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Constructive procedure for perturbations of spacetimes

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The method of Debye scalar superpotentials has previously been extended by the authors to curved spaces to yield a constructive procedure for neutrino, electromagnetic, and gravitational perturbations of algebraically special spacetimes. The solution of a decoupled scalar wave equation is differentiated to give the solution of the corresponding spinor or tensor perturbation field equations. In this paper covariant formulations and proofs are given. The results are derived in a general spinor formalism framework which extends earlier exterior differential form and tensor treatments of the electromagnetic case.

I. INTRODUCTION

Recent advances in the explicit computation of perturbations of spacetimes have stemmed from the study of congruences of null geodesics and from the development of a formal calculus based upon null frames. These basic investigations were done largely by Sachs,[1] Goldberg and Sachs,[2] and Newman and Penrose.[3] The Price[4] and Bardeen-Press[5] equations for perturbations of Schwarzschild[6] space represent the first use of the null-frame formalism to obtain decoupled, separable equations governing spacetime perturbations and the first change in approach to the problem since the earlier Regge-Wheeler[7-9] methods. The essential similarities between the Schwarzschild and Kerr[10] spacetimes when viewed from suitably chosen null frames (they are of the same Petrov[11] type) led very soon to an extension of the Price and Bardeen-Press approach to yield a treatment of perturbations of the Kerr rotating-black-hole spacetime. Fackerell and Ipser[12] derived the first decoupled equation for a Newmann-Penrose (NP) null-frame component of the Maxwell tensor, although it failed to separate variables. Teukolsky,[13] by choosing to work with the radiative NP components of the fields, obtained decoupled equations for these two components of neutrino, electromagnetic, and gravitational perturbations which did, in fact, separate variables in the Kerr metric.

This line of development, while making tractable the computation of zero-rest-mass fields around a rotating black hole, has left open the following aspects of the problem of constructing spacetime perturbations: (a) a treatment of perturbations of nonvacuum spacetimes, (b) computation of the full perturbation of the Kerr spacetime, i.e., of all components of the perturbing Maxwell tensor, metric tensor, or Weyl tensor in terms of a single complex scalar, (c) a covariant formulation of the scalar wave equation and covariant proof that the field components given by the scalar indeed satisfy the perturbation field equations, and (d) a demonstration that the perturbation components of the Weyl tensor are "metric," i.e., that appropriate integrability conditions are satisfied which ensure that they are derivable from a perturbation of the metric tensor.

During this period other investigators were using the techniques of Hertz[14] and Debye[15] potentials to calculate perturbation fields. Mo and Papas[16] treated Maxwell fields in spherical spacetimes by a generalized three-vector analysis extension of the Debye method. Hertz and Debye potential approaches were used by Sachs and Bergmann[17] and by Campbell and Morgan[18] for linearized or weak-field gravitational perturbations. Penrose[19] gave a Hertz potential treatment for arbitrary-spin zero-rest-mass fields on a Minkowski background. Cohen and Kegeles[20] utilized the machinery of exterior differential forms to generalize electromagnetic Hertz potentials to all curved spacetimes. This curved-space Hertz potential treatment, in conjunction with the null-frame formalism, has been shown to yield a curved-space extension of the scalar Debye potential method whereby all of the perturbation field components are given in terms of the solutions of a single separable scalar wave equation for each value of spin.[20,21]

The main result of this paper is the presentation of a spinor framework for Hertz and Debye potentials for zero-rest-mass field perturbations of the generalized Goldberg-Sachs[2,22] spacetimes (i.e., all algebraically special spacetimes admitting a shear-free congruence of null geodesics along the repeated principal null direction of the Weyl tensor). This includes an alternate formulation of the earlier differential form and tensor

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treatments of electrodynamics,[20] as well as spinor proofs for the analogous treatments of neutrino and gravitational perturbations.[21] The gravitational results are valid in all vacuum algebraically special spaces, while the neutrino formulation covers the full (nonvacuum) class treated earlier[20] in the electrodynamic case.

The spinor-formalism proof of this result which is presented in this paper provides the following answers to the above questions regarding spacetime perturbations: (a) For spins $\frac{1}{2}$ and 1, the method applies to a class of spacetimes which includes not only the vacuum black-hole spaces, but also the matter-filled cosmological models of Friedmann, Gödel, Kantowski and Sachs, and all other perfect-fluid models with local rotational symmetry.[23,24] (b) All $2s+1$ of the field components are given directly in terms of the scalar superpotential. (c) The basic spinor framework leads to covariant formulations and proofs for the wave equations and field-component expressions. (d) Metric perturbation expressions are given by construction as derivatives of the superpotential, establishing that the Weyl tensor perturbations are "metric."

In Sec. II we review the arbitrary-spin Hertz potentials introduced by Penrose[19] in flat space. We present a further aspect of these potentials: A treatment of gauge transformations of the third kind[25] is given for arbitrary spin,[26] which is essential for the subsequent reduction of the $2s+1$ components of the Hertz potential in curved space to a single scalar Debye potential.

Section III presents a curved-space spinor Hertz and Debye potential treatment of electromagnetism.[26] It may be viewed either as an extension of the spin-1 results of Sec. II to curved space, or as a translation of the earlier exterior form and tensor formulations[20] into the language of spinor analysis. The final results, when summarized in the NP[3] formalism, are identical to those of Ref. 20.

A curved-space version of the neutrino Hertz and Debye potentials[26] is given in Sec. IV.

The analogy between the flat-space spinor formulas for the lower spins and their curved-space generalizations leads in Sec. V to a curved-space treatment of gravitational perturbations. Proofs are given for metric and Weyl tensor perturbations in all algebraically special vacuum spaces. The results are as referred to in Refs. 20 and 21, although the proofs are given here for the first time.

For ready reference, the Appendix contains a brief recapitulation of the exterior form Hertz and Debye potential formalism for electromagnetism, as well as those formulas and identities of spinor analysis which are essential in the above presentation.

II. FLAT-SPACE HERTZ POTENTIALS FOR ARBITRARY SPIN

A. Arbitrary-spin Hertz potentials of Penrose

In investigating the asymptotic properties of zero-rest-mass fields, Penrose[19] introduced a class of potentials for arbitrary-spin fields in Minkowski space. One of these types of potentials is a natural extension to arbitrary spin of the Hertzian electromagnetic or spin-1 potential formalism. Since it is this formalism which is extended in the present work to curved space for spins $s=\frac{1}{2}$, 1, and 2, this section is devoted to a sketch of Penrose's treatment.

The minimal gravitational coupling rule by which partial derivatives are replaced by covariant derivatives is used to write the covariant zero-mass field equation for a free spin-s field as

$$\nabla^{AX'}\phi_{AB\cdots K}=0, \tag{2.1}$$

where $\phi_{AB\cdots K}$ is a totally symmetric spinor with $2s$ indices.[19,27-31]

The generalized Hertzian potential of Penrose is a spinor of the same type as the physical field, that is, a totally symmetric $2s$-spinor $P_{AB\cdots K}$, which is assumed to satisfy

$$\Box\overline{P}^{M'\cdots W'}=0, \tag{2.2}$$

where the spinor d'Alembertian operator denotes $\Box\equiv\nabla_{AX'}\nabla^{AX'}$, for, if the physical field is given by

$$\phi_{AB\cdots K}=\nabla_{AM'}\nabla_{BN'}\cdots\nabla_{KW'}\overline{P}^{M'N'\cdots W'}, \tag{2.3}$$

then the field operator expression (2.1) becomes

$$\begin{aligned}\nabla^{AX'}\phi_{AB\cdots K}&=\nabla^{AX'}\nabla_{AM'}\nabla_{BN'}\cdots\nabla_{KW'}\overline{P}^{M'N'\cdots W'}\\&=\nabla_{BN'}\cdots\nabla_{KW'}\nabla^{AX'}\nabla_{AM'}\overline{P}^{M'N'\cdots W'}\end{aligned}$$

(by the commuting of covariant derivatives in flat space)

$$\begin{aligned}&=\tfrac{1}{2}\delta^{X}{}_{M'}\nabla_{BN}\cdots\nabla_{KW'}\Box\overline{P}^{M'N'\cdots W'}\quad\text{by Eq. (A8)}\\&=0\quad\text{by Eq. (2.2)},\end{aligned}$$

which establishes that expression (2.3) is indeed a solution of Eq. (2.1).

B. Gauge freedom

Gauge transformations of the third kind[25,26] are now considered in spinor notation. These are inhomogeneous terms which appear on the right-hand side of Eq. (2.2) but which, with a suitable modification of Eq. (2.3), preserve the source-free (i.e., free-field) character of Eq. (2.1).

