doing so one is led to an infinity of counterterms to be added, as expected from a nonrenormalizable theory. This way of removing infinities is, of course, paid by the arbitrariness of the constants introduced with the counterterms. Moreover, it is not sure at all that another such "renormalization" scheme would lead to the same physical results. It should be added that, as is well known, the infinities brought by the scalar field, are easily removed by adding to the Lagrangian counterterms of the form $\alpha\phi + \beta\phi^2 + \gamma\phi^3 + \lambda\phi^4$ so that the choice made by Chin[56] (the constants $\alpha, \beta, \gamma, \lambda$ once renormalized are chosen to be zero; i.e., the infinities are exactly removed; in fact, they must be determined from experiments, at least in principle) is somewhat arbitrary. Therefore, it seems preferable either to ignore them *or* to keep arbitrary constants (a finite number at each step of the approximation scheme) in the model, with—in this last case—no possibility of numerical conclusion.

Consequently we refer—for these questions of quantum fluctuations—to the very interesting article by Chin.[56] We should also add that, insofar as our model is merely phenomenological, this problem is perhaps not so acute as it would be in a complete theory. There is, however, another possibility considered elsewhere[23]: one can look at the massive vector field A^μ_B as an approximation of a *non-Abelian* gauge field; the renormalization process is then linked to the one of gauge fields.

(3) The formalism used throughout this paper is very close to the one used in classical plasma physics.[58] This similarity suggests strongly approximation schemes generally not considered in quantum field theory, such as the neglect of correlations from a given order, the use of cluster expansions, etc.

In particular the relativistic quantum BBGKY hierarchy for the quark plasma was obtained and truncated (i) in the Hartree approximation as to the scalar gluons and (ii) neglecting the three-body correlations as to the vector gluons. In fact, the approximation (i) was based on the idea that the confined state of quarks was essentially a collective state while two-quark correlations were taken into account in (ii). It should be remarked that the quark–scalar-gluon or scalar-gluon–scalar-gluon correlations could also be taken into account as is done in Ref. 34: in order not to complicate the algebra too much, it has been preferred to take account of the collective aspect only. As to the vector gluons, the lowest orders in g_V^2 have been considered (since $g_V<1$) for the sake of illustration (see Appendix B).

Finally, with the approximations used, our quark plasma was very similar to a QED plasma with the following modifications: $m\to M$, $e^2\to g_V^2$, $\Omega_p^2\to 2\Omega_p^2$. It follows that one can use previous studies by Tsytovich,[40] Bezzerides and Dubois,[59] or Melrose.[60]

(4) In Sec. V the above model and formalism have tentatively been applied—illustrated—on the case of neutron stars. Here again it should be stressed that our fit of the constants of the model is extremely rough and thus our numerical results are quite uncertain. However—although numerically crude—our results give perhaps a correct idea of the *qualitative* behavior of the quark plasma. In fact the main qualitative conclusion (in the framework of this model) is the existence of a very large plateau that links the nuclear and the quark regimes. This large plateau leads to a massive quark core, in neutron stars, separated from neutron matter by an abrupt jump. This typical characteristic is due to the scalar field and more particularly to the large value of Γ.

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APPENDIX A

From Dirac's equations obtained from Eq. (2.1) one gets

$$i\partial_\mu \tilde{J}^\mu_B = g_V \bar{\psi}[\Lambda_D, \Lambda_B]\gamma^\mu \psi A^D_\mu\,, \tag{A1}$$

where the color current $\tilde{J}^\mu_B$ is given by Eq. (2.18). Using now[61]

$$[\Lambda_D, \Lambda_B] = f_{DBC}\Lambda_C\,, \tag{A2}$$

where the f_{DBC}'s are the structure constants[61] of SU_3, Eq. (A1) can be rewritten as

$$\partial_\mu \tilde{J}^\mu_B = -ig_V \mathrm{Tr}\left(\int d^4p \tilde{F}(x,p) A^\mu_D(x)\gamma_\mu \Lambda_C f_{DBC}\right). \tag{A3}$$

