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Quantum limits on resonant-mass gravitational-radiation detectors

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The methods of quantum detection theory are applied to a resonant-mass gravitational-radiation antenna. Quantum sensitivity limits are found which depend strongly on the quantum state in which the antenna is prepared. Optimum decision strategies and their corresponding sensitivities are derived for some important initial states. The linear detection limit ($E_{\min} \sim \hbar\omega$) is shown to apply when the antenna is prepared in a coherent state. Preparation of the antenna in an excited energy eigenstate or in a state highly localized in position or momentum space leads to increased sensitivity. A set of minimum-uncertainty states for phase-sensitive detection is introduced.

I. INTRODUCTION

The problem of detection of gravitational radiation is a difficult one. Even optimistic calculations of possible sources suggest extreme sensitivity requirements for resonant detectors.[1] The state of the experimental science is now beginning to approach fundamental limits imposed by quantum-mechanical effects. These considerations have led to considerable interest in methods for optimizing the measurement strategy in order to obtain the highest sensitivity within the quantum limits. In view of this it is useful to perform explicit calculations of the quantum limits under various conditions.

All past resonant antenna systems and apparently all those presently under development use linear amplification to detect the state of the antenna. It is well known that linear amplifiers have a sensitivity limit imposed by quantum mechanics.[2-5] We will refer to this limit as the linear detection limit. Braginskiǐ and co-workers[6-9] have shown by uncertainty-principle arguments that the linear detection limit may be surpassed by performing measurements of the energy eigenstate of an antenna and have suggested devices which might perform the desired measurement. These and other energy measuring devices have been analyzed further by Unruh.[10] Moncrief[11] and Unruh[12] have suggested the possible usefulness of coherent states in evading the linear limit. Thorne and co-workers[13] have described conceptually a phase-sensitive device for improving on the linear limit. Braginskiǐ and co-workers[14] have also proposed a phase-sensitive or "stroboscopic" technique.

In this paper we do not attempt to describe any specific device for measuring the state of an antenna. Rather, we attempt to consider all possible measurements in order to find the fundamental limits which arise once an initial state for the antenna has been chosen, using the techniques of quantum detection theory. This theory has been developed along the lines of classical detection theory and a large body of literature exists on the subject.[15-17] The results we obtain are consistent with those obtained, in specific cases, by the previously cited authors, but are of general applicability.

A resonant-mass gravitational-radiation antenna is a damped harmonic oscillator which couples to the Riemann tensor via the nonvanishing mass quadrupole moment of a vibrational eigenmode. As a simple model of such an antenna one may consider two point particles each of mass $m/2$ connected by a spring of length l along the x axis. The classical equation of motion is[18,19]

$$m\frac{d^2x}{dt^2}+\frac{m\omega}{Q}\frac{dx}{dt}+m\omega^2x=-c^2lmR_{x0x0}(t)\equiv -F(t)\,, \qquad (1)$$

where x is the change in separation of the masses, R_{x0x0} is a component of the Riemann tensor, c is the speed of light, and Q is the quality factor which characterizes the damping of the oscillator. The motion induced by a burst of gravitational radiation which occurs in the interval $-\tau \le t \le 0$ is

$$x(t)=\mathrm{Re}\left[(U_i+U_s)\exp\left(-\frac{\omega t}{2Q}-i\omega t\right)\right] \text{ for } t\ge 0\,, \qquad (2)$$

where U_s is a complex amplitude given by

$$U_s\equiv -\frac{i}{m\omega}\int_{-\tau}^{0}F(t')e^{i\omega t'}e^{\omega t'/2Q}dt' \qquad (3)$$

and U_i is the complex amplitude of the antenna before the pulse arrives. We have assumed that the Q is sufficiently high that the frequency shift due to finite Q may be neglected. We see that the pulse merely displaces the complex amplitude of the antenna by the amount U_s. A convenient measure of the signal energy available from the antenna is the quantity E_s given by

$$E_s = \tfrac{1}{2}m\omega^2 |U_s|^2 = \frac{1}{2m} \left| \int_{-\tau}^{0} F(t') e^{i\omega t'} e^{\omega t'/2Q} dt' \right|^2 . \quad (4)$$

E_s will be called the signal energy and is the energy that would be deposited in an antenna originally at rest. It is important to realize that the energy change imparted to an excited antenna ($U_i \neq 0$) may be much greater than E_s and may be either positive or negative. The signal causes a displacement $U_s = U_f - U_i$ while the energy change is

$$\tfrac{1}{2}m\omega^2(|U_f|^2 - |U_i|^2) = E_s + m\omega^2 \mathrm{Re}(U_s U_i^*).$$

The interference term may easily be larger than E_s if U_i is sufficiently large and has the proper phase.

The quantum limit for linear amplifiers has been well studied. Specific models for linear amplifiers have been given a full quantum treatment.[2] Model-independent calculations which assume only linearity have been studied with both the uncertainty principle[3] and with a full quantum treatment.[4] The application of these results to resonant gravitational-radiation detectors has also been studied.[5] The result is that in order to detect a pulse with an antenna whose state is measured with a linear amplifier the signal energy must satisfy $E_s \gtrsim \hbar\omega$. This is what we have called the linear detection limit.

II. QUANTUM DETECTION THEORY

We will consider a gravity-wave antenna to consist of a single-mode harmonic oscillator with no coupling to any other modes. We are thus assuming an infinite mechanical Q and no thermal noise. We treat the gravitational-radiation field as a classical force which couples to the oscillator. This assumption is very well justified since the coupling between the radiation field and the antenna is so weak. We are interested in detecting short pulses of gravitational radiation with a large spectral density at the oscillator frequency.

The technique is as follows. The antenna is prepared in some initial state. Next the antenna is allowed to interact with the radiation field for some length of time. A measurement is then performed on the antenna. Finally a detection algorithm is used to make one of two conclusions. One conclusion is that no gravity wave pulse has arrived; we will refer to this as the null conclusion. Alternatively the conclusion will be that a pulse has arrived; this will be called the alternative or positive conclusion. In general, conclusions as to the size of the pulse, its time of arrival, and so forth may be desired. We will restrict our attention to the more primitive question of whether or not a pulse has arrived at all. The combination of measurement and detection algorithm will be referred to as a decision strategy. Our task is to determine what the optimum decision strategy is and what sensitivity limit it leads to. We will find that this will depend on the initial state of the antenna.

In order to make analytical progress a criterion must be found for assessing the value of a given decision strategy. To facilitate this we define two probabilities. The detection probability Q_D is the probability that a given decision strategy will result in the positive conclusion under the hypothesis that a pulse has in fact arrived. Q_D is thus the "efficiency" of the detector. The false-alarm probability Q_0 is the probability that the decision strategy will result in the positive conclusion under the hypothesis that no pulse has arrived. Q_0 is thus the probability of "accidentals." Clearly it is desirable to make Q_D high and Q_0 low.