The gauge terms in question are given by specified derivatives acting upon an arbitrary gauge spinor $G_{AN'\cdots W'}=G_{A(N'\cdots W')}$ with one unprimed and $2s-1$ symmetrized primed indices. With such terms included, the gauge transformed versions of Eqs. (2.2) and (2.3) are[26]

$$\Box \bar{P}^{X'N'\cdots W'} = 2\nabla^{A(X'}G_A{}^{N'\cdots W')} \tag{2.4}$$

(where the symmetrization applies to the $2s$ primed indices on the right side) and

$$\phi_{AB\cdots K} = \nabla_{AM'}\nabla_{BN'}\cdots\nabla_{KW'}\bar{P}^{M'N'\cdots W} - \nabla_{(BN'}\cdots\nabla_{KW'}G_{A)}{}^{N'\cdots W'} \tag{2.5}$$

(where the symmetrization applies to the $2s$ unprimed indices in the last term). That $\phi_{AB\cdots K}$ given by Eq. (2.5) is indeed a solution to Eq. (2.1) is now shown:

$$\begin{aligned}\nabla^{AX'}\phi_{ABC\cdots JK} &= \nabla^{AX'}\nabla_{AM'}\nabla_{BN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}\bar{P}^{M'N'S'\cdots V'W'} - \nabla^{AX'}\nabla_{(BN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}G_{A)}{}^{N'S'\cdots V'W'}\\ &= \nabla^{AX'}\nabla_{AM'}\nabla_{BN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}\bar{P}^{M'N'S'\cdots V'W'} - \frac{1}{2s}\nabla^{AX'}(\nabla_{BN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}G_A{}^{N'S'\cdots V'W'}\\ &\quad + \nabla_{AN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}G_B{}^{N'S'\cdots V'W'} + \cdots\\ &\quad + \nabla_{BN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{AW'}G_K{}^{N'S'\cdots V'W'})\end{aligned}$$

(with $2s$ terms contained in parentheses, which arise from the definition of symmetrization of indices)

$$\begin{aligned}&= \nabla_{BN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}[\nabla^{AX'}\nabla_{AM'}\bar{P}^{M'N'S'\cdots V'W'} - \frac{1}{2s}(\nabla^{AX'}G_A{}^{N'S'\cdots V'W'} + \nabla^{AN'}G_A{}^{X'S'\cdots V'W'}\\ &\quad + \cdots + \nabla^{AW'}G_A{}^{N'S'\cdots V'X'})]\\ &\quad + \frac{1}{2s}[(\nabla_{BN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}\nabla^{AN'}G_A{}^{X'S'\cdots V'W'} - \nabla^{AX'}\nabla_{AN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}G_B{}^{N'S'\cdots V'W'}) + \cdots\\ &\quad + (\nabla_{BN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}\nabla^{AW'}G_A{}^{N'S'\cdots V'X'} - \nabla^{AX'}\nabla_{BN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{AW'}G_K{}^{N'S'\cdots V'W'})]\end{aligned}$$

(where there are $2s$ terms in the first set of parentheses, $2s-1$ pairs of terms in the second set of square brackets, and $2s-1$ terms have been added and subtracted, making use of the commuting of covariant derivatives in flat space)

$$\begin{aligned}&= \nabla_{BN'}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}(\tfrac{1}{2}\Box\bar{P}^{X'N'S'\cdots V'W'} - \nabla^{A(X'}G_A{}^{N'S'\cdots V'W')})\\ &\quad + \frac{1}{2s}[(\tfrac{1}{2}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}\delta_B{}^A\Box G_A{}^{X'S'\cdots V'W'} - \tfrac{1}{2}\nabla_{CS'}\cdots\nabla_{JV'}\nabla_{KW'}\delta^{X'}{}_{N'}\Box G_B{}^{N'S'\cdots V'W'}) + \cdots\\ &\quad + (\tfrac{1}{2}\nabla_{BN'}\nabla_{CS'}\cdots\nabla_{JV'}\delta_K{}^A\Box G_A{}^{N'S'\cdots V'X'} - \tfrac{1}{2}\nabla_{BN'}\nabla_{CS'}\cdots\nabla_{JV'}\delta^{X'}{}_{W'}\Box G_K{}^{N'S'\cdots V'W'})]\end{aligned}$$

[where there are $2s-1$ pairs of terms in the square brackets and repeated use has been made of the identity, Eq. (A8)] = 0 by the wave equation (2.4) for the first term and by pairwise cancellation for the remaining $2s-1$ terms.

This section has established Eqs. (2.4) and (2.5) as the flat-space arbitrary-spin analogs of the electromagnetic theory equations (A1) and (A4).

The chief results of this paper consist of an extension of Eqs. (2.4) and (2.5) to curved space for the cases $s=\frac{1}{2}$, 1, and 2 and a subsequent Debye complex scalar reduction of the potential for these cases.

C. Vector potential and metric perturbations

Potentials intermediate between the Hertzian superpotentials and their corresponding zero-mass fields for spins 1 and 2 are well known—they are the vector potential and metric fields, respectively. These potentials are "half-way" between the Hertz potentials and the zero-rest-mass fields, in the sense that (1) the Hertz potential for spin 1 is differentiated once to obtain the vector potential, which in turn is differentiated once to produce the Maxwell field, while (2) the spin-2 Hertz potential is differentiated twice to yield the metric, which is differentiated twice again to give the Weyl tensor.

(1) For the spin-1 case, this result is proved in exterior form and tensor notation in Ref. 20 and the relevant formulas are repeated here in the Appendix, Eqs. (A1)–(A4). An equivalent flat-space spinor treatment is as follows.

The spin-1 or 2-index case of Sec. II A above states that

$$\nabla^{AX'}\phi_{AB} = 0\,, \tag{2.6}$$

that is, Maxwell's source-free equations are identically satisfied, for any ϕ_{AB} given by

$$\phi_{AB} = \nabla_{AX'}\nabla_{BY'}\bar{P}^{X'Y'}\,, \tag{2.7}$$

provided that the Hertz potential $\bar{P}^{X'Y'}$ obeys

$$\Box\bar{P}^{X'Y'} = 0\,. \tag{2.8}$$

If now the vector potential is defined by

$$A_{BY'}=\nabla_{BZ'}\overline{P}_{Y'}{}^{Z'}+\text{c.c.} \tag{2.9}$$

[cf. Eq. (A2)], then the tensor $f_{\mu\nu}$ which is defined in the usual way in terms of this potential, i.e., $f_{\mu\nu}=\nabla_\mu A_\nu-\nabla_\nu A_\mu$, is related under the standard skew 2-tensor–symmetric 2-spinor correspondence [Eq. (A10)] to ϕ_{AB} given by Eq. (2.7) and therefore is a Maxwell field. (Here and below, c.c. denotes complex conjugate.) To see this, consider the tensor

$$f_{AX'BY'}=\nabla_{AX'}A_{BY'}-\nabla_{BY'}A_{AX'};$$

we wish to show, for ϕ_{AB} given by Eq. (2.7), that

$$f_{AX'BY'}=\epsilon_{AB}\overline{\phi}_{X'Y'}+\epsilon_{X'Y'}\phi_{AB},$$

which is done by direct substitution:

$$\begin{aligned}f_{AX'BY'}&=\nabla_{AX'}\nabla_{BZ'}\overline{P}_{Y'}{}^{Z'}-\nabla_{BY'}\nabla_{AZ'}\overline{P}_{X'}{}^{Z'}+\text{c.c.}\\&=\nabla_{AX'}\nabla_{BZ'}\overline{P}_{Y'}{}^{Z'}-\nabla_{BX'}\nabla_{AZ'}\overline{P}_{Y'}{}^{Z'}\\&\quad+\nabla_{BX'}\nabla_{AZ'}\overline{P}_{Y'}{}^{Z'}-\nabla_{BY'}\nabla_{AZ'}\overline{P}_{X'}{}^{Z'}\end{aligned}$$

(by addition and subtraction of terms)

$$=\epsilon_{AB}\nabla_{HX'}\nabla^{H}{}_{Z'}\overline{P}_{Y'}{}^{Z'}+\epsilon_{X'Y'}\nabla_{BR'}\nabla_{AZ'}\overline{P}^{R'Z'}\quad\text{[by Eq. (A6)]}.$$

But Eq. (A8) and the wave equation (2.8) for $\overline{P}^{X'Y'}$ imply that the first term vanishes, leaving

$$f_{AX'BY'}=\epsilon_{X'Y'}\nabla_{BR'}\nabla_{AZ'}\overline{P}^{R'Z'}=\epsilon_{X'Y'}\phi_{AB}+\text{c.c.}$$

as claimed. This establishes that $A_{BY'}$ as given by Eq. (2.9) is indeed the vector potential in the standard sense and is given by a simple differential operation on the Hertzian superpotential. We see, furthermore, from Eqs. (2.7) and (2.9) that

$$\phi_{AB}=\nabla_{AX'}A_{B}{}^{X'}, \tag{2.10}$$

since the conjugate term in Eq. (2.9) contributes nothing to Eq. (2.10) by virtue of the wave condition (2.8).

In the presence of gauge transformations, the 2-index case of Sec. II B becomes

$$\nabla^{AX'}\phi_{AB}=0, \tag{2.11}$$

i.e., the source-free Maxwell equations are still satisfied, for ϕ_{AB} of the form

$$\phi_{AB}=\nabla_{AX'}\nabla_{BY'}\overline{P}^{X'Y'}-\nabla_{(AX'}G_{B)}{}^{X'}, \tag{2.12}$$

where $G_B{}^{X'}$ is an arbitrary spinor of indicated type, provided that the wave equation

$$\Box\overline{P}^{X'Y'}=2\nabla^{A(X'}G_A{}^{Y')} \tag{2.13}$$

is obeyed by the potential and gauge spinors. Here it is shown that the expression

$$A_{BY'}=\nabla_{BZ'}\overline{P}_{Y'}{}^{Z'}-G_{BY'}+\text{c.c.} \tag{2.14}$$

is the gauge-transformed vector potential corresponding to Eq. (A2) from which the field ϕ_{AB} given by Eq. (2.12) is obtained by the standard rule. The proof is carried out much as above in the absence of gauge terms:

$$\begin{aligned}f_{AX'BY'}&=\nabla_{AX'}A_{BY'}-\nabla_{BY'}A_{AX'}\\&=\nabla_{AX'}(\nabla_{BZ'}\overline{P}_{Y'}{}^{Z'}-G_{BY'})-\nabla_{BY'}(\nabla_{AZ'}\overline{P}_{X'}{}^{Z'}-G_{AX'})+\text{c.c.}\\&=\nabla_{AX'}\nabla_{BZ'}\overline{P}_{Y'}{}^{Z'}-\nabla_{BX'}\nabla_{AZ'}\overline{P}_{Y'}{}^{Z'}+\nabla_{BX'}\nabla_{AZ'}\overline{P}_{Y'}{}^{Z'}-\nabla_{BY'}\nabla_{AZ'}\overline{P}_{X'}{}^{Z'}\\&\quad+\nabla_{BY'}G_{AX'}-\nabla_{AX'}G_{BY'}\\&=\epsilon_{AB}\nabla_{HX'}\nabla^{H}{}_{Z'}\overline{P}_{Y'}{}^{Z'}+\epsilon_{X'Y'}\nabla_{BR'}\nabla_{AZ'}\overline{P}^{R'Z'}+\nabla_{BY'}G_{AX'}-\nabla_{AX'}G_{BY'}.\end{aligned}$$

