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General relativity as a limit of the de Sitter gauge theory

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The gauge character of the vierbein field is investigated by introducing the Yang-Mills fields associated with the de Sitter gauge group and by contracting the de Sitter group to the Poincaré group. The vierbein field can indeed be seen as a gauge field consisting of the Yang-Mills fields associated with both the Lorentz and the translational symmetry. The local Lorentz invariance is found essential to deduce Einstein's theory of general relativity.

I. INTRODUCTION

The gauge theory of Yang and Mills,[1] as formulated by Utiyama,[2] stipulates a general way to introduce a set of vector potentials $A_\mu{}^a$ ($a=1,2,\ldots,n$, $\mu=1,2,3,0$) which couple minimally with a multiplet field Q carrying internal degrees of freedom. First, one demands that the system of Q be invariant under a linear homogeneous transformation,

$$\delta Q=\epsilon^a T_a Q, \tag{1.1}$$

where ϵ^a are n constant parameters and T_a are representatives of the n generators of a Lie group satisfying the commutation relations

$$[T_a, T_b]=f_{ab}{}^c T_c, \tag{1.2}$$

with the structure constants $f_{ab}{}^c$ characteristic to the group. Then one modifies the theory by introducing n vector fields $A_\mu{}^a$ so that the entire system remains invariant when the parameters ϵ^a become position-dependent functions. This local gauge invariance is ensured if the gauge fields $A_\mu{}^a$ transform as

$$\delta A_\mu{}^a=f_{bc}{}^a\epsilon^b A_\mu{}^c-\partial_\mu\epsilon^a. \tag{1.3}$$

Usually (1.1) and (1.3) are referred to as the gauge transformations of the first and second kind, respectively.

In generalizing the Yang-Mills theory, Utiyama[2] and later Kibble[3] attempted to formulate the gauge theory of gravitation. Utiyama employed, as a gauge group, the Lorentz group of six parameters ϵ_{ij} ($i,j=1,2,3,0$):

$$\delta x^i=\epsilon^i{}_j x^j,\quad \epsilon_{ij}+\epsilon_{ji}=0, \tag{1.4}$$

with which are associated the six gauge potentials $A_\mu{}^{ij}$. In order to identify the gauge fields with the local affine connection pertaining to gravitation, however, it is necessary to introduce another set of four vector fields or a vierbein field $b_\mu{}^i$. Kibble proposed to replace the Lorentz group by the Poincaré group for which

$$\delta x^i=\epsilon^i{}_j x^j+\epsilon^i, \tag{1.5}$$

and to interpret the vierbein field as a gauge field transforming according to the rule

$$\delta b_\mu{}^i=\epsilon^i{}_j b_\mu{}^j-\partial_\mu(\delta x^\nu)b_\nu{}^i, \tag{1.6}$$

which we shall call the gauge transformation of the third kind.

The gauge fields of the Yang-Mills type undergo transformations of the second kind (1.3), whereas the vierbein field transforms in the third way (1.6). The action of the standard gauge group is isotropic; it leaves the contact point of spacetime intact. In contrast, the Poincaré transformation (1.5) is inhomogeneous; the translation shifts one reference point to another. Therefore, even though the vierbein field is introduced in conjunction with the Poincaré invariance, its gauge character is still ambiguous.

It has also been argued by Hayashi and Nakano,[4] and very recently by Cho[5,6] that the vierbein field can be formed from the gauge fields $B_\mu{}^i$ associated with the translation group of four parameters,

$$\delta x^i=\epsilon^i. \tag{1.7}$$

In fact, after the replacement of the group parameters by arbitrary position-dependent functions, the Lorentz transformation (1.4), the Poincaré transformation (1.5), and the translation (1.7) all appear to become the same general coordinate transformation

$$\delta x^\mu=f^\mu(x), \tag{1.8}$$

which is after all the transformation basic to the formulation of general relativity. Furthermore, Cho has claimed that the gauge theory of the translation group yields uniquely Einstein's theory of gravitation and that the Poincaré group leads to the Einstein-Cartan theory of spacetime endowed with curvature and torsion.

In the present paper, we investigate the gauge character of the vierbein field.[7] To this end, we choose the de Sitter group SO(4,1) of ten parameters as the starting gauge group. We note that the de Sitter group is reducible to the Poincaré group by contraction. Introducing ten Yang-Mills fields and relating the internal-coordinate transformation with the external-coordinate transformation, we observe that upon contraction the ten gauge fields are reduced to the local affine connection $A_\mu{}^{ij}$ and the vierbein field $b_\mu{}^i$. The gauge transformation of the third kind (1.6) is obtained from the transformation rule of the second kind (1.3). Thus, we propose that the vierbein field is indeed a gauge field which consists of the Yang-Mills fields associated with both the Lorentz and the translational symmetry. Contrary to Cho's claim,[5] the local Lorentz invariance is found essential to Einstein's theory of general relativity. Although the fiber-bundle formulation of gauge theory is possible[8] and elegant, we follow the conventional procedure[1-3] in order to retain the original spirit of gauge theory.