The optimum strategy will be found in the following way. A tolerable false-alarm probability Q_0 is prescribed. The decision strategy which maximizes Q_D is then found. A decision strategy which maximizes Q_D for a prescribed Q_0 is said to satisfy the Neyman-Pearson criterion.

Binary decision theory has been studied for quantum-mechanical systems by several groups[15,16,17]; we follow closely the book by Helstrom.[15] In order to optimize the decision strategy one must define a set of possible measurements. The set of measurements considered in the theory is the set of "probability operator measures."[15,20] This set includes not only all conventional "projection valued measures" such as energy and position measurements, but also more general types of measurements. For example, we can imagine a measurement which is made by allowing a second quantum system to interact with a primary system. A "projection valued" measurement on the second system will not in general be describable as a "projection valued" measurement on the first. Such a measurement will be a member of the class of "probability operator measures" on the primary system, however, and thus is considered in our optimization.

To set up the problem we must first compute the density operator which describes the state of the system at the time of measurement under the hypothesis that no gravity wave pulse has been received. This density operator will be labeled ρ_0. ρ_1 will refer to the density operator under the alternative hypothesis. We wish to find the set of Neyman-Pearson decision strategies for distinguishing between ρ_0 and ρ_1. This complex problem has been reduced to the following eigenvalue problem[15]:

$$(\rho_1 - \lambda\rho_0)|\eta_i\rangle = \eta_i|\eta_i\rangle . \qquad (5)$$

We must solve for the eigenvalues η_i and eigenstates $|\eta_i\rangle$ of the operator $\rho_1 - \lambda\rho_0$ where λ is an arbitrary, real Lagrange multiplier. Each value of λ will correspond to a different decision strategy which satisfies the Neyman-Pearson criterion. Thus, λ will parametrize a curve of optimal strategies in the Q_0, Q_D domain. Once the eigenvalue problem is solved, Q_0 and Q_D may be computed according to

$$Q_0 = \sum_{\eta_i > 0} \langle\eta_i|\rho_0|\eta_i\rangle ,$$
$$Q_D = \sum_{\eta_i > 0} \langle\eta_i|\rho_1|\eta_i\rangle . \qquad (6)$$

(Here we have assumed for simplicity that $\rho_1 - \lambda\rho_0$ has no zero eigenvalues.) The decision strategy to be followed for each λ is to measure the operator $\rho_1 - \lambda\rho_0$; the measurement will give one of the eigenvalues η_i, if η_i is negative the null conclusion is made, if η_i is positive the alternative conclusion is made.

The eigenvalue problem is easily solved if ρ_0 and ρ_1 represent pure states. Suppose $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and $\rho_1 = |\psi_1\rangle\langle\psi_1|$. We try to find states $|\eta_i\rangle$ which are linear combinations of $|\psi_0\rangle$ and $|\psi_1\rangle$. After diagonalizing the two by two matrix representing $|\psi_1\rangle\langle\psi_1| - \lambda|\psi_0\rangle\langle\psi_0|$ we solve for $Q_0(\lambda)$ and $Q_D(\lambda)$. Eliminating λ we obtain[21]

$$Q_D = \begin{cases} [\sqrt{Q_0}\,|\gamma| + (1-Q_0)^{1/2}(1-|\gamma|^2)^{1/2}]^2, & 0 \le Q_0 \le |\gamma|^2 \\ 1, \quad |\gamma|^2 \le Q_0 \le 1 , & \end{cases} \qquad (7)$$

where $\gamma \equiv \langle\psi_0|\psi_1\rangle$ is just the overlap between the two possible states. Figure 1 is a plot of equation (7) for several values of the overlap. Note that if the overlap is small it is easy to distinguish states and we can find decision strategies with small Q_0 and large Q_D. For large overlaps $|\gamma| \lesssim 1$ it is hard to distinguish the two states and the locus approaches a straight line which characterizes a decision strategy based on random guessing. In the case of gravity wave reception we expect a very low event rate. It will thus be necessary to maintain a very low false-alarm probability. Consequently the case where $Q_0 = 0$ will be of special interest to us.

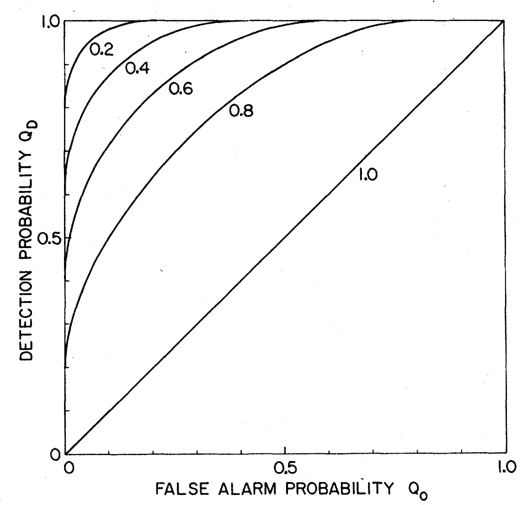

FIG. 1. Loci of Neyman-Pearson strategies in Q_0, Q_D domain for binary decisions between pure states $|\psi_0\rangle$ and $|\psi_1\rangle$. The curves are labeled by $|\gamma|^2 = |\langle\psi_0|\psi_1\rangle|^2$.

III. COMPUTATION OF DENSITY OPERATORS

The Hamiltonian describing the system is

$$H = \tfrac{1}{2}m\omega^2\hat{x}^2 + \frac{\hat{p}^2}{2m} + F(t)\hat{x} - \tfrac{1}{2}\hbar\omega$$

or

$$H = \hbar\omega a^\dagger a + F(t)(\hbar/2m\omega)^{1/2}(a^\dagger + a) , \qquad (8)$$

where $a^\dagger = (m\omega/2\hbar)^{1/2}\hat{x} - i(2m\hbar\omega)^{-1/2}\hat{p}$. The term $(\hbar/2m\omega)^{1/2}F(t)(a^\dagger + a)$ is a classical driving term representing the interaction with the gravitational-radiation field. For convenience we have subtracted away the zero-point energy.