In this case Eq. (A8) and the wave equation (2.13) allow the substitution $\epsilon_{AB}\nabla_{H(X'}G^{H}{}_{Y')}$ for the first term, while Eq. (A13) gives for the last two terms the alternate expression $-\epsilon_{AB}\nabla_{H(X'}G^{H}{}_{Y')}-\epsilon_{X'Y'}\nabla_{(AP'}G_{B)}{}^{P'}$. With these substitutions, we have

$$\begin{aligned}f_{AX'BY'}&=\epsilon_{AB}(\nabla_{H(X'}G^{H}{}_{Y')}-\nabla_{H(X'}G^{H}{}_{Y')})\\&\quad+\epsilon_{X'Y'}(\nabla_{BR'}\nabla_{AZ'}\overline{P}^{R'Z'}-\nabla_{(AZ'}G_{B)}{}^{Z'})\\&=\epsilon_{X'Y'}\phi_{AB}+\text{c.c.}\end{aligned}$$

Thus we have shown that Eq. (2.14) is the standard vector potential for the Maxwell field (2.2) in terms of a Hertz potential and gauge spinor obeying Eq. (2.13). From Eqs. (2.12) and (2.14), it is seen that the relation between the vector potential and the Maxwell spinor is

$$\phi_{AB}=\nabla_{(AX'}A_{B)}{}^{X'}, \tag{2.15}$$

since the contributions to this expression from the conjugate terms in Eq. (2.14) vanish by the wave equation (2.13).

(2) In the spin-2 or linearized gravitational case, it is claimed that the metric perturbations of flat space are given in the absence of gauge terms by[19]

$$h_{CD}{}^{M'N'}=\nabla_{CP'}\nabla_{DQ'}\overline{P}^{M'N'P'Q'}+\text{c.c.} \tag{2.16}$$

That is, the 4-index case of Sec. II A,

$$\nabla^{AX'}\psi_{ABCD}=0 \tag{2.17}$$

(interpreted as the free spin-2 field equation for the perturbed Weyl tensor), is satisfied identically provided that the potential obeys

$$\Box\bar{P}^{M'N'P'Q'}=0 \tag{2.18}$$

and that the perturbed Weyl tensor ψ_{ABCD} is related through the standard formula[32]

$$2R_{\alpha\beta\gamma\delta}=h_{\alpha\gamma;\beta\delta}+h_{\beta\delta;\alpha\gamma}-h_{\beta\gamma;\alpha\delta}-h_{\alpha\delta;\beta\gamma}+R^{(0)}_{\alpha\sigma\gamma\delta}h^{\sigma}{}_{\beta}-R^{(0)}_{\beta\sigma\gamma\delta}h^{\sigma}{}_{\alpha} \tag{2.19}$$

to the perturbed metric and hence via Eq. (2.16) to the potential (the background curvature terms are included for the discussion below of perturbations of curved space, although of course they are absent in the present considerations).

This result is shown in two ways, one of which focuses attention on the perturbed curvature and the other on the metric. The first approach shows by direct calculation that the metric (2.16) is a solution of the perturbed Einstein vacuum field equation $R_{\alpha\beta}=0$, which in terms of the metric becomes[7,32,33]

$$2R_{\alpha\beta}=h_{;\alpha\beta}+h_{\alpha\beta;\rho}{}^{\rho}-h^{\rho}{}_{\alpha;\beta\rho}-h^{\rho}{}_{\beta;\alpha\rho}=0\,, \tag{2.20}$$

where h denotes the trace of $h_{\mu\nu}$ (indices always being raised and lowered with the background metric). The second proof establishes that Eq. (2.19) in spinor notation becomes simply

$$\psi_{ABCD}=\nabla_{AW'}\nabla_{BX'}h_{CD}{}^{W'X'}\,, \tag{2.21}$$

which in combination with the formula (2.16) and the arbitrary-spin proof of Sec. IIA gives the desired result.

The proof which shows that $R_{\alpha\beta}=0$ is satisfied by the perturbed metric (2.16) is immediate and involves only the repeated use of Eq. (A8) and the wave equation (2.18) for the potential. Substitution of (2.16) into (2.20) gives

$$\begin{aligned}2R_{AW'BX'}={}&\nabla_{BX'}\nabla_{AW'}\nabla_{EP'}\nabla^{E}{}_{Q'}\bar{P}_{R'}{}^{R'P'Q'}\\&+\nabla^{ER'}\nabla_{ER'}\nabla_{AP'}\nabla_{BQ'}\bar{P}_{W'X'}{}^{P'Q'}\\&-\nabla_{ER'}\nabla_{BX'}\nabla^{E}{}_{P'}\nabla_{AQ'}\bar{P}^{R'}{}_{W'}{}^{P'Q'}\\&-\nabla_{ER'}\nabla_{AW'}\nabla^{E}{}_{P'}\nabla_{BQ'}\bar{P}^{R'}{}_{X'}{}^{P'Q'}+\text{c.c.}\end{aligned}$$

Each term vanishes by application of the identity $\nabla_{ER'}\nabla^{E}{}_{P'}=\frac{1}{2}\epsilon_{R'P'}\Box$ [Eq. (A8)] in combination with the wave equation (2.18) and the commuting of covariant derivatives in flat space, except for the second term which contains the d'Alembertian operator directly.

Alternatively, the proof which concentrates on the perturbed Weyl tensor and mades use of the results of Sec. IIA proceeds by substitution of Eq. (2.16) into Eq. (2.19), which leads eventually (up to a constant factor) to

$$R_{AW'BX'CY'DZ'}=\epsilon_{W'X'}\epsilon_{Y'Z'}\psi_{ABCD}+\text{c.c.}$$

[cf. Eq. (A14)]

with ψ_{ABCD} given by Eq. (2.21) or by the 4-index case of Eq. (2.3), as claimed. To see this, one writes Eq. (2.19) as

$$\begin{aligned}2R_{AW'BX'CY'DZ'}={}&\nabla_{BX'}\nabla_{DZ'}h_{AW'CY'}+\nabla_{AW'}\nabla_{CY'}h_{BX'DZ'}-\nabla_{AW'}\nabla_{DZ'}h_{BX'CY'}-\nabla_{BX'}\nabla_{CY'}h_{AW'DZ'}\\={}&\nabla_{AW'}\nabla_{CY'}h_{BX'DZ'}-\nabla_{BW'}\nabla_{CY'}h_{AX'DZ'}+\nabla_{BW'}\nabla_{CY'}h_{AX'DZ'}-\nabla_{BX'}\nabla_{CY'}h_{AW'DZ'}\\&+\nabla_{BX'}\nabla_{DZ'}h_{AW'CY'}-\nabla_{AX'}\nabla_{DZ'}h_{BW'CY'}+\nabla_{AX'}\nabla_{DZ'}h_{BW'CY'}-\nabla_{AW'}\nabla_{DZ'}h_{BX'CY'}\end{aligned}$$

(by addition and subtraction of terms)

$$=\epsilon_{AB}\nabla_{EW'}\nabla_{CY'}h^{E}{}_{X'DZ'}+\epsilon_{W'X'}\nabla_{BR'}\nabla_{CY'}h_{A}{}^{R'}{}_{DZ'}+\epsilon_{BA}\nabla_{EX'}\nabla_{DZ'}h^{E}{}_{W'CY'}+\epsilon_{X'W'}\nabla_{AR'}\nabla_{DZ'}h_{B}{}^{R'}{}_{CY'}$$

[by application of Eq. (A6)]

$$\begin{aligned}={}&\epsilon_{AB}\nabla_{EW'}\nabla_{CY'}h^{E}{}_{X'DZ'}-\epsilon_{AB}\nabla_{EW'}\nabla_{DY'}h^{E}{}_{X'CZ'}+\epsilon_{AB}\nabla_{EW'}\nabla_{DY'}h^{E}{}_{X'CZ'}\\&-\epsilon_{AB}\nabla_{EX'}\nabla_{DY'}h^{E}{}_{W'CZ'}+\epsilon_{AB}\nabla_{EX'}\nabla_{DY'}h^{E}{}_{W'CZ'}-\epsilon_{AB}\nabla_{EX'}\nabla_{DZ'}h^{E}{}_{W'CY'}\\&+\epsilon_{W'X'}\nabla_{BR'}\nabla_{CY'}h_{A}{}^{R'}{}_{DZ'}-\epsilon_{W'X'}\nabla_{BR'}\nabla_{CZ'}h_{A}{}^{R'}{}_{DY'}+\epsilon_{W'X'}\nabla_{BR'}\nabla_{CZ'}h_{A}{}^{R'}{}_{DY'}\\&-\epsilon_{W'X'}\nabla_{AR'}\nabla_{CZ'}h_{B}{}^{R'}{}_{DY'}+\epsilon_{W'X'}\nabla_{AR'}\nabla_{CZ'}h_{B}{}^{R'}{}_{DY'}-\epsilon_{W'X'}\nabla_{AR'}\nabla_{DZ'}h_{B}{}^{R'}{}_{CY'}\end{aligned}$$

(by addition and subtraction of terms)

$$\begin{aligned}={}&\epsilon_{AB}\epsilon_{CD}\nabla_{EW'}\nabla_{FY'}h^{EF}{}_{X'Z'}+\epsilon_{AB}\epsilon_{W'X'}\nabla_{ER'}\nabla_{DY'}h^{ER'}{}_{CZ'}+\epsilon_{AB}\epsilon_{Y'Z'}\nabla_{EX'}\nabla_{DR'}h^{E}{}_{W'C}{}^{R'}\\&+\epsilon_{W'X'}\epsilon_{Y'Z'}\nabla_{BR'}\nabla_{CS'}h_{AD}{}^{R'S'}+\epsilon_{W'X'}\epsilon_{BA}\nabla_{ER'}\nabla_{CZ'}h^{ER'}{}_{DY'}+\epsilon_{W'X'}\epsilon_{CD}\nabla_{AR'}\nabla_{EZ'}h_{B}{}^{R'E}{}_{Y'}\end{aligned} \tag{2.22}$$

[by application of Eq. (A6)]. If now the first term of Eq. (2.16) is substituted in this expression for the perturbed Riemann tensor, all terms except the fourth vanish by application of Eq. (A8) and the wave equation (2.18). Similarly, when the conjugate term in Eq. (2.16) is substituted, only the first term in this expression contributes. Together, they yield

$$2R_{AW'BX'CY'DZ'}=\epsilon_{AB}\epsilon_{CD}\overline{\psi}_{W'X'Y'Z'}+\epsilon_{W'X'}\epsilon_{Y'Z'}\psi_{ABCD},$$

with ψ_{ABCD} given by the spin-2 case of Eq. (2.3). This result, together with the proof of Sec. II A that ψ_{ABCD} satisfies the zero-rest-mass spin-2 wave equation, establishes that Eq. (2.16) gives the metric perturbations of Minkowski space.