II. DE SITTER GAUGE INVARIANCE

Following the standard prescription of gauge theory, we start with a Lagrangian

$$L(x)=L\{Q(x),Q_i(x)\} \tag{2.1}$$

given as a scalar function of a multiplet field $Q(x)$ and its derivative $Q_i(x)$ at a point x^μ ($\mu=1,2,3,0$) in Minkowskian space-time. Here, for convenience, we employ an orthonormal set $\partial_i=(\partial_i x^\mu)\partial_\mu$ ($i=1,2,3,0$), defined at the point x^μ. The de Sitter gauge invariance means

$$\delta_0 L=\frac{\partial L}{\partial Q}\,\delta_0 Q+\frac{\partial L}{\partial Q_i}\delta_0 Q_i=0 \tag{2.2}$$

under the fixed-point field transformations

$$\delta_0 Q=Q'(x)-Q(x)=\epsilon^{ab}T_{ab}Q(x)\,, \tag{2.3}$$

$$\delta_0 Q_i=(\delta_0 Q_i)=\epsilon^{ab}(T_{ab}Q)_i\,, \tag{2.4}$$

where $\epsilon^{ab}=-\epsilon^{ba}$ ($a,b=1,2,3,0,5$) are ten infinitesimal parameters and T_{ab} are the representatives of the SO(4,1) generators. See the Appendix where a short account of the de Sitter group, appropriate to the present discussions, is given.

If the values of ϵ^{ab} are all independent of the choice of the reference point, then $Q_i=\partial_i Q$ satisfies the condition (2.4). If ϵ^{ab} are functions of the space-time variables x^μ, then we have to choose Q_i of the form

$$Q_i=\nabla_i Q\equiv\partial_i Q+A_i{}^{ab}T_{ab}Q \tag{2.5}$$

by introducing the ten potential fields $A_i{}^{ab}(x)$ which transform, along with (2.3), according to

$$\delta_0 A_i{}^{ab}=\epsilon^a{}_c A_i{}^{cb}-\epsilon^b{}_c A_i{}^{ca}-\partial_i\epsilon^{ab}. \tag{2.6}$$

The potential fields introduced above are the typical Yang-Mills gauge fields. The commutator of the gauge-invariant differential operators is given by

$$[\nabla_i,\nabla_j]=F_{ij}{}^{ab}T_{ab}+A_j{}^{ab}\partial_i T_{ab}-A_i{}^{ab}\partial_j T_{ab}\,, \tag{2.7}$$

where

$$F_{ij}{}^{ab}=\partial_i A_j{}^{ab}-\partial_j A_i{}^{ab}+A_i{}^a{}_c A_j{}^{cb}-A_j{}^a{}_c A_i{}^{cb}. \tag{2.8}$$

The free Lagrangian for the gauge fields $A_i{}^{ab}$ may be formed out of $F_{ij}{}^{ab}$ given by (2.8).

Suppose that $Q(x)$ is a function of the internal de Sitter variables z^α ($\alpha=1,2,3,0,5$) and that the gauge transformation (2.3) is associated with the transformation,

$$\delta z^\alpha=\epsilon^{ab}\zeta_{ab}{}^\alpha(z) \tag{2.9}$$

where $\zeta_{ab}{}^\alpha$ are functions characteristic to the de Sitter transformation, whose explicit forms are given in the Appendix. Then the group generators can be represented by

$$T_{ab}=S_{ab}-\zeta_{ab}{}^\alpha\partial_\alpha\,, \tag{2.10}$$

where the spin matrices S_{ab} act on the multiplet components of Q, while the second term refers to the internal variables of each component. In this case, the gauge invariance means

$$\delta L=\delta_0 L+\delta z^\alpha\partial_\alpha L=0 \tag{2.11}$$

under the transformations

$$\delta Q=\epsilon^{ab}S_{ab}Q, \tag{2.12}$$

$$\delta Q_i=\epsilon^{ab}S_{ab}Q_i-(\delta z^\alpha\partial_\alpha)_i Q. \tag{2.13}$$

The last expression for the variation of Q_i is formal in the sense that the role of the operator $(\delta z^\alpha\partial_\alpha)_i$ in the second term is left unspecified. If the transformation (2.9) of the internal variables z^α does not affect on the external coordinates x^μ at all, the second term in (2.13) vanishes. If the internal and external variables are related by $\delta z^\alpha\partial_\alpha=\delta x^i\partial_i$, but if the gauge parameters ϵ^{ab} are global, then we can write (2.13) explicitly as

$$\delta(\partial_i Q)=\epsilon^{ab}S_{ab}\partial_i Q-\partial_i(\partial x^j)\partial_j Q. \tag{2.13'}$$

For the local gauge invariance, we need a separate treatment which we shall discuss in Sec. III.