Suppose the oscillator is prepared at $t = -\tau$ in a state described by the density operator ρ_i. Further, suppose the oscillator is allowed to interact with the radiation field for a time τ. If $F(t)$ is zero in this interval then at $t = 0$ the state of the oscillator will be

$$\rho_0 = e^{-i\omega\tau a^\dagger a}\rho_i e^{i\omega\tau a^\dagger a} . \qquad (9)$$

Alternatively, if $F(t) \ne 0$ in the interaction interval then the state will evolve differently. Fortunately, the quantum harmonic oscillator with classical driving force is exactly soluble. The density operator at $t = 0$ is given by[22]

$$\rho_1 = e^{-i\omega\tau a^\dagger a}D(\mu e^{i\omega\tau})\rho_i D^\dagger(\mu e^{i\omega\tau})e^{i\omega\tau a^\dagger a} , \qquad (10)$$

where $D(\mu)$ is referred to as the displacement operator and is given by

$$D(\mu) \equiv \exp(\mu a^\dagger - \mu^* a) \qquad (11)$$

and μ is a normalized complex amplitude given by

$$\mu = -\frac{i}{(2m\hbar\omega)^{1/2}} \int_{-T}^{0} F(t')e^{i\omega t'}\,dt' . \tag{12}$$

By comparison with Eq. (3) we see that $\mu = (m\omega/2\hbar)^{1/2} U_s$ where U_s is the classical displacement amplitude for $Q \to \infty$. The signal energy E_s is given by

$$E_s = \tfrac{1}{2} m\omega^2 |U_s|^2 = \hbar\omega |\mu|^2 . \tag{13}$$

In the Appendix it is shown that

$$e^{-i\omega\tau a^\dagger a} D(\mu) e^{i\omega\tau a^\dagger a} = D(\mu e^{-i\omega\tau}) , \tag{14}$$

so we may rewrite Eq. (10) as

$$\rho_1 = D(\mu) e^{-i\omega\tau a^\dagger a} \rho_i e^{i\omega\tau a^\dagger a} D^\dagger(\mu) \tag{15}$$

and recalling Eq. (9) we have

$$\rho_0 = e^{-i\omega\tau a^\dagger a} \rho_i e^{i\omega\tau a^\dagger a} , \qquad \rho_1 = D(\mu)\rho_0 D^\dagger(\mu) . \tag{16}$$

IV. COHERENT STATES

An important set of initial states is the set of coherent states.[23] These are minimum-uncertainty states for the position and momentum operators. They are the closest analogs of classical oscillator states. In the position representation the coherent states are Gaussian wave packets with the same width as the ground-state wave function moving about in the oscillator along classical trajectories. We may generate the set of coherent states by displacing the ground state by the complex amplitude β. We define the coherent state $|\beta\rangle$ by

$$|\beta\rangle \equiv D(\beta)|0\rangle . \tag{17}$$

An oscillator could be prepared in a coherent state by starting in the ground state and driving it with a classical source. In the energy representation the coherent state $|\beta\rangle$ is given by

$$|\beta\rangle = \exp(-\tfrac{1}{2}|\beta|^2) \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle .$$

Suppose the oscillator has been prepared in an initial state $|\psi_i\rangle = |\beta'\rangle$, a coherent state. Under the null hypothesis we will have

$$\begin{aligned} |\psi_0\rangle = e^{-i\omega\tau a^\dagger a}|\beta'\rangle &= e^{-i\omega\tau a^\dagger a} D(\beta') e^{i\omega\tau a^\dagger a}|0\rangle \\ &= D(\beta' e^{-i\omega\tau})|0\rangle \\ &= |\beta' e^{-i\omega\tau}\rangle . \end{aligned}$$

Thus the state evolves into a new coherent state $|\beta\rangle$ where $\beta \equiv \beta' e^{-i\omega\tau}$. Under the null hypothesis we will have $\rho_0 = |\beta\rangle\langle\beta|$. Under the alternative hypothesis we will have $\rho_1 = D(\mu)|\beta\rangle\langle\beta|D^\dagger(\mu)$. In this example both ρ_0 and ρ_1 are pure states. To solve the problem we need only compute the overlap between the two states. We have

$$\begin{aligned} \gamma \equiv \langle\psi_0|\psi_1\rangle &= \langle\beta|D(\mu)|\beta\rangle \\ &= \langle 0|D^\dagger(\beta)D(\mu)D(\beta)|0\rangle . \end{aligned}$$

But two important properties of the displacement operator are

$$D(\mu)D(\beta) = D(\mu+\beta)\exp[\tfrac{1}{2}(\beta^*\mu - \beta\mu^*)] \tag{18}$$

and

$$D^\dagger(\beta) = D(-\beta) . \tag{19}$$

Using these we find

$$D^\dagger(\beta)D(\mu)D(\beta) = D(\mu)\exp(\beta^*\mu - \beta\mu^*) , \tag{20}$$

so $\gamma = \langle 0|D(\mu)|0\rangle \exp[2i\,\mathrm{Im}(\beta^*\mu)]$. In the Appendix it is shown that $\langle 0|D(\mu)|0\rangle = \exp(-\frac{1}{2}|\mu|^2)$ so we conclude $\gamma = \exp[-\frac{1}{2}|\mu|^2 + 2i\,\mathrm{Im}(\beta^*\mu)]$ and $|\gamma|^2 = \exp(-|\mu|^2) = \exp(-E_s/\hbar\omega)$. Now Eq. (7) may be used to compute Q_D in terms of Q_0. For the case $Q_0 = 0$ we have

$$Q_D = 1 - \exp(-E_s/\hbar\omega) \quad \text{(coherent states, } Q_0 = 0\text{)}. \tag{21}$$

We note that the expression for Q_D is independent of β. This immediately tells us that if the oscillator is prepared in a coherent state, there is no improvement in sensitivity obtained by preparing it in a highly excited state. In Fig. 2 we have plotted Eq. (21). We will define a minimum detectable pulse energy $E_{\min}$ as follows: For $Q_0 = 0$ we will maximize Q_D and find the minimum signal energy such that the detection probability is at least 50% for all $E_s \geq E_{\min}$. For coherent initial states the result is

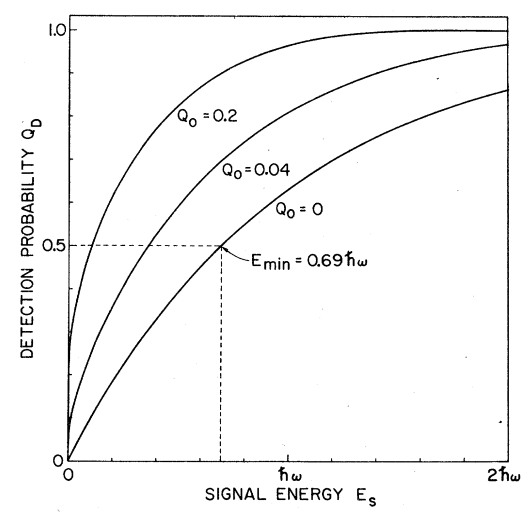

FIG. 2. Detection probability Q_D vs signal energy E_s for an oscillator prepared in a coherent state. Curves are shown for several values of the false-alarm probability Q_0.

$$E_{min} = \hbar\omega \ln 2 \quad \text{(coherent states)}. \tag{22}$$

This is very close to the quantum limit for linear amplification. We conclude that the linear detection limit cannot be improved on by using coherent initial states.