The final results to be shown in this section consist of a generalization of these two proofs for the metric perturbation formula (2.16) to include gauge transformations of the third kind. That is, it is now shown that

$$h_{CD}{}^{M'N'}=\nabla_{CP'}\nabla_{DQ'}\overline{P}^{M'N'P'Q'}-\nabla_{(CP'}G_{D)}{}^{M'N'P'}+\text{c.c.} \tag{2.23}$$

gives the metric perturbations of flat space, provided that the 4-index case of Sec. II B holds, i.e., that the potential obeys

$$\Box\overline{P}^{M'N'P'Q'}=2\nabla^{A(M'}G_{A}{}^{N'P'Q')} \tag{2.24}$$

and that the Weyl tensor is given by Eq. (2.19) which becomes

$$\psi_{ABCD}=\nabla_{(AM'}\nabla_{BN'}h_{CD)}{}^{M'N'}, \tag{2.25}$$

or, in terms of the potential,

$$\psi_{ABCD}=\nabla_{AM'}\nabla_{BN'}\nabla_{CP'}\nabla_{DQ'}\overline{P}^{M'N'P'Q'}-\nabla_{(BN'}\nabla_{CP'}\nabla_{DQ'}G_{A)}{}^{N'P'Q'} \tag{2.26}$$

[the conjugate terms in Eq. (2.23) not contributing to ψ_{ABCD} because of Eq. (2.24)].

For the first proof, direct substitution of the perturbed metric (2.23) into the perturbed vacuum field equation $R_{\alpha\beta}=0$ [Eq. (2.20)] gives

$$\begin{aligned}2R_{AW'BX'}={}&\nabla_{BX'}\nabla_{AW'}(\nabla_{EP'}\nabla^{E}{}_{Q'}\overline{P}_{R'}{}^{R'P'Q'}-\nabla_{(EP'}G^{E)}{}_{R'}{}^{R'P'})+\nabla^{ER'}\nabla_{ER'}(\nabla_{AP'}\nabla_{BQ'}\overline{P}_{W'X'}{}^{P'Q'}-\nabla_{(AP'}G_{B)W'X'}{}^{P'})\\&-\nabla_{ER'}\nabla_{BX'}(\nabla^{E}{}_{P'}\nabla_{AQ'}\overline{P}^{R'}{}_{W'}{}^{P'Q'}-\nabla^{(E}{}_{P'}G_{A)}{}^{R'}{}_{W'}{}^{P'})-\nabla_{ER'}\nabla_{AW'}(\nabla^{E}{}_{P'}\nabla_{BQ'}\overline{P}^{R'}{}_{X'}{}^{P'Q'}-\nabla^{(E}{}_{P'}G_{B)}{}^{R'}{}_{X'}{}^{P'})+\text{c.c.}\end{aligned}$$

The two terms in the first parentheses each vanish since contraction on a pair of symmetrized spinor indices gives zero (which shows that $h_{CD}{}^{M'N'}$ is trace-free). The remaining terms may be written as

$$\begin{aligned}2R_{AW'BX'}={}&\nabla_{AP'}\nabla_{BQ'}(\Box\overline{P}_{W'X'}{}^{P'Q'}-\tfrac{1}{2}\nabla^{E}{}_{W'}G_{EX'}{}^{P'Q'}-\tfrac{1}{2}\nabla^{E}{}_{X'}G_{EW'}{}^{P'Q'}-\tfrac{1}{2}\nabla^{EP'}G_{EW'X'}{}^{Q'}-\tfrac{1}{2}\nabla^{EQ'}G_{EW'X'}{}^{P'})\\&+\nabla_{AP'}\nabla_{BQ'}(\tfrac{1}{2}\nabla^{E}{}_{W'}G_{EX'}{}^{P'Q'}+\tfrac{1}{2}\nabla^{E}{}_{X'}G_{EW'}{}^{P'Q'}+\tfrac{1}{2}\nabla^{EP'}G_{EW'X'}{}^{Q'}+\tfrac{1}{2}\nabla^{EQ'}G_{EW'X'}{}^{P'})\\&-\tfrac{1}{2}\Box(\nabla_{AP'}G_{BW'X'}{}^{P'}+\nabla_{BP'}G_{AW'X'}{}^{P'})+\tfrac{1}{2}\nabla_{ER'}\nabla_{BX'}(\nabla^{E}{}_{P'}G_{A}{}^{R'}{}_{W'}{}^{P'}+\nabla_{AP'}G^{ER'}{}_{W'}{}^{P'})\\&+\tfrac{1}{2}\nabla_{ER'}\nabla_{AW'}(\nabla^{E}{}_{P'}G_{B}{}^{R'}{}_{X'}{}^{P'}+\nabla_{BP'}G^{ER'}{}_{X'}{}^{P'})-\tfrac{1}{2}\nabla_{BX'}\epsilon_{R'P'}\Box\nabla_{AQ'}\overline{P}^{R'}{}_{W'}{}^{P'Q'}-\tfrac{1}{2}\nabla_{AW'}\epsilon_{R'P'}\Box\nabla_{BQ'}\overline{P}^{R'}{}_{X'}{}^{P'Q'},\end{aligned}$$

by addition and subtraction of terms. The last two terms are obtained by application of Eq. (A8) and each vanishes since the potential is symmetric in R' and P' while $\epsilon_{R'P'}$ is skew. Recombination of these terms gives

$$\begin{aligned}2R_{AW'BX'}={}&\nabla_{AP'}\nabla_{BQ'}(\Box\overline{P}_{W'X'}{}^{P'Q'}-2\nabla^{E}{}_{(W'}G_{EX'}{}^{P'Q')})+\tfrac{1}{2}\nabla_{AP'}\nabla_{BQ'}\nabla^{E}{}_{W'}G_{EX'}{}^{P'Q'}\\&+\tfrac{1}{2}\nabla_{AP'}\nabla_{BQ'}\nabla^{E}{}_{X'}G_{EW'}{}^{P'Q'}+\tfrac{1}{2}\nabla_{AP'}\nabla_{BQ'}\nabla^{EP'}G_{EW'X'}{}^{Q'}+\tfrac{1}{2}\nabla_{AP'}\nabla_{BQ'}\nabla^{EQ'}G_{EW'X'}{}^{P'}-\tfrac{1}{2}\Box\nabla_{AP'}G_{BW'X'}{}^{P'}-\tfrac{1}{2}\Box\nabla_{BP'}G_{AW'X'}{}^{P'}\\&+\tfrac{1}{2}\nabla_{ER'}\nabla_{BX'}\nabla^{E}{}_{P'}G_{A}{}^{R'}{}_{W'}{}^{P'}+\tfrac{1}{2}\nabla_{ER'}\nabla_{BX'}\nabla_{AP'}G^{ER'}{}_{W'}{}^{P'}+\tfrac{1}{2}\nabla_{ER'}\nabla_{AW'}\nabla^{E}{}_{P'}G_{B}{}^{R'}{}_{X'}{}^{P'}+\tfrac{1}{2}\nabla_{ER'}\nabla_{AW'}\nabla_{BP'}G^{ER'}{}_{X'}{}^{P'}.\end{aligned}$$

The terms in parentheses add up to zero by the wave equation (2.24). Of the remaining ten terms, the second and eighth may be combined using Eq. (A6) to yield

$$-\tfrac{1}{2}\nabla_{EX'}\nabla_{BQ'}\nabla_{AP'}G^{E}{}_{W'}{}^{P'Q'}+\tfrac{1}{2}\nabla_{EQ'}\nabla_{BX'}\nabla_{AP'}G^{EQ'}{}_{W'}{}^{P'}=\tfrac{1}{2}\epsilon_{Q'X'}\nabla_{EZ'}\nabla_{B}{}^{Z'}\nabla_{AP'}G^{E}{}_{W'}{}^{P'Q'},$$

which may in turn by use of Eq. (A8) by written as $\frac{1}{4}\epsilon_{Q'X'}\epsilon_{EB}\nabla_{AP'}\Box G^{E}{}_{W'}{}^{P'Q'}$ or $\frac{1}{4}\nabla_{AP'}\Box G_{BW'X'}{}^{P'}$. Similarly, the first and tenth terms combine to give $\frac{1}{4}\nabla_{BP'}\Box G_{AW'X'}{}^{P'}$. Furthermore, Eq. (A8) may be used to generate the d'Alembertian operator in each of the remaining terms. With all of these substitutions, we have

$$\begin{aligned}2R_{AW'BX'}={}&\tfrac{1}{4}\nabla_{BP'}\Box G_{AW'X'}{}^{P'}+\tfrac{1}{4}\nabla_{AP'}\Box G_{BW'X'}{}^{P'}+\tfrac{1}{4}\nabla_{BQ'}\Box G_{AW'X'}{}^{Q'}\\&+\tfrac{1}{4}\nabla_{AP'}\Box G_{BW'X'}{}^{P'}-\tfrac{1}{2}\nabla_{AP'}\Box G_{BW'X'}{}^{P'}-\tfrac{1}{2}\nabla_{BP'}\Box G_{AW'X'}{}^{P'}+\tfrac{1}{4}\nabla_{BX'}\Box\epsilon_{R'P'}G_{A}{}^{R'}{}_{W'}{}^{P'}+\tfrac{1}{4}\nabla_{AW'}\Box\epsilon_{R'P'}G_{B}{}^{R'}{}_{X'}{}^{P'}.\end{aligned}$$

The last two terms each vanish by contraction of symmetric indices, while the remaining terms cancel, thus completing the proof.