III. IMPRINTING OF THE DE SITTER GAUGE ON SPACE-TIME

Let us now demand that the internal transformation (2.9) induce the external coordinate transformation,

$$\delta x^\mu(z) = \delta z^\alpha \partial_\alpha x^\mu = \epsilon^{ab}(x)\zeta_{ab}{}^\alpha(z)\partial_\alpha x^\mu . \qquad (3.1)$$

Apparently the background spacetime can no longer be Minkowskian. In general it is curved. In particular, if the parameters ϵ^{ab} are to assume global values, one will find the de Sitter world of constant curvature.

As a result of imprinting (3.1), the gauge-invariant derivative (2.5) can be written in the form

$$\nabla_i Q = \partial_i Q + A_i{}^{ab} S_{ab} Q - A_i{}^{ab}\zeta_{ab}{}^\alpha \partial_\alpha x^\mu \partial_\mu Q, \qquad (3.2)$$

or

$$\nabla_i Q = h_i{}^\mu \partial_\mu Q + A_i{}^{ab} S_{ab} Q, \qquad (3.3)$$

where

$$h_i{}^\mu = (\partial_i - A_i{}^{ab}\zeta_{ab}{}^\alpha \partial_\alpha) x^\mu . \qquad (3.4)$$

Furthermore, introducing functions $b_\mu{}^i$ by

$$h_i{}^\mu b_\mu{}^j = \delta_i{}^j, \quad h_i{}^\mu b_\nu{}^i = \delta^\mu{}_\nu \ , \qquad (3.5)$$

or by

$$b_\mu{}^i \partial_i z^\alpha = \partial_\mu z^\alpha + A_\mu{}^{ab}\zeta_{ab}{}^\alpha , \qquad (3.6)$$

we can define new gauge fields

$$A_\mu{}^{ab} = b_\mu{}^i A_i{}^{ab} \qquad (3.7)$$

and rewrite (3.4) as

$$\nabla_i Q = h_i{}^\mu \nabla_\mu Q, \qquad (3.8)$$

with

$$\nabla_\mu Q = \partial_\mu Q + A_\mu{}^{ab} S_{ab} Q. \qquad (3.9)$$

Thus we substitute $h_i{}^\mu \nabla_\mu Q$ for Q_i in the Lagrangian (2.1) and replace (2.11), (2.12), and (2.13) by

$$\delta L = \delta_0 L + \delta x^\mu \partial_\mu L = 0, \qquad (3.10)$$

$$\delta Q = \epsilon^{ab} S_{ab} Q \ , \qquad (3.11)$$

$$\delta(\nabla_\mu Q) = \epsilon^{ab} S_{ab} \nabla_\mu Q - \partial_\mu(\delta x^\nu)\nabla_\nu Q. \qquad (3.12)$$

The variation of $\nabla_\mu Q$ is explicit and also consistent with (2.13′). The variation of the gauge fields (3.7), pertinent to (3.11) and (3.12), is given by

$$\delta A^{ab} = \epsilon^a{}_c A^{cb} - \epsilon^b{}_c A^{ca} - \partial_\mu \epsilon^{ab} - \partial_\mu(\delta x^\nu) A_\nu{}^{ab}. \qquad (3.13)$$

IV. CONTRACTION OF THE DE SITTER GAUGE TO THE POINCARÉ GAUGE

The variation (2.9) of the internal variables z^α can be expressed by using (A12) as

$$\delta z^\alpha = \epsilon^\alpha{}_\beta z^\beta = \epsilon^\beta [\delta_\beta{}^\alpha - \tfrac{1}{4}\lambda^2(\delta_\beta{}^\alpha z_\gamma z^\gamma - 2z^\alpha z_\beta)]. \qquad (4.1)$$

The contraction of the de Sitter group to the Poincaré group is achieved by the limiting procedure $\lambda \to 0$. Certainly, in the limit $\lambda \to 0$, the de Sitter transformation (4.1) becomes the Poincaré transformation,

$$\delta z^\alpha = \epsilon^\alpha{}_\beta z^\beta + \epsilon^\alpha . \qquad (4.2)$$

Now it is possible to adjust the orthonormal set ∂_i so that $\partial_i z^\alpha = \delta_i{}^\alpha$. Thus, we replace after contraction the Greek indices $\alpha, \beta, \gamma, \ldots$ by the Latin indices $i, j, k, \ldots$.