V. ENERGY EIGENSTATES

Several schemes have been proposed for gravity-wave detection using energy measurements.[6-10] It is therefore natural to consider the case in which the oscillator is prepared in an energy eigenstate. Assume $\rho_i = |n\rangle\langle n|$, where $|n\rangle$ is the energy eigenstate with eigenvalue $n\hbar\omega$. We have

$$\rho_0 = e^{-i\omega\tau a^\dagger a}|n\rangle\langle n|e^{i\omega\tau a^\dagger a} = |n\rangle\langle n|$$

and

$$\rho_1 = D(\mu)|n\rangle\langle n|D^\dagger(\mu).$$

Again the problem involves only pure states and we need $\gamma = \langle n|D(\mu)|n\rangle$. It is shown in the Appendix that

$$\langle n|D(\mu)|n\rangle = L_n(|\mu|^2)e^{-|\mu|^2/2},$$

where $L_n(x)$ is the Laguerre polynomial of order n. For the case $Q_0 = 0$ we have

$$Q_D = 1 - e^{-E_s/\hbar\omega}[L_n(E_s/\hbar\omega)]^2 \quad (n \text{ states, } Q_0 = 0). \tag{23}$$

Recalling that $L_0 = 1$ we see that this result reduces to Eq. (21) in the ground state. This is as it must be since the ground state is a coherent state. In order to do better than the linear detection limit one must prepare the oscillator in an excited state. Figure 3 is a plot of Q_D vs E_s for the cases $n = 10$ and $n = 0$. For large n and small argument we may approximate the Laguerre polynomial by

$$L_n(x) \approx e^x J_0(2\sqrt{nx}), \quad 0 \leq x \ll n.$$

To find E_{min} we set $1 - e^{-E_{min}/\hbar\omega}[L_n(E_{min}/\hbar\omega)]^2 = \frac{1}{2}$. Using the above approximation we get

$$E_{min} \approx \frac{0.32}{n}\hbar\omega \quad (n \text{ states}). \tag{24}$$

We see that it is possible to reduce the quantum limit as far as we like by preparing the oscillator in a highly excited energy eigenstate. The increased sensitivity is due to the classical interference term mentioned in the introduction. In quantum language we say that the incident radiation field has stimulated emission or absorption of quanta by the antenna. This result is in agreement with the result obtained by Braginskiĭ using uncertainty-principle arguments.[6]

Now we wish to examine what the decision strategy is for this case, that is, how does one actually achieve the optimum sensitivity. As discussed in Sec. II one measures the operator

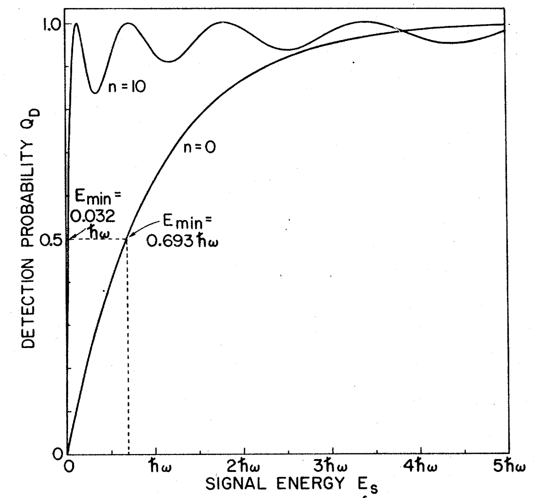

FIG. 3. Detection probability vs signal energy E_s for an oscillator prepared in the energy states with $n = 10$ and $n = 0$.

$$\rho_1 - \lambda\rho_0 = D(\mu)|n\rangle\langle n|D^\dagger(\mu) - \lambda|n\rangle\langle n|.$$

Let us examine this for the case $Q_0 = 0$. The appropriate value of λ for $Q_0 = 0$ is $\lambda \to \infty$. As $\lambda \to \infty$, ρ_1 may be ignored in the operator $\rho_1 - \lambda\rho_0$. The optimum measurement for $Q_0 = 0$ is a measurement of the projection operator $\rho_0 = |n\rangle\langle n|$. This may be accomplished by performing an energy measurement. If the measured eigenvalue is different from n (the initial value) then the positive conclusion is made, otherwise the null conclusion is made. It is clear that the false-alarm probability is zero since if no pulse occurs the measurement will always give the eigenvalue n. On the other hand, the probability that the measurement will yield the value n after a pulse has arrived will be $p = |\langle n|\psi_1\rangle|^2$ or $p = |\langle n|D(\mu)|n\rangle|^2 = |\gamma|^2$ leading to a detection probability $Q_D = 1 - p = 1 - |\gamma|^2$ in agreement with the above result.

VI. WAVE-PACKET STATES

Perhaps the most ubiquitous nonlinear detection scheme used in laboratory practice is synchronous or phase-sensitive detection. In classical linear detection one measures the displacement of the oscillator $x(t)$. In phase-sensitive detection one measures $X_1(t)$ or $X_2(t)$, where $x(t) = X_1(t)\cos\omega t + X_2(t)\sin\omega t$. It has been realized for some time that it is possible to escape the linear detection limit by doing a "quantum-mechanical" phase-sensitive measurement.[24]

To examine this possibility we introduce operators which correspond to the classical variables $X_1(t)$ and $X_2(t)$. We follow Thorne *et al.*[13] by defining

$$\begin{aligned} \hat{X}_1 &\equiv a^\dagger e^{-i\omega t} + a e^{i\omega t} \\ &= (2m\omega/\hbar)^{1/2}[\hat{x}\cos\omega t - (\hat{p}/m\omega)\sin\omega t], \\ \hat{X}_2 &\equiv i(a^\dagger e^{-i\omega t} - a e^{i\omega t}) \\ &= (2m\omega/\hbar)^{1/2}[\hat{x}\sin\omega t + (\hat{p}/m\omega)\cos\omega t]. \end{aligned} \tag{25}$$

Note that $\hat{X}_1\cos\omega t + \hat{X}_2 \sin\omega t = (2m\omega/\hbar)^{1/2}\hat{x}$. These operators are explicitly time dependent and it must be remembered that we are working in the Shrödinger picture. A measurement of the $\hat{X}_1$ operator corresponds to a position measurement at $\omega t = 0$ and to a momentum measurement at $\omega t = \pi/2$. Since $[\hat{X}_1, \hat{X}_2] = 2i$ there is an uncertainty principle which reads $\Delta\hat{X}_1 \Delta\hat{X}_2 \geq 1$ where $\Delta\hat{X}_1 \equiv (\langle \hat{X}_1^2\rangle - \langle \hat{X}_1\rangle^2)^{1/2}$.

Now let us introduce a set of states which will turn out to be minimum-uncertainty states for $\hat{X}_1$ and $\hat{X}_2$. First, let us consider the unitary operator

$$S(z) \equiv \exp[\tfrac{1}{2}z(a^\dagger)^2 - \tfrac{1}{2}z^* a^2].$$

In the Appendix it is shown that if $|\psi\rangle$ is the state of a system and r is a real number, then $S(r)|\psi\rangle$ represents the same system compressed in position space by the factor $\alpha = e^{-r}$ and expanded in momentum space by the factor $1/\alpha = e^r$. For this reason we call $S(z)$ the "squeeze" operator. Let us define the state $|0, r\rangle$ according to

$$|0, r\rangle \equiv S(r)|0\rangle. \tag{26}$$

Since the ground state $|0\rangle$ is a Gaussian wave packet with position spread $\Delta x = (\hbar/2m\omega)^{1/2}$, we know that $|0, r\rangle$ is a Gaussian packet with $\Delta x = e^r(\hbar/2m\omega)^{1/2} = (1/\alpha)(\hbar/2m\omega)^{1/2}$. For very-large positive values of r, the state $|0, r\rangle$ is highly localized in momentum space. For very-large negative values of r $|0, r\rangle$ is highly localized in position space. The state $|0, 0\rangle$ with $r = 0$ is the ground state.