Finally, the proof of the perturbed metric expression (2.23) which concentrates on the perturbed Weyl tensor is generalized to include gauge transformations of the third kind. To do this it suffices to show that with the perturbed metric given by Eq. (2.23), the perturbed Weyl tensor which follows from Eq. (2.19) is in fact just Eq. (2.25), or equivalently, Eq. (2.26). The proof itself then follows from the arbitrary-spin results of Sec. II B.

A spinor computation which shows directly that Eq. (2.19) does in fact yield Eq. (2.25) for the Weyl tensor is now sketched. As shown above, Eq. (2.19) may be transformed to the spinor expression (2.22) for the perturbed curvature spinor. Substitution of the first term of Eq. (2.23) into Eq. (2.22), with repeated application of Eq. (A8) yields

$$
\begin{aligned}
2R_{AW'BX'CY'DZ'} = {} & \epsilon_{AB}\epsilon_{CD}\left(\tfrac{1}{4}\epsilon_{W'P'}\epsilon_{Y'Q'}\Box\Box\bar{P}_{X'Z'}{}^{P'Q'} - \tfrac{1}{4}\epsilon_{W'P'}\Box\nabla_{FY'}G^{F}{}_{X'Z'}{}^{P'} - \tfrac{1}{4}\epsilon_{Y'P'}\Box\nabla_{FW'}G^{F}{}_{X'Z'}{}^{P'}\right) \\
& + \epsilon_{AB}\epsilon_{W'X'}\left(\tfrac{1}{2}\epsilon_{R'P'}\Box\nabla_{DY'}\nabla_{CQ'}\bar{P}^{R'}{}_{Z'}{}^{P'Q'} - \tfrac{1}{4}\epsilon_{R'P'}\Box\nabla_{DY'}G_{C}{}^{R'}{}_{Z'}{}^{P'} - \tfrac{1}{2}\nabla_{ER'}\nabla_{DY'}\nabla_{CP'}G^{ER'}{}_{Z'}{}^{P'}\right) \\
& + \epsilon_{AB}\epsilon_{Y'Z'}\left(\tfrac{1}{2}\epsilon_{X'P'}\Box\nabla_{DR'}\nabla_{CQ'}\bar{P}^{R'}{}_{W'}{}^{P'Q'} - \tfrac{1}{4}\epsilon_{X'P'}\Box\nabla_{DR'}G_{C}{}^{R'}{}_{W'}{}^{P'} - \tfrac{1}{2}\nabla_{EX'}\nabla_{DR'}\nabla_{CP'}G^{ER'}{}_{W'}{}^{P'}\right) \\
& + \epsilon_{W'X'}\epsilon_{Y'Z'}\left(\nabla_{BR'}\nabla_{CS'}\nabla_{AP'}\nabla_{DQ'}\bar{P}^{R'S'P'Q'} - \tfrac{1}{2}\nabla_{BR'}\nabla_{CS'}\nabla_{AP'}G_{D}{}^{R'S'P'} - \tfrac{1}{2}\nabla_{BR'}\nabla_{CS'}\nabla_{DP'}G_{A}{}^{R'S'P'}\right) \\
& + \epsilon_{W'X'}\epsilon_{BA}\left(\tfrac{1}{2}\epsilon_{R'P'}\Box\nabla_{CZ'}\nabla_{DQ'}\bar{P}^{R'}{}_{Y'}{}^{P'Q'} - \tfrac{1}{4}\epsilon_{R'P'}\Box\nabla_{CZ'}G_{D}{}^{R'}{}_{Y'}{}^{P'} - \tfrac{1}{2}\nabla_{ER'}\nabla_{CZ'}\nabla_{DP'}G^{ER'}{}_{Y'}{}^{P'}\right) \\
& + \epsilon_{W'X'}\epsilon_{CD}\left(\tfrac{1}{2}\epsilon_{Z'Q'}\Box\nabla_{AR'}\nabla_{BP'}\bar{P}^{R'}{}_{Y'}{}^{P'Q'} - \tfrac{1}{2}\nabla_{AR'}\nabla_{EZ'}\nabla_{BP'}G^{ER'}{}_{Y'}{}^{P'} - \tfrac{1}{2}\epsilon_{Z'P'}\Box\nabla_{AR'}G_{B}{}^{R'}{}_{Y'}{}^{P'}\right).
\end{aligned}
$$

In the second and fifth set of parentheses, the first two terms each vanish by contraction of symmetric spinor indices. The strategy at this point is to regroup the remaining terms by adding and subtracting suitable terms in such a way as to generate the expression (2.26) for ψ_{ABCD} and also to generate the difference between the potential and gauge terms in Eq. (2.24) [which then cancel by the wave equation (2.24)]. This procedure gives

$$
\begin{aligned}
2R_{AW'BX'CY'DZ'} = {} & \epsilon_{W'X'}\epsilon_{Y'Z'}\left(\nabla_{AM'}\nabla_{BN'}\nabla_{CP'}\nabla_{DQ'}\bar{P}^{M'N'P'Q'} - \nabla_{(BN'}\nabla_{CP'}\nabla_{DQ'}G_{A)}{}^{N'P'Q'}\right) \\
& - \tfrac{1}{4}\epsilon_{W'X'}\epsilon_{Y'Z'}\left(\epsilon_{CD}\nabla_{BR'}\nabla_{AP'}\nabla_{ES'}G^{ER'S'P'} + \epsilon_{BA}\nabla_{CS'}\nabla_{DP'}\nabla_{ER'}G^{ER'S'P'}\right) \\
& + \tfrac{1}{4}\epsilon_{AB}\epsilon_{CD}\Box\left(\tfrac{1}{2}\epsilon_{X'Y'}\nabla^{E}{}_{R'}G_{EW'}{}^{R'}{}_{Z'} + \tfrac{1}{2}\epsilon_{Z'W'}\nabla^{E}{}_{R'}G_{E}{}^{R'}{}_{X'Y'}\right) \\
& - \tfrac{1}{2}\epsilon_{AB}\epsilon_{Y'Z'}\nabla_{DR'}\nabla_{CQ'}\left(\tfrac{1}{2}\epsilon_{W'X'}\nabla^{E}{}_{S'}G_{E}{}^{R'S'Q'} + \tfrac{1}{2}\nabla^{ER'}G_{EW'X'}{}^{Q'} + \tfrac{1}{2}\nabla^{EQ'}G_{E}{}^{R'}{}_{W'X'}\right) \\
& - \tfrac{1}{2}\epsilon_{CD}\epsilon_{W'X'}\nabla_{AR'}\nabla_{BP'}\left(\tfrac{1}{2}\epsilon_{Y'Z'}\nabla^{E}{}_{S'}G_{E}{}^{R'S'P'} + \tfrac{1}{2}\nabla^{ER'}G_{EZ'Y'}{}^{P'} + \tfrac{1}{2}\nabla^{EP'}G_{E}{}^{R'}{}_{Y'Z'}\right) \\
& - \tfrac{1}{2}\epsilon_{AB}\epsilon_{W'X'}\nabla_{ER'}\nabla_{DY'}\nabla_{CP'}G^{ER'}{}_{Z'}{}^{P'} + \tfrac{1}{2}\epsilon_{AB}\epsilon_{W'X'}\nabla_{ER'}\nabla_{CZ'}\nabla_{DP'}G^{ER'}{}_{Y'}{}^{P'} \\
& + \tfrac{1}{4}\epsilon_{AB}\epsilon_{Y'Z'}\Box\nabla_{DR'}G_{C}{}^{R'}{}_{W'X'} + \tfrac{1}{4}\epsilon_{W'X'}\epsilon_{CD}\Box\nabla_{AR'}G_{B}{}^{R'}{}_{Y'Z'},
\end{aligned}
$$

where terms canceling by Eq. (2.24) have been omitted. All terms beyond the first may be shown to cancel one another by suitably combining them and applying Eq. (A8), as well as by addition and subtraction of terms followed by use of Eq. (A6).

The whole procedure outlined here may be repeated for the complex-conjugate term in the metric (2.23), giving finally

$$2R_{AW'BX'CY'DZ'} = \epsilon_{AB}\epsilon_{CD}\bar{\psi}_{W'X'Y'Z'} + \epsilon_{W'X'}\epsilon_{Y'Z'}\psi_{ABCD},$$

with ψ_{ABCD} given by Eq. (2.26). This result, with vanishing perturbed Ricci tensor, establishes that Eq. (2.25) [or equivalently, Eq. (2.26)] indeed gives the Weyl tensor as claimed. The results of Sec. II B then complete the second proof that Eq. (2.23) gives the metric perturbations of flat space with gauge transformations of the third kind.