From the expression (3.6) for the b field, we obtain

$$b_\mu{}^i = \partial_\mu z^i + A_\mu{}^i{}_j z^j + B_\mu{}^i, \qquad (4.3)$$

where

$$B_\mu{}^i = \lim_{\lambda\to 0} A^{i5}/\lambda .$$

In the same limit, the gauge transformations (3.11) are written as

$$\delta Q = \epsilon^{ij} S_{ij} Q, \qquad (4.4)$$

$$\delta A_\mu{}^{ij} = \epsilon^i{}_k A_\mu{}^{kj} + \epsilon^j{}_k A_\mu{}^{ik} - \partial_\mu \epsilon^{ij} - \partial_\mu(\delta x^\nu) A_\nu{}^{ij}, \qquad (4.5)$$

$$\delta B_\mu{}^i = \epsilon_k A_\mu{}^{ki} + \epsilon^i{}_k B_\mu{}^k - \partial_\mu(\delta x^\nu) B_\nu{}^i . \qquad (4.6)$$

Since the variation of $\partial_\mu z^i$ is given by

$$\delta(\partial_\mu z^i) = \partial_\mu \epsilon^i{}_k z^k + \epsilon^i{}_k \partial_\mu z^k - \partial_\mu(\delta x^\nu)\partial_\nu z^k, \qquad (4.7)$$

we can determine, by using (4.2), (4.5), and (4.6), the transformation rule of the b field:

$$\delta b_\mu{}^i = \epsilon^i{}_j b_\mu{}^j - \partial_\mu(\delta x^\nu) b_\nu{}^i . \qquad (4.8)$$

Correspondingly, the inverse of $b_\mu{}^i$ transforms as

$$\delta h_i{}^\mu = \epsilon_i{}^j h_j{}^\mu + \partial_\nu(\delta x^\mu) h_i{}^\nu, \qquad (4.9)$$

which is precisely the form for the variation of the vierbein field obtained by Kibble. Therefore, we identify the set of fields $b_\mu{}^i$ and $h_i{}^\mu$ with the vierbein system, and the new gauge fields $A_\mu{}^{ij}$ in (3.7) with a local affine connection.

V. CONCLUDING REMARKS

In Kibble's formulation, the b field and the A fields are rather independent. Now we have seen that the b field can be derived from the gauge fields of the Yang-Mills type and that it undergoes the gauge transformation of the third kind (4.9).

If $\nabla_\mu Q$ defined by (3.9) is interpreted as the covariant derivative of Q under the general coordinate transformation (3.1), then it must be reducible for $Q = h_\lambda{}^k v^\lambda$ to the standard expression

$$\nabla_\mu v^\lambda = \partial_\mu v^\lambda + \Gamma^\lambda{}_{\mu\nu} v^\nu \qquad (5.1)$$

with the help of the relation $(S_{ij})^{km} = \delta_i{}^k \delta_j{}^m - \delta_j{}^k \delta_j{}^m$. This is true provided that

$$A_\mu{}^k{}_j = b_\lambda{}^k h_j{}^\nu \Gamma^\lambda{}_{\mu\nu} - h_j{}^\nu \partial_\mu b_\nu{}^k, \qquad (5.2)$$

or

$$\nabla_\mu b_\nu{}^k \equiv \partial_\mu b_\nu{}^k - \Gamma^\lambda{}_{\mu\nu} b_\lambda{}^k - A_\mu{}^k{}_j b_\nu{}^j = 0. \quad (5.3)$$

From (5.2) it follows that

$$A_i{}^k{}_j - A_j{}^k{}_i = 2b_\lambda{}^k h_i{}^\mu h_j{}^\nu S_{\mu\nu}{}^\lambda - h_i{}^\mu h_j{}^\nu (\partial_\mu b_\nu{}^k - \partial_\nu b_\mu{}^k), \quad (5.4)$$

where $S_{\mu\nu}{}^\lambda$ is the torsion tensor defined by

$$S_{\mu\nu}{}^\lambda = \tfrac{1}{2}(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}). \quad (5.5)$$

Using (5.2) and (5.4), we can analyze the geometries associated with the gauge symmetries.