We may generalize this set of wave-packet states by defining

$$|\beta, z\rangle \equiv D(\beta)S(z)|0\rangle, \tag{27}$$

where β is a complex displacement and z is a complex squeeze factor. The state $|\beta, r\rangle$ is a Gaussian packet with the same shape as $|0, r\rangle$ but displaced from the origin in position and momentum space. In the Appendix we show that these states develop in time according to

$$e^{-i\omega t a^\dagger a}|\beta, z\rangle = |\beta e^{-i\omega t}, z e^{-2i\omega t}\rangle. \tag{28}$$

That is, they remain wave-packet states with the complex amplitudes following the classical trajectory and the complex squeeze factor z rotating at twice the resonant frequency.

Since the set of wave-packet states is unitarily equivalent to the set of coherent states, many of the useful properties of the coherent states such as the overcompleteness relation may be generalized for these states. It is not surprising that these states and the techniques we have developed for dealing with them have appeared in the literature many times before. The first use of these states for their low noise properties is, to our knowledge, in the paper by Takahasi[24] in which he discusses the degenerate parametric amplifier, which is a type of phase-sensitive amplifier. A very clear presentation of a unitary transformation technique just like that used in this paper may be found in the papers by Stoler.[25,26] These states are again introduced in a paper by Yuen[27] in which a more complete set of references may be found.

To get a better understanding of the wave-packet states we quote some expectation values for the state $|\beta e^{-i\omega t}, r e^{-2i\omega t}\rangle$ with $\alpha = e^{-r}$ and $\beta = \beta_1 + i\beta_2$:

$$\begin{aligned} \langle\hat{x}\rangle &= (2\hbar/m\omega)^{1/2}(\beta_1\cos\omega t + \beta_2\sin\omega t), \\ \langle\hat{p}\rangle &= (2m\hbar\omega)^{1/2}(-\beta_1\sin\omega t + \beta_2\cos\omega t), \\ \Delta\hat{x} &= (\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2)^{1/2} \\ &= (\hbar/2m\omega)^{1/2}[\alpha^2\sin^2\omega t + (1/\alpha^2)\cos^2\omega t]^{1/2}, \\ \Delta\hat{p} &= (m\hbar\omega/2)^{1/2}[(1/\alpha^2)\sin^2\omega t + \alpha^2\cos^2\omega t]^{1/2}, \\ \Delta\hat{x}\Delta\hat{p} &= \tfrac{1}{2}\hbar\{1 + [\tfrac{1}{2}(\alpha^2 - 1/\alpha^2)\sin 2\omega t]^2\}, \\ \langle\hat{E}\rangle &= \hbar\omega\{|\beta|^2 + [(1-\alpha^2)/2\alpha]^2\}, \\ \langle\hat{X}_1\rangle &= 2\beta_1, \quad \langle\hat{X}_2\rangle = 2\beta_2, \\ \Delta\hat{X}_1 &= 1/\alpha, \quad \Delta\hat{X}_2 = \alpha, \\ \Delta\hat{X}_1\Delta\hat{X}_2 &= 1. \end{aligned} \tag{29}$$

Recall that for $r = 0$ $(\alpha = 1)$ the states $|\beta, 0\rangle$ are just the coherent states. The last five equations make clear the connection to the $\hat{X}_1$ and $\hat{X}_2$ operators. We see that the states $|\beta e^{-i\omega t}, r e^{-2i\omega t}\rangle$ are minimum-uncertainty states $(\Delta\hat{X}_1\Delta\hat{X}_2 = 1)$ for $\hat{X}_1$ and $\hat{X}_2$, that α gives the spread in $\hat{X}_2$, $1/\alpha$ gives the spread in $\hat{X}_1$, and $\mathrm{Re}\{\beta\}$ and $\mathrm{Im}\{\beta\}$ give the time-independent expectation values of $\hat{X}_1$ and $\hat{X}_2$, respectively.

Now let us consider preparing the system initially in a wave-packet state. For notational convenience we will suppose that the initial state is $|\psi_i\rangle = e^{i\omega\tau a^\dagger a}|\beta, r\rangle$. At $t = 0$ we have $|\psi_0\rangle = e^{-i\omega\tau a^\dagger a}|\psi_i\rangle = |\beta, r\rangle$, thus $\rho_0 = |\beta, r\rangle\langle\beta, r|$ and $\rho_1 = D(\mu)|\beta, r\rangle\langle\beta, r|D^\dagger(\mu)$. We need to compute $\gamma = \langle\beta, r|D(\mu)|\beta, r\rangle = \langle 0, r|D^\dagger(\beta)D(\mu)D(\beta)|0, r\rangle$. Recalling Eq. (20) we may write this as $\gamma = \langle 0, r|D(\mu)|0, r\rangle \exp[2i\,\mathrm{Im}(\beta^*\mu)]$.

The value of this matrix element is shown in the Appendix to be

$$\gamma = \exp\{-\tfrac{1}{2}|\mu|^2[\alpha^2\cos^2\phi + (1/\alpha^2)\sin^2\phi] + 2i\,\mathrm{Im}(\beta^*\mu)\}$$

and

$$|\gamma|^2 = \exp\{-(E_s/\hbar\omega)[\alpha^2\cos^2\phi + (1/\alpha^2)\sin^2\phi]\},$$

where ϕ is defined by $\mu = |\mu|e^{i\phi}$. Note that for $\alpha = 1$ the result reduces to the coherent state result as it must. As in the coherent state result we find that $|\gamma|^2$ is independent of the initial displacement β. A new feature has appeared in this result, namely phase dependence. The overlap depends on the phase ϕ which is a function of the arrival time and shape of the gravity wave pulse. To this point we have assumed that this phase is known. In practice this assumption is incorrect and we must in principle do a more difficult calculation as described in Sec. VII. However, let us examine the present result in more detail. For $Q_0 = 0$ we have

$$Q_D = 1 - \exp\{-(E_s/\hbar\omega)[\alpha^2\cos^2 + (1/\alpha^2)\sin^2\phi]\}$$

$$\text{(wave packets, } Q_0 = 0, \text{ known phase)} \qquad (30)$$

and

$$E_{\min} = \frac{\hbar\omega \ln 2}{[\alpha^2\cos^2\phi + (1/\alpha^2)\sin^2\phi]}$$