D. Summary of Sec. II

It has been shown that in flat space, the arbitrary-spin zero-rest-mass field equations

$$\nabla^{AX'}\phi_{AB\cdots K} = 0 \tag{2.1}$$

are satisfied by Hertzian superpotentials $P_{AB\cdots K}$ of the same spinor type as the fields themselves, provided that (1) the fields are given by

$$\phi_{AB\cdots K} = \nabla_{AM'}\nabla_{BN'}\cdots\nabla_{KW'}\bar{P}^{M'N'\cdots W'} - \nabla_{(BN'}\cdots\nabla_{KW'}G_{A)}{}^{N'\cdots W'}, \tag{2.5}$$

with $G_{A}{}^{N'\cdots W'} = G_{A}{}^{(N'\cdots W')}$ an arbitrary spinor of indicated type, and that (2) a wave condition

$$\Box\bar{P}^{X'N'\cdots W'} = 2\nabla^{A(X'}G_{A}{}^{N'\cdots W')} \tag{2.4}$$

is satisfied by the potential and the gauge spinor $G_{A}{}^{N'\cdots W'}$.

Furthermore, the flat-space vector potential and metric perturbations are obtained by simple differentiations of the corresponding Hertzian potentials, which in turn yield the Maxwell field or Weyl tensor, respectively, by further differentiation. These relations are

$$A_{BY'} = \nabla_{BZ'}\bar{P}_{Y'}{}^{Z'} - G_{BY'} + \text{c.c.} \tag{2.14}$$

and

$$\phi_{AB} = \nabla_{(AX'}A_{B)}{}^{X'} \tag{2.15}$$

for spin 1 or electrodynamics, and

$$h_{CD}{}^{M'N'} = \nabla_{CP'}\nabla_{DQ'}\bar{P}^{M'N'P'Q'} - \nabla_{(CP'}G_{D)}{}^{M'N'P'} + \text{c.c.} \qquad (2.23)$$

and

$$\psi_{ABCD} = \nabla_{(AW'}\nabla_{BX'}h_{CD)}{}^{W'X'} \qquad (2.25)$$

for spin 2 or linearized gravitation.

III. SPIN-1 HERTZ, DEBYE, AND VECTOR POTENTIALS IN CURVED SPACE

The results of this section may be viewed either as a translation of the results of Ref. 20 into spinor notation, or as a generalization of the spin-1 results of the last section to curved spacetimes. Either viewpoint leads in a fairly natural way to the formalism below, although the first provides a more systematic development.

The correspondence of spinors and tensors given by Eq. (A10) is now used to show that Eq. (2.1) for $s=1$,

$$\nabla^{AX'}\phi_{AB} = 0\,, \qquad (3.1)$$

is indeed the spinor version of Maxwell's equations, which for convenience are repeated here in tensor notation:

$$\nabla_\mu f_{\nu\lambda} + \nabla_\nu f_{\lambda\mu} + \nabla_\lambda f_{\mu\nu} = 0\,, \qquad (3.2a)$$

$$\nabla^\mu f_{\mu\nu} = 0\,. \qquad (3.2b)$$

Writing

$$f_{AW'BX'} = \epsilon_{AB}\bar{\phi}_{W'X'} + \epsilon_{W'X'}\phi_{AB} \qquad (3.3)$$

according to the bivector-spinor correspondence of Eq. (A10) and expressing Eq. (3.2b) in spinor notation gives

$$\nabla^{AX'}(\epsilon_{AB}\bar{\phi}_{W'X'} + \epsilon_{W'X'}\phi_{AB}) = 0\,. \qquad (3.4)$$

Moreover, Eq. (3.2a) states that $\nabla^\mu(*f_{\mu\nu}) = 0$, which by Eq. (A15) becomes

$$\nabla^{AX'}(\epsilon_{AB}\bar{\phi}_{W'X'} - \epsilon_{W'X'}\phi_{AB}) = 0\,. \qquad (3.5)$$

Subtracting Eq. (3.5) from Eq. (3.4) yields Eq. (3.1) as claimed.

A. Hertz potentials

Similarly a direct transcription of the harmonic operator of Eq. (A1) via the correspondence Eq. (A10) into spinor notation yields

$$\Box P_{AB} + \nabla_{(AR'}\nabla^{CR'}P_{B)C} - \nabla^{CR'}\nabla_{(BR'}P_{A)C} = 0\,, \qquad (3.6)$$

where the free unprimed indices A and B are symmetrized and the symmetric spinor P_{AB} corresponds to the Hertz potential by the standard relation (A10), $P_{\mu\nu} \to \epsilon_{AB}\bar{P}_{W'X'} + \epsilon_{W'X'}P_{AB}$. It is to be noted that the harmonic operator on a second-rank spinor does not correspond to the d'Alembertian operator [the first term of Eq. (3.6)], except in flat space where the remaining terms of Eq. (3.6) vanish by the Ricci identities.

The same procedure when applied to the prescriptions (A4) for the Maxwell field yields two expressions for ϕ_{AB} in terms of $\bar{P}^{W'X'}$, which become identical by virtue of the wave condition, Eq. (3.6). The expressions are

$$\phi_{AB} = \nabla_{(AR'}\nabla_{B)Z'}\bar{P}^{R'Z'} - \nabla_{(AR'}\nabla^{DR'}P_{B)D} \qquad (3.7a)$$

$$= \Box P_{AB} + \nabla_{(AR'}\nabla_{B)Z'}\bar{P}^{R'Z'} - \nabla^{DZ'}\nabla_{(BZ'}P_{A)D}\,, \qquad (3.7b)$$

where the symmetrization applies to A and B. To show that these are in fact the same one uses the spinor identity

$$\Box\eta_{AB} = \nabla_{(AR'}\nabla^{CR'}\eta_{B)C} + \nabla^{CR'}\nabla_{(BR'}\eta_{A)C}\,, \qquad (3.8)$$

which is given in the Appendix as Eq. (A16) and is proved there. The first application of this identity will be to Eq. (3.6), where direct substitution shows that the wave operator becomes

$$\nabla_{(AR'}\nabla^{CR'}P_{B)C} = 0\,. \qquad (3.9)$$

Next Eq. (3.7b) becomes, by virtue of Eq. (3.8),

$$\phi_{AB} = \nabla_{(AR'}\nabla^{CR'}P_{B)C} + \nabla_{(AR'}\nabla_{B)Z'}\bar{P}^{R'Z'}\,. \qquad (3.10)$$

Finally, the harmonic condition (3.9) is used in Eqs. (3.7a) and (3.10) to show that they both become

$$\phi_{AB} = \nabla_{(AR'}\nabla_{B)Z'}\bar{P}^{R'Z'}\,. \qquad (3.11)$$

This equation together with the complex conjugate of Eq. (3.9),

$$\nabla^{A(X'}\nabla_{AZ'}\bar{P}^{R')Z'} = 0\,, \qquad (3.12)$$

are the desired curved-space extensions of Eqs. (2.3) and (2.2) for spin $s=1$. It should be remarked that Eq. (2.3) has been generalized to curved space in the most natural and obvious way, by mere symmetrization of the derivatives, the flat-space expression being automatically symmetric in the derivatives by the flat-space commutation property.

That ϕ_{AB} given by Eq. (3.11) satisfies the Maxwell equation (3.1) by virtue of the wave condition (3.12) is now proved in spinor notation. One approach would be to transcribe the tensor proof of Ref. 20 into spinor notation, but a more direct, purely spinor approach will be adopted instead. The basic strategy of the proof is to break up $\nabla^{AX'}\phi_{AB}$ with ϕ_{AB} given by Eq. (3.11) into a piece which vanishes because of Eq. (3.12), and other terms which are commutators and vanish by the Ricci identities, Eqs. (A19) and (A20). The computation is presented here:

$$\begin{aligned}\nabla^{AX'}\phi_{AB} &= \nabla^{AX'}(\nabla_{(AR'}\nabla_{B)S'}\bar{P}^{R'S'}) \quad \text{[by Eq. (3.11)]}\\
&= \tfrac{1}{2}(\nabla^{AX'}\nabla_{AR'}\nabla_{BS'}\bar{P}^{R'S'} + \nabla^{AX'}\nabla_{BR'}\nabla_{AS'}\bar{P}^{R'S'})\\
&= \nabla^{AX'}\nabla_{BS'}\nabla_{AR'}\bar{P}^{R'S'} + \tfrac{1}{2}\nabla^{AX'}(\nabla_{AR'}\nabla_{BS'} - \nabla_{BS'}\nabla_{AR'})\bar{P}^{R'S'} \quad \text{(using the symmetry of } \bar{P}^{R'S'})\\
&= \nabla_{BS'}\nabla^{AX'}\nabla_{AR'}\bar{P}^{R'S'} + (\nabla^{AX'}\nabla_{BS'} - \nabla_{BS'}\nabla^{AX'})\nabla_{AR'}\bar{P}^{R'S'} + \tfrac{1}{2}\nabla^{AX'}(\nabla_{AR'}\nabla_{BS'} - \nabla_{BS'}\nabla_{AR'})\bar{P}^{R'S'}\\
&= \tfrac{1}{2}\nabla_{BS'}\nabla^{AX'}\nabla_{AR'}\bar{P}^{R'S'} + \tfrac{1}{2}\nabla_{BS'}\nabla_{AR'}\nabla^{AX'}\bar{P}^{R'S'} + \tfrac{1}{2}\nabla_{BS'}(\nabla^{AX'}\nabla_{AR'} - \nabla_{AR'}\nabla^{AX'})\bar{P}^{R'S'}\\
&\quad + (\nabla^{AX'}\nabla_{BS'} - \nabla_{BS'}\nabla^{AX'})\nabla_{AR'}\bar{P}^{R'S'} + \tfrac{1}{2}\nabla^{AX'}(\nabla_{AR'}\nabla_{BS'} - \nabla_{BS'}\nabla_{AR'})\bar{P}^{R'S'}\,.\end{aligned}$$