For the translational gauge invariance, we may set $A_\mu{}^{ij} = 0$. As a result, we obtain

$$\Gamma^\lambda{}_{\mu\nu} = h_k{}^\lambda \partial_\mu b_\lambda{}^k \quad (5.6)$$

which characterizes a spacetime of teleparallelism.[9] This contradicts Cho's claim that the vierbein field (4.3) lacking the A field leads uniquely to Einstein's theory of general relativity. In order to obtain Einstein's theory we must assume no torsion, $S_{\mu\nu}{}^\lambda = 0$, hence imposing the condition

$$A_i{}^k{}_j - A_j{}^k{}_i = - h_i{}^\mu h_j{}^\nu (\partial_\mu b_\nu{}^k - \partial_\nu b_\mu{}^k). \quad (5.7)$$

Without the restriction (5.7), we are led, as Kibble suggested, to the Einstein-Cartan theory.

After completing this manuscript, we became aware of Hayashi's paper[10] which points out, in accordance with our result, that the gauge theory of the translation group leads to a spacetime of teleparallelism.

APPENDIX: DE SITTER GROUP

The de Sitter world is a spacetime with uniform curvature

$$R_{\alpha\beta\gamma\delta} = \lambda^2 (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}), \quad (A1)$$

which can be realized as a hypersphere,[11]

$$\eta_{ab} q^a q^b = \lambda^{-2}, \quad (A2)$$

in a five-dimensional flat space with real coordinates q^a $(a = 1, 2, 3, 0, 5)$ and with the metric tensor η_{ab} ($\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{00} = \eta_{55} = 1$ and $\eta_{ab} = 0$ for $a \neq b$). The set of real linear homogeneous transformations which leave (A2) invariant form the de Sitter group SO(4, 1) whose generators $J_{ab} = -J_{ba}$ satisfy the commutation relation

$$[J_{ab}, J_{cd}] = \eta_{ad} J_{bc} - \eta_{ac} J_{bd} + \eta_{bc} J_{ad} - \eta_{bd} J_{ac}. \quad (A3)$$

By the stereographic projection

$$q^\alpha = \phi(z) z^\alpha \quad (\alpha = 1, 2, 3, 0), \quad (A4)$$

$$q^5 = \lambda^{-1} \phi(z) / \tilde{\phi}(z), \quad (A5)$$

with

$$\phi(z) = (1 + \tfrac{1}{4}\lambda^2 z_\alpha z^\alpha)^{-1} = (1 + \lambda q^5)/2, \quad (A6)$$

$$\tilde{\phi}(z) = (1 - \tfrac{1}{4}\lambda^2 z_\alpha z^\alpha)^{-1} = (1 + \lambda q^5)/(2\lambda q^5), \quad (A7)$$

one can induce on the hypersphere (A2) a conformally flat metric

$$g_{\alpha\beta}(z) = \phi^2(z) \eta_{\alpha\beta}. \quad (A8)$$

The infinitesimal de Sitter transformation with ten parameters, $\epsilon^{ab} = -\epsilon^{ba}$,

$$q^a = \epsilon^a{}_b q^b, \quad (A9)$$

is translated in the language of z variables as

$$\delta z^\alpha = \epsilon^{ab} \zeta_{ab}{}^\alpha (z), \quad (A10)$$

where

$$\zeta_{\alpha\beta}{}^\gamma = \tfrac{1}{2}(\delta_\alpha{}^\gamma z_\beta - \delta_\beta{}^\gamma z_\alpha), \quad (A11a)$$

$$\zeta_{\alpha 5}{}^\gamma = \lambda^{-1}[\delta_\alpha{}^\gamma + \tfrac{1}{4}\lambda^2(\delta_\alpha{}^\gamma z_\beta z^\beta - 2 z_\alpha z^\gamma)], \quad (A11b)$$

or more simply,

$$\delta z^\alpha = \epsilon^\alpha{}_\beta z^\beta + \epsilon^\beta f_\beta{}^\alpha(z), \quad (A12)$$

with

$$\epsilon^{\alpha 5} = \lambda \epsilon^\alpha, \quad f_\beta{}^\alpha = \lambda \zeta_{\beta 5}{}^\alpha. \quad (A13)$$

It is quite clear that in the limit $\lambda \to 0$, the de Sitter coordinate transformation (A10) or (A12) becomes the Poincaré transformation. This limiting process, called *contraction*, reduces the algebra (A3) to that of the Poincaré group with the generator of the translation,

$$P_\alpha = \lim_{\lambda \to 0} J_{\alpha 5}/\lambda. \quad (A14)$$

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