$$\text{(wave packets, known phase)}. \qquad (31)$$

Figure 4 is a plot of Eq. (31) vs α for various

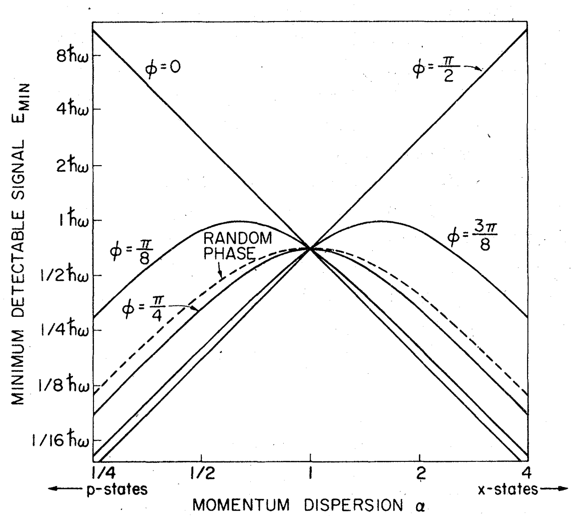

FIG. 4. Minimum detectable signal for detection of gravity waves of known phase for an oscillator prepared in the initial state $|\beta, r\rangle$ vs the momentum dispersion $\alpha = e^{-r}$. Also shown is the curve for the case of random phase.

values of the phase. We see that for favorable values of ϕ we can make the minimum detectable energy very small either by making $\alpha \ll 1$ or $\alpha \gg 1$. It turns out that for the case $Q_0 = 0$, the optimum decision strategy for pulses of unknown phase is identical to the strategy for signals of known phase. In this case Q_D is gotten by integrating over the phase, that is

$$Q_D = \int_0^{2\pi} (d\phi/2\pi)(1 - \exp\{-(E_s/\hbar\omega) \times [\alpha^2\cos^2\phi + (1/\alpha^2)\sin^2\phi]\})$$

and we obtain

$$Q_D = 1 - \exp[-(E_s/2\hbar\omega)(\alpha^2 + 1/\alpha^2)] \times I_0[(E_s/2\hbar\omega)(\alpha^2 - 1/\alpha^2)]$$

$$\text{(wave packets, } Q_0 = 0, \text{ random phase)}, \qquad (32)$$

where $I_0(x)$ is the modified Bessel function of order zero. We can approximate Q_D in two limits: $Q_D \approx 1 - (\hbar\omega/\pi E_s\alpha^2)^{1/2}$ for $\hbar\omega/E_s\alpha^2 \ll 1$ and $Q_D \approx 1 - (\hbar\omega\alpha^2/\pi E_s)^{1/2}$ for $\hbar\omega\alpha^2/E_s \ll 1$. We also conclude

$$E_{\min} \approx 1.8\hbar\omega/\alpha^2, \quad \text{for } \alpha^2 \gg 1,$$

$$E_{\min} \approx 1.8\alpha^2\hbar\omega, \quad \text{for } \alpha^2 \ll 1 \qquad (33)$$

$$\text{(wave packets, } Q_0 = 0, \text{ random phase)}.$$

We see that as in the case of n states we can obtain high sensitivity by preparing the system in a highly excited state. We may choose to localize the oscillator either in position space ($\alpha \gg 1$) or in momentum space ($\alpha \ll 1$).

We now consider another scheme which is analogous to a two-channel phase-sensitive detector.[28] One builds two gravity-wave antennas. One of them is prepared in a state $|0, r\rangle$ with $\alpha = e^{-r} \gg 1$ which is highly localized in position space. The other is prepared in the state $|0, -r\rangle$ which is highly localized in momentum space. The optimum decision strategy is used separately on each antenna. If both antennas give null results we make the null conclusion, otherwise we make the alternative conclusion. With this strategy we still have $Q_0 = 0$, but now,

$$Q_D = \int_0^{2\pi} \frac{d\phi}{2\pi}\left\{1 - \exp\left[-\frac{E_s}{\hbar\omega}\left(\alpha^2\cos^2\phi + \frac{1}{\alpha^2}\sin^2\phi\right)\right] \times \exp\left[-\frac{E_s}{\hbar\omega}\left(\frac{1}{\alpha^2}\cos^2\phi + \alpha^2\sin^2\phi\right)\right]\right\}$$

$$= \int_0^{2\pi} \frac{d\phi}{2\pi}\left\{1 - \exp\left[-\frac{E_s}{\hbar\omega}\left(\alpha^2 + \frac{1}{\alpha^2}\right)\right]\right\},$$

so

$$Q_D = 1 - \exp[-(E_s/\hbar\omega)(\alpha^2 + 1/\alpha^2)] \quad \text{(two-phase detector, } Q_0 = 0\text{)}. \tag{34}$$

This implies

$$E_{\min} = \frac{\hbar\omega \ln 2}{(\alpha^2 + 1/\alpha^2)} \quad \text{(two-phase detector, } Q_0 = 0\text{)}. \tag{35}$$

The advantage of this technique is that the detection efficiency depends exponentially on the signal strength. With a single antenna the detection efficiency is a weak function of the signal energy for random phase signals. This can be seen in Fig. 5.

The optimum measurement for $Q_0 = 0$ is a measurement of the projection operator $\rho_0 = |\beta, r\rangle\langle r, \beta|$. For large positive r the optimum measurement can be approximated by measuring $\hat{X}_1$. The almost optimum decision strategy (for a single antenna) is as follows. The oscillator is prepared in the state that evolves to $|0, r\rangle$ at $t = 0$ with $\alpha \gg 1$. This state is highly localized in $\hat{X}_1$ since $\Delta\hat{X}_1 = (1/\alpha) \ll 1$. The expectation value of $\hat{X}_1$ is $\langle\hat{X}_1\rangle = 0$. If a gravity wave pulse arrives, the state at $t = 0$ will be $D(\mu)|0, r\rangle = |\mu, r\rangle$ which is also highly localized in $\hat{X}_1$ but at a shifted value $\langle\hat{X}_1\rangle = 2\mu$, with $\Delta\hat{X}_1 = (1/\alpha) \ll 1$. A measurement of $\hat{X}_1$ is performed and compared to zero, if a shift much bigger than $\Delta\hat{X}_1 = 1/\alpha$ is observed we conclude that a gravity wave has been received.

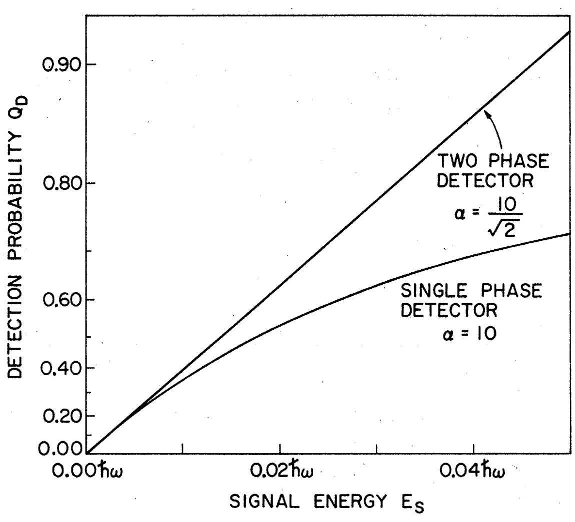

FIG. 5. Detection probability vs signal energy for single-phase and two-phase detection schemes with $\alpha = 10$ and $\alpha = 10/\sqrt{2}$, respectively.