But $\nabla^{AX'}\bar{P}^{R'S'} - \nabla^{AS'}\bar{P}^{R'X'} = \epsilon^{X'S'}\nabla^{A}{}_{Z'}\bar{P}^{R'Z'}$, so the second term may be replaced by $\frac{1}{2}\nabla_{BS'}\nabla_{AR'}(\nabla^{AS'}\bar{P}^{R'X'} + \epsilon^{X'S'}\nabla^{A}{}_{Z'}\bar{P}^{R'Z'})$. Furthermore, $\nabla_{AR'}\nabla^{AS'}\bar{P}^{R'X'} = \nabla^{AS'}\nabla_{AR'}\bar{P}^{R'X'} + (\nabla_{AR'}\nabla^{AS'} - \nabla^{AS'}\nabla_{AR'})\bar{P}^{R'X'}$, so the expression for the second term becomes $\frac{1}{2}\nabla_{BS'}\nabla^{AS'}\nabla_{AR'}\bar{P}^{R'X'} + \frac{1}{2}\nabla_{BS'}(\nabla_{AR'}\nabla^{AS'} - \nabla^{AS'}\nabla_{AR'})\bar{P}^{R'X'} + \frac{1}{2}\nabla_{B}{}^{X'}\nabla_{AR'}\nabla^{A}{}_{Z'}\bar{P}^{R'Z'}$. Hence, substituting,

$$\begin{aligned}\nabla^{AX'}\phi_{AB} &= \nabla_{BS'}\nabla^{A(X'}\nabla_{AR'}\bar{P}^{S')R'} + \tfrac{1}{2}\nabla_{BS'}(\nabla_{AR'}\nabla^{AS'} - \nabla^{AS'}\nabla_{AR'})\bar{P}^{R'X'} + \tfrac{1}{2}\nabla_{B}{}^{X'}\nabla_{AR'}\nabla^{A}{}_{Z'}\bar{P}^{R'Z'}\\
&\quad + \tfrac{1}{2}\nabla_{BS'}(\nabla^{AX'}\nabla_{AR'} - \nabla_{AR'}\nabla^{AX'})\bar{P}^{R'S'} + (\nabla^{AX'}\nabla_{BS'} - \nabla_{BS'}\nabla^{AX'})\nabla_{AR'}\bar{P}^{R'S'} + \tfrac{1}{2}\nabla^{AX'}(\nabla_{AR'}\nabla_{BS'} - \nabla_{BS'}\nabla_{AR'})\bar{P}^{R'S'}\,.\end{aligned} \tag{3.13}$$

The first term vanishes by the wave condition Eq. (3.12). The second and fourth terms combine by Eq. (A6) to yield $\frac{1}{2}\epsilon^{S'X'}\nabla_{BS'}(\nabla_{AR'}\nabla^{A}{}_{Z'} - \nabla^{A}{}_{Z'}\nabla_{AR'})\bar{P}^{R'Z'}$, which in turn combines with the third term, leaving

$$\begin{aligned}\nabla^{AX'}\phi_{AB} &= \tfrac{1}{2}\nabla_{B}{}^{X'}\nabla^{A}{}_{Z'}\nabla_{AR'}\bar{P}^{R'Z'}\\
&\quad + (\nabla^{AX'}\nabla_{BS'} - \nabla_{BS'}\nabla^{AX'})\nabla_{AR'}\bar{P}^{R'S'}\\
&\quad + \tfrac{1}{2}\nabla^{AX'}(\nabla_{AR'}\nabla_{BS'} - \nabla_{BS'}\nabla_{AR'})\bar{P}^{R'S'}\,.\end{aligned} \tag{3.14}$$

These three terms are now expressed via the Ricci identities in terms of curvature quantities.

In the first term, the last two operators $\nabla^{A}{}_{Z'}\nabla_{AR'}\bar{P}^{R'Z'} = \nabla^{A}{}_{(Z'}\nabla_{AR')}\bar{P}^{R'Z'}$ may be expressed, using the complex conjugate of Eq. (A19) and the Leibnitz product rule, as

$$\begin{aligned}\nabla^{A}{}_{(Z'}\nabla_{AR')}\bar{P}^{R'Z'} &= -\bar{\Psi}_{R'Z'}{}^{Z'}{}_{X'}\bar{P}^{R'X'} + 2\Lambda\bar{P}^{R'}{}_{(R'}\epsilon_{Z')}{}^{Z'}\\
&\quad - \bar{\Psi}_{R'Z'}{}^{R'}{}_{X'}\bar{P}^{Z'X'} + 2\Lambda\bar{P}^{Z'}{}_{(R'}\epsilon_{Z')}{}^{R'}\,.\end{aligned}$$

But the Weyl spinor is totally symmetric and hence the contractions vanish. The terms involving the curvature scalar are

$$\begin{aligned}&\Lambda(2\bar{P}^{R'}{}_{R'} + \bar{P}^{R'}{}_{Z'}\delta_{R'}{}^{Z'}) + \Lambda(\bar{P}^{Z'}{}_{R'}\delta_{Z'}{}^{R'} + 2\bar{P}^{Z'}{}_{Z'})\\
&\quad = 6\Lambda\bar{P}^{R'}{}_{R'} = 0\end{aligned}$$

by the symmetry of the potential. Hence the first term vanishes.

The second term may be rewritten as

$$(-\delta^{A}{}_{B}\nabla_{E}{}^{(X'}\nabla^{E}{}_{S')} - \delta^{X'}{}_{S'}\nabla^{(A}{}_{Z'}\nabla_{B)}{}^{Z'})\xi_{A}{}^{S'}\,, \tag{3.15}$$

where $\xi_{A}{}^{S'} \equiv \nabla_{AR'}\bar{P}^{R'S'}$, and where use has been made of Eq. (A12). In evaluating the first term in Eq. (3.15), one finds by the conjugates of Eqs. (A19) and (A20) and the Leibnitz rule that

$$\begin{aligned}\nabla^{E}{}_{(X'}\nabla_{ES')}\xi_{AY'} &= -\bar{\Psi}_{X'S'Y'Z'}\xi_{A}{}^{Z'}\\
&\quad + 2\Lambda\xi_{A(X'}\epsilon_{S')Y'} - \Phi_{ABX'S'}\xi^{B}{}_{Y'}\,.\end{aligned}$$

Hence

$$\begin{aligned}\nabla_{E}{}^{(X'}\nabla^{E}{}_{S')}\xi_{A}{}^{S'} &= \bar{\Psi}^{X'}{}_{S'}{}^{S'}{}_{Z'}\xi_{A}{}^{Z'}\\
&\quad - 2\Lambda\xi_{A}{}^{(X'}\epsilon_{S')}{}^{S'} + \Phi_{AE}{}^{X'}{}_{S'}\xi^{ES'}\,.\end{aligned}$$

The contracted Weyl tensor vanishes and the remaining terms work out to be

$$\begin{aligned}&-\Lambda(\xi_{A}{}^{X'}\cdot 2 + \xi_{AS'}\epsilon^{X'S'}) + \Phi_{AE}{}^{X'}{}_{S'}\xi^{ES'}\\
&\quad = -3\Lambda\xi_{A}{}^{X'} + \Phi_{AE}{}^{X'}{}_{S'}\xi^{ES'}\,.\end{aligned}$$

Hence the first term in Eq. (3.15) is

$$\begin{aligned}&-\delta^{A}{}_{B}\nabla_{E}{}^{(X'}\nabla^{E}{}_{S')}\nabla_{AR'}\bar{P}^{R'S'}\\
&\quad = 3\Lambda\nabla_{BR'}\bar{P}^{R'X'} - \Phi_{BE}{}^{X'}{}_{S'}\nabla^{E}{}_{R'}\bar{P}^{R'S'}\,.\end{aligned} \tag{3.16}$$

The second term in Eq. (3.15) is evaluated similarly, making use again of Eqs. (A19) and (A20), the Leibnitz rule, and the Hermiticity of $\Phi_{ABW'X'}$ and reality of Λ. The result is just the negative of Eq. (3.16), so the second term in Eq. (3.14) vanishes.

The commutator in the third term in Eq. (3.14) may be rewritten by Eq. (A12) as

$$\epsilon_{AB}\nabla_{E(R'}\nabla^{E}{}_{S')}\bar{P}^{R'S'} + \epsilon_{R'S'}\nabla_{(AZ'}\nabla_{B)}{}^{Z'}\bar{P}^{R'S'}\,.$$

The second of these vanished by the antisymmetry of $\epsilon_{R'S'}$ and the symmetry of $\bar{P}^{R'S'}$. The first term contains the factor $\nabla_{E(R'}\nabla^{E}{}_{S')}\bar{P}^{R'S'}$, which was shown above to vanish in the discussion of the first term of Eq. (3.14).

Hence it has been established that if the potential obeys the harmonic condition (3.12), then ϕ_{AB} given by the second-order operation (3.11) is a solution to Eq. (3.1), i.e., ϕ_{AB} obeys Maxwell's equations.

The formalism for gauge transformations of the third kind is now expressed in spinor notation for the curved-space case. The analogs of the flat-space Eqs. (2.4) and (2.5) of the last section for $s=1$ are

$$\nabla^{A(X'}\nabla_{AZ'}\overline{P}^{R')Z'}=\nabla^{A(X'}G_A{}^{R')} \tag{3.17}$$

and

$$\phi_{AB}=\nabla_{(AR'}\nabla_{B)Z'}\overline{P}^{R'Z'}-\nabla_{(AR'}G_{B)}{}^{R'}. \tag{3.18}$$

These are the spinor translations of Eqs. (A1) and (A4), i.e., of Eqs. (3.7) and (3.8) or Eqs. (4.6) and (4.7) of Ref. 20. They are the spinor version of the Hertz potential formalism as generalized to all curved spacetimes.