Once multiplied from the right by $A^K_\lambda(x')$, averaging and using the approximation (4.21), Eq. (A3) provides

$$\partial_\mu \chi^{\mu\lambda}_{BK} = -ig_V f_{DBC}\mathrm{Tr}\left(\int d^4p F(x,p)\Lambda_C \gamma_\mu \mathfrak{A}^{\mu\lambda}_{DK}\right). \tag{A4}$$

Owing to the fact that (i) $F(x,p)$ is necessarily proportional to the unit matrix in color space and (ii) $\mathrm{Tr}\Lambda_c=0$, it follows that the right-hand side of this last equation vanishes identically,

$$\partial_\mu \chi^{\mu\lambda}_{BK} = 0\,. \tag{A5}$$

Taking now the average value of the Klein-Gordon equation satisfied by A^μ_B [after it has been multiplied by $A^\lambda_K(x')$ from the right], one obtains

$$(\Box + \mu_V^2)\partial_\mu \mathfrak{A}^{\mu\lambda}_{BK} = 0\,, \tag{A6}$$

a solution of which is

$$\partial_\mu \mathfrak{A}^{\mu\lambda}_{BK} = 0\,. \tag{A7}$$

Thus we have shown the consistency of this Lorentz-type condition with our model, within the approximations considered.

APPENDIX B

In this appendix the various statistical quantities are calculated in a state of thermodynamical equilibrium, neglecting three-body correlations and at orders g_V^2.

Let us start with Eq. (4.18) for $F_{\mathrm{eq}}(p)$, the zeroth-order Wigner function given by Eqs. (4.28) and (4.29). Inserting Eq. (4.24) into Eq. (4.18), one obtains

$$F_{eq}(p)=\frac{\gamma\cdot p+M}{4M}f_{eq}(p)-g_V{}^2\frac{\gamma\cdot p+M}{p^2-M^2}\int d^4p'\gamma^\mu\Lambda_B\left[\frac{\gamma\cdot(p-p')+M}{(p-p')^2-M^2+i\epsilon}\Lambda_A\gamma^\nu\frac{\gamma\cdot(p-p'/2)+M}{4M}f_{eq}(p-p'/2)+\frac{\gamma\cdot(p-p')+M}{4M}f_{eq}(p-p')\Lambda_A\gamma^\nu\frac{\gamma\cdot(p-p'/2)+M}{(p-p'/2)^2-M^2-i\epsilon}\right]\times\hat{\mathcal{Q}}^{AB}_{\mu\nu}(p'), \quad \text{(B1)}$$

where $\hat{\mathcal{Q}}^{AB}_{\mu\nu}$ must be evaluated at order zero in g_V, i.e.,

$$\hat{\mathcal{Q}}^{AB(0)}_{\mu\nu}(p')=\frac{\delta^{AB}\delta(p'^2-\mu_V{}^2)\Delta_{\mu\nu}(p')}{\exp(\beta u^\mu p'_\mu)-1}. \quad \text{(B2)}$$

The expression for $\hat{\chi}^\lambda(k,p)$ at order $g_V{}^2$ involves the knowledge of $F_{eq}(p)$ at order zero and the one of $\hat{\mathcal{Q}}$ at order one. On the other hand, from Eq. (4.25) $\hat{\mathcal{Q}}$ can be obtained at order $g_V{}^2$

$$\hat{\mathcal{Q}}^{\mu\lambda}_{BC}(k)=\hat{\mathcal{Q}}^{\mu\lambda(0)}_{BC}(k)-\frac{4\pi g_V{}^2}{k^2-\mu_V{}^2}\mathrm{Tr}\left\{\int d^4p\gamma^\mu\Lambda_B\left[\frac{\gamma\cdot(p-k/2)+M}{(p-k/2)^2-M^2+i\epsilon}\Lambda_A\gamma^\nu\frac{\gamma\cdot(p+k/2)+M}{4M}f_{eq}(p+k/2)+\frac{\gamma\cdot(p-k/2)+M}{4M}\Lambda_A\gamma^\nu\frac{\gamma\cdot(p+k/2)+M}{(p+k/2)^2-M^2-i\epsilon}f_{eq}(p-k/2)\right]\right\}\times\hat{\mathcal{Q}}^{\lambda A(0)}_{\nu C}(k). \quad \text{(B3)}$$

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