VII. RANDOM PHASE

In order to treat the more realistic problem of gravity waves of unknown phase we no longer have a simple pure-state problem. The density matrix ρ_1 is given by

$$\rho_1 = \int_0^{2\pi} \frac{d\phi}{2\pi} D(|\mu| e^{i\phi}) \rho_0 D^\dagger(|\mu| e^{i\phi}),$$

where we have assigned equal probability to each value of phase. Although we do not present the results here we have solved this problem for coherent states and n states. In the case $Q_0 = 0$ the results are exactly the same as we have calculated for the assumption of known phase. However, as Q_0 is allowed to differ from zero the value of Q_D does not improve as fast for random phase as it does for known phase. For the wave-packet states the result with $Q_0 = 0$ is obtained by integrating over the phase as we have done in Sec. VI.

VIII. CONCLUSION

We have seen that for an infinite-Q gravity-wave antenna interacting with a classical radiation field quantum mechanics imposes no ultimate sensitivity limit. However, for any given initial state of the antenna there is a sensitivity limit. We have found this limit for several important classes of initial states by finding the optimum detection strategy in each case. When the antenna is prepared in a coherent state the sensitivity limit is of the same order as the limit arising in a detection scheme employing a linear amplifier. Higher sensitivity may be obtained by preparing the antenna initially in an excited energy eigenstate or in a highly localized wave-packet state. The wave-packet states look particularly interesting since they are minimum-uncertainty states and it is not hard to imagine devices which will prepare an antenna in such a state and subsequently read it out. An example of such a device is the degenerate parametric amplifier.

Although the state of the art of gravity-wave astronomy has not yet reached the linear detection limit, we may hope that a future generation of detectors will reach and surpass it.

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APPENDIX

In this appendix we will establish some results which are used in the body of the paper. First we recall the operator identity

$$e^{\xi A}Be^{-\xi A}=B+\xi[A,B]+\frac{\xi^2}{2!}[A,[A,B]]+\cdots . \quad \text{(A1)}$$

Letting $A=a^\dagger a$ and $B=a^\dagger$ we find $e^{\xi a^\dagger a}a^\dagger e^{-\xi a^\dagger a}=a^\dagger e^{\xi}$. Similarly we find $e^{\xi a^\dagger a}ae^{-\xi a^\dagger a}=ae^{-\xi}$. If $f(a, a^\dagger)$ is any function of a and $a^\dagger$ which may be written as a power series we have

$$\begin{aligned} e^{\xi a^\dagger a}f(a,a^\dagger)e^{-\xi a^\dagger a}&=f(e^{\xi a^\dagger a}ae^{-\xi a^\dagger a},\ e^{\xi a^\dagger a}a^\dagger e^{-\xi a^\dagger a})\\ &=f(ae^{-\xi},a^\dagger e^{\xi}). \end{aligned} \quad \text{(A2)}$$

Recalling the definition of the displacement operator $D(\mu)=\exp(\mu a^\dagger-\mu^* a)$ we use (A2) to obtain

$$\begin{aligned} e^{-i\omega t a^\dagger a}&D(\mu)e^{i\omega t a^\dagger a}\\ &=\exp[\mu e^{-i\omega t}a^\dagger-\mu^* e^{i\omega t}a]=D(\mu e^{-i\omega t}). \end{aligned} \quad \text{(A3)}$$

Another important operator identity is the Baker-Hausdorff formula,[29]

$$e^{A+B}=e^Ae^Be^{-[A,B]/2}, \quad \text{(A4)}$$

for any operators A and B which commute with their commutator $[A,B]$. Applying this formula to the displacement operator we can easily show

$$D(\mu)D(\beta)=D(\mu+\beta)\exp[\tfrac{1}{2}(\beta^*\mu-\beta\mu^*)]. \quad \text{(A5)}$$

We may also use this identity to rewrite $D(\mu)$ in normal-ordered form. We get

$$D(\mu)=\exp(-\tfrac{1}{2}|\mu|^2)\exp(\mu a^\dagger)\exp(-\mu^* a). \quad \text{(A6)}$$

It follows that

$$\langle 0|D(\mu)|0\rangle=\exp(-\tfrac{1}{2}|\mu|^2). \quad \text{(A7)}$$

Equation (A6) may be used to obtain the number-state representation of the coherent state $|\alpha\rangle$. We have $|\alpha\rangle=D(\alpha)|0\rangle=\exp(-\frac{1}{2}|\alpha|^2)\exp(\mu a^\dagger)|0\rangle$ and thus

$$|\alpha\rangle=\exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \quad \text{(A8)}$$

If B is any operator, we define its coherent-state representation $B(\alpha^*,\beta)$ by $B(\alpha^*,\beta)\equiv\langle\alpha|B|\beta\rangle \times\exp(\frac{1}{2}|\alpha|^2+\frac{1}{2}|\beta|^2)$. Using Eq. (A8) we can write this as

$$B(\alpha^*,\beta)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}\beta^n}{(m!n!)^{1/2}}\langle m|B|n\rangle. \quad \text{(A9)}$$

We see that $B(\alpha^*,\beta)$ is a generating function for the matrix elements of B in the number representation. The coherent-state representation of $D(\mu)$ is

$$\begin{aligned} D(\alpha^*,\beta;\mu)&=\langle\alpha|D(\mu)|\beta\rangle\exp(\tfrac{1}{2}|\alpha|^2+\tfrac{1}{2}|\beta|^2)\\ &=\langle 0|D^\dagger(\alpha)D(\mu)D(\beta)|0\rangle\exp(\tfrac{1}{2}|\alpha|^2+\tfrac{1}{2}|\beta|^2)\\ &=\exp(-\tfrac{1}{2}|\mu|^2+\alpha^*\beta+\alpha^*\mu-\mu^*\beta). \end{aligned}$$

We may therefore write

$$\begin{aligned} \exp(-\tfrac{1}{2}|\mu|^2)&\exp(\alpha^*\beta+\alpha^*\mu-\mu^*\beta)\\ &=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}\beta^n}{(m!n!)^{1/2}}\langle m|D(\mu)|n\rangle. \end{aligned} \quad \text{(A10)}$$

Now a generating function for the associated Laguerre polynomials is[30]

$$\exp(\lambda w+\kappa z+\nu wz)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{w^m}{m!}\frac{z^n}{n!}D_{mn}(\lambda,\kappa,\nu), \quad \text{(A11)}$$

with

$$D_{mn}(\lambda,\kappa,\nu)=\begin{cases} m!\,\nu^m\kappa^{n-m}L_m^{(n-m)}\left(-\dfrac{\lambda\kappa}{\nu}\right), & \text{for } m\le n\\ n!\,\nu^n\lambda^{m-n}L_n^{(m-n)}\left(-\dfrac{\lambda\kappa}{\nu}\right), & \text{for } m\ge n. \end{cases}$$

Setting $\lambda=\mu$, $\kappa=-\mu^*$, $w=\alpha^*$, $z=\beta$, and $\nu=1$ we obtain from Eq. (A10)

$$\langle m|D(\mu)|n\rangle=\exp(-\tfrac{1}{2}|\mu|^2)\left(\frac{m!}{n!}\right)^{1/2}(-\mu^*)^{n-m}L_m^{(n-m)}(|\mu|^2),\quad \text{for } m\le n. \quad \text{(A12)}$$

This gives the matrix element for a transition from the state $|n\rangle$ to the state $|m\rangle$ under the influence of a gravity wave. For the case $m=n$ we have

$$\langle n|D(\mu)|n\rangle=\exp(-\tfrac{1}{2}|\mu|^2)L_n(|\mu|^2). \quad \text{(A13)}$$

To facilitate calculations with the wave-packet states let us introduce the "squeeze" operator

$$S(z)\equiv\exp\left[\frac{z}{2}(a^\dagger)^2-\frac{z^*}{2}a^2\right].$$

Now

$$S^\dagger(z)=\exp\left[-\frac{z}{2}(a^\dagger)^2+\frac{z^*}{2}a^2\right]=S^{-1}(z)=S(-z).$$

Thus $S^\dagger(z)S(z)=S(z)S^\dagger(z)=1$ so $S(z)$ is a unitary operator. S induces a canonical transformation of the creation and annihilation operators. Using Eq. (A1) we may show

$$b^\dagger(z)\equiv S^\dagger a^\dagger S=\cosh|z|a^\dagger+\frac{z^*}{|z|}\sinh|z|a\,,$$
$$b(z)\equiv S^\dagger a S=\cosh|z|a+\frac{z}{|z|}\sinh|z|a^\dagger\,. \qquad \text{(A14)}$$

Since the transformation is canonical we have $[b,b^\dagger]=[a,a^\dagger]=1$. If $f(a,a^\dagger)$ is any power-series function of a and $a^\dagger$, then

$$S^\dagger f(a,a^\dagger)S=f(b,b^\dagger)\,.$$

Let us examine the effect of this transformation on the position and momentum operators for the case where z is a real number r:

$$S^\dagger(r)\hat{x}S(r)=\left(\frac{2\hbar}{m\omega}\right)^{1/2}(b^\dagger+b)$$
$$=\left(\frac{2\hbar}{m\omega}\right)^{1/2}(\cosh r+\sinh r)(a^\dagger+a)\,,$$

so $S^\dagger(r)\hat{x}S(r)=\hat{x}e^{-r}=(1/\alpha)\hat{x}$, where we have defined $\alpha=e^{-r}$. Similarly we have $S^\dagger(r)\hat{p}S(r)=e^r\hat{p}=\alpha\hat{p}$.

If $|\psi\rangle$ is the state vector of a system then $S(r)|\psi\rangle$ represents the same system compressed in position space by the factor α and expanded in momentum space by the factor $1/\alpha$. This is seen since for any power k we have

$$\langle\psi|S^\dagger(\hat{x})^kS|\psi\rangle=\langle\psi|(\hat{x}/\alpha)^k|\psi\rangle$$

and

$$\langle\psi|S^\dagger(\hat{p})^kS|\psi\rangle=\langle\psi|(\alpha\hat{p})^k|\psi\rangle\,.$$

We define the class of states $|\beta,z\rangle$ where both z and β may be complex by

$$|\beta,z\rangle\equiv D(\beta)S(z)|0\rangle,\quad\text{with } \alpha=e^{-z}, \qquad \text{(A15)}$$

$S(z)$ propagates in time according to

$$e^{-i\omega t a^\dagger a}\exp\left(\frac{z}{2}a^{\dagger 2}-\frac{z^*}{2}a^2\right)e^{i\omega t a^\dagger a}$$
$$=\exp\left(\frac{z}{2}e^{-2i\omega t}a^{\dagger 2}-\frac{z^*}{2}e^{2i\omega t}a^2\right)$$
$$=S(ze^{-2i\omega t})\,.$$

Thus the state $|\beta,z\rangle$ develops in time according to

$$e^{-i\omega t a^\dagger a}|\beta,z\rangle=D(\beta e^{-i\omega t})S(ze^{-2i\omega t})|0\rangle$$
$$=|\beta e^{-i\omega t},ze^{-2i\omega t}\rangle\,.$$

The displacement β has the classical time dependence, while the complex spreading factor z rotates in the complex plane at twice the resonant frequency.

Expectation values in the wave-packet states are easily calculated using the transformation equations (A14). We will illustrate this by computing $\langle 0,r|D(\mu)|0,r\rangle$ for real r. We have

$$\langle 0,r|D(\mu)|0,r\rangle=\langle 0|S^\dagger(r)D(\mu)S(r)|0\rangle$$
$$=\langle 0|\exp(\mu b^\dagger-\mu^*b)|0\rangle$$
$$=\langle 0|\exp(\eta a^\dagger-\eta^*a)|0\rangle\,,$$

where $\eta=(\mu\cosh r-\mu^*\sinh r)$. Thus

$$\langle 0,r|D(\mu)|0,r\rangle=\langle 0|D(\eta)|0\rangle=\exp(-\tfrac{1}{2}|\eta|^2)$$
$$=\exp[-\tfrac{1}{2}|\mu|^2\times(\cosh 2r-\sinh 2r\cos^2\phi)]\,,$$

where $\mu=|\mu|e^{i\phi}$. Recalling that $\alpha=e^{-r}$ we may write this as

$$\langle 0,r|D(\mu)|0,r\rangle=\exp\left[-\tfrac{1}{2}|\mu|^2\left(\alpha^2\cos^2\phi+\frac{1}{\alpha^2}\sin^2\phi\right)\right]. \qquad \text{(A16)}$$

Although we do not need it here, it is useful to have an expression for $S(z)$ which is in normal-ordered form. We find that

$$S(z)=(\cosh|z|)^{1/2}\exp\left(\frac{z}{2|z|}\tanh|z|a^{\dagger 2}\right)\times\sum_{n=0}^{\infty}\frac{(\operatorname{sech}|z|-1)^n}{n!}(a^\dagger)^n a^n\times\exp\left(-\frac{z}{2|z|}\tanh|z|a^2\right). \qquad \text{(A17)}$$

This form may be used to find the $|n\rangle$ state representation of the state $|0,z\rangle$:

$$|0,z\rangle=S(z)|0\rangle$$
$$=(\cosh|z|)^{1/2}\times\sum_{n=0}^{\infty}\left(\frac{z}{2|z|}\tanh|z|\right)^n\frac{[(2n)!]^{1/2}}{n!}|2n\rangle\,.$$

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