The proof is now presented that ϕ_{AB} given by Eq. (3.18) is a source-free Maxwell field, i.e., obeys Eq. (3.1):

$$\begin{aligned}\nabla^{AX'}\phi_{AB}&=\nabla^{AX'}\nabla_{(AR'}\nabla_{B)Z'}\overline{P}^{R'Z'}-\nabla^{AX'}\nabla_{(AR'}G_{B)}{}^{R'}\\&=\nabla^{AX'}\nabla_{(AR'}\nabla_{B)Z'}\overline{P}^{R'Z'}-\tfrac{1}{2}\nabla_{BR'}\nabla^{AX'}G_A{}^{R'}-\tfrac{1}{2}(\nabla^{AX'}\nabla_{BR'}-\nabla_{BR'}\nabla^{AX'})G_A{}^{R'}\\&\quad-\tfrac{1}{2}\nabla_{BR'}\nabla^{AR'}G_A{}^{X'}+\tfrac{1}{2}(\nabla_{BR'}\nabla^{AR'}G_A{}^{X'}-\nabla^{AX'}\nabla_{AR'}G_B{}^{R'}).\end{aligned} \tag{3.19}$$

The second and fourth terms combine to yield $-\nabla_{BR'}\nabla^{A(X'}G_A{}^{R')}$. The commutator in the third term is seen to vanish by the Ricci identities:

$$\begin{aligned}(\nabla^{AX'}\nabla_{BR'}-\nabla_{BR'}\nabla^{AX'})G_A{}^{R'}=(&-\delta^A{}_B\nabla_H{}^{(X'}\nabla^H{}_{R')}\\&-\delta^{X'}{}_{R'}\nabla^{(A}{}_{P'}\nabla_{B)}{}^{P'})G_A{}^{R'}\end{aligned}$$

[by Eq. (A12)].

This is the same as expression (3.15) which was shown above to be zero. If Eq. (3.13) is now substituted for the first term in Eq. (3.19), making use of the fact that all terms in Eq. (3.13) except for the first have been shown to vanish identically by the Ricci identities, Eq. (3.19) becomes

$$\begin{aligned}\nabla^{AX'}\phi_{AB}=&\nabla_{BS'}\nabla^{A(X'}\nabla_{AR'}\overline{P}^{S')R'}-\nabla_{BS'}\nabla^{A'(X'}G_A{}^{S')}\\&+\tfrac{1}{2}(\nabla_{BR'}\nabla^{AR'}G_A{}^{X'}-\nabla^{AX'}\nabla_{AR'}G_B{}^{R'}).\end{aligned} \tag{3.20}$$

The first two terms cancel by the assumed potential wave equation (3.17). The last term is now shown to vanish. The first term in the parentheses, $\nabla_{BR'}\nabla^{AR'}G_A{}^{X'}$, may be rewritten as follows: Since

$$\nabla_{BR'}\nabla^{AR'}G_A{}^{X'}-\nabla^A{}_{R'}\nabla_B{}^{R'}G_A{}^{X'}=\delta_B{}^A\Box G_A{}^{X'}=\Box G_B{}^{X'},$$

which may be written as

$$2\nabla_{BR'}\nabla^{AR'}G_A{}^{X'}+(\nabla^{AR'}\nabla_{BR'}-\nabla_{BR'}\nabla^{AR'})G_A{}^{X'}=\Box G_B{}^{X'},$$

we have for the first term

$$\nabla_{BR'}\nabla^{AR'}G_A{}^{X'}=\tfrac{1}{2}\Box G_B{}^{X'}-\tfrac{1}{2}(\nabla^{AR'}\nabla_{BR'}-\nabla_{BR'}\nabla^{AR'})G_A{}^{X'}. \tag{3.21}$$

Similarly, the second term is shown to be

$$-\nabla^{AX'}\nabla_{AR'}G_B{}^{R'}=-\tfrac{1}{2}\Box G_B{}^{X'}+\tfrac{1}{2}(\nabla_{AR'}\nabla^{AX'}-\nabla^{AX'}\nabla_{AR'})G_B{}^{R'}. \tag{3.22}$$

Hence in the sum of Eqs. (3.21) and (3.22) the d'Alembertian terms cancel one another. The remaining commutators cancel as well, as may be seen by applying the Ricci identities: From Eq. (3.21), the commutator is

$$\begin{aligned}(\nabla^{AR'}\nabla_{BR'}-\nabla_{BR'}\nabla^{AR'})G_A{}^{X'}=&-\delta^A{}_B\nabla_H{}^{(R'}\nabla^H{}_{R')}G_A{}^{X'}\\&-2\nabla^{(A}{}_{P'}\nabla_{B)}{}^{P'}G_A{}^{X'}.\end{aligned}$$

The first term contains a contracted symmetrized pair of indices and hence vanishes. The second is

$$2(-3\Lambda G_B{}^{X'}+\Phi_{BE}{}^{X'}{}_{S'}G^{ES'}) \tag{3.23}$$

as indicated in the discussion of Eq. (3.15) where the same expression occurs. Similarly the commutator in Eq. (3.22) is

$$\begin{aligned}&(\nabla_{AR'}\nabla^{AX'}-\nabla^{AX'}\nabla_{AR'})G_B{}^{R'}\\&\qquad=(2\nabla_{H(R'}\nabla^{HX')}+\delta_{X'}{}^{R'}\nabla_{(AP'}\nabla^{A)P'})G_B{}^{R'}.\end{aligned}$$

The second term vanishes because of the contracted symmetrized indices; the first is equal to (3.23) as shown by the Ricci identities and discussed above in connection with Eq. (3.16). Hence the sum of Eqs. (3.21) and (3.22) vanishes, and the entire right-hand side of Eq. (3.20) has been shown to be zero. This establishes the spinor form of the electrodynamic Hertz formalism, Eqs. (3.17) and (3.18), with gauge transformations of the third kind.

B. Vector potentials

Equation (2.14) for the vector potential in flat space expressed as a covariant derivative of the Hertz potential is in fact equally valid in curved space. The proof that this is so requires only a minor modification of the proof of Eq. (2.14). Here it is shown that if the Hertz potential obeys Eq. (3.17), then the quantity

$$A_{BY'}=\nabla_{BZ'}\overline{P}_{Y'}{}^{Z'}-G_{BY'}+\text{c.c.} \tag{3.24}$$

generates the field spinor ϕ_{AB} given by Eq. (3.18) via the usual relation between the field tensor and the vector potential. Since ϕ_{AB} obeys Maxwell's equations as just shown, $A_{BY'}$ is therefore the usual vector potential. To see this, note that the field tensor

$$f_{AX'BY'} = \nabla_{AX'} A_{BY'} - \nabla_{BY'} A_{AX'}$$

becomes

$$f_{AX'BY'} = \epsilon_{AB} \nabla_{HX'} \nabla^{H}{}_{Z'} \overline{P}_{Y'}{}^{Z'} + \epsilon_{X'Y'} \nabla_{BR'} \nabla_{AZ'} \overline{P}^{R'Z'} + \nabla_{BY'} G_{AX'} - \nabla_{AX'} G_{BY'} + \text{c.c.} \qquad (3.25)$$

on substituting from Eq. (3.24). An analogous computation which involves the addition and subtraction of $\nabla_{AY'} \nabla_{BZ'} \overline{P}_{X'}{}^{Z'}$ [as opposed to $\nabla_{BX'} \nabla_{AZ'} \overline{P}_{Y'}{}^{Z'}$ which was added and subtracted to obtain (3.25)] gives

$$f_{AX'BY'} = \epsilon_{AB} \nabla_{HY'} \nabla^{H}{}_{Z'} \overline{P}_{X'}{}^{Z'} + \epsilon_{X'Y'} \nabla_{AR'} \nabla_{BZ'} \overline{P}^{R'Z'} + \nabla_{BY'} G_{AX'} - \nabla_{AX'} G_{BY'} . \qquad (3.26)$$

Taking half the sum of Eqs. (3.25) and (3.26) and applying Eq. (A13) to the last two terms, just as in the flat-space case, gives

$$f_{AX'BY'} = \epsilon_{AB} (\nabla_{H(X'} G^{H}{}_{Y')} - \nabla_{H(X'} G^{H}{}_{Y')}) + \epsilon_{X'Y'} (\nabla_{(AR'} \nabla_{B)Z'} \overline{P}^{R'Z'} - \nabla_{(AP'} G_{B)}{}^{P'}) ,$$

where use has been made of the wave equation (3.17). Equation (3.18) for the field spinor shows that this expression is just

$$f_{AX'BY'} = \epsilon_{X'Y'} \phi_{AB} + \text{c.c.} ,$$

which shows that Eq. (3.24) is the curved-space vector potential with guage transformations of the third kind.

Furthermore, the relation of Eq. (2.15) between the field spinor and the vector potential,

$$\phi_{AB} = \nabla_{(AX'} A_{B)}{}^{X'} , \qquad (3.27)$$

holds as well in curved space; this is seen by observing that substitution of Eq. (3.24) for $A_B{}^{X'}$ into this expression generates Eq. (3.18), the conjugate terms not contributing by virtue of the wave condition (3.17).

C. Debye potentials

The basic equations (3.17) and (3.18) of the Hertz potential formalism provide the framework for a one-component Debye reduction of the potential in the generalized Goldberg-Sachs class of spacetimes, as in Ref. 20. (For other approaches to the formulation of electromagnetic Debye potentials in certain classes of spacetimes, see e.g. Refs. 16 and 34.) This procedure is now outlined for the spinor case.

A spin dyad frame is chosen so that one of the basis elements, say o^A, is oriented along the repeated principal null direction of the Weyl tensor [one orients a spinor along a null vector in the sense of Eq. (A11)]. Alternatively, one accomplishes the same result by choosing o^A to coincide with the repeated principal spinor of the Weyl spinor.[35,36] (For a discussion of the motivation for this choice, see Ref. 20, Sec. V.) Next the notation ψ is introduced for the dyad component P_{22} of the Hertz spinor; other components are zero with respect to the chosen dyad. The gauge spinor is chosen to be $G_{22'} = -2\overline{\tau}\overline{\psi}$, $G_{12'} = -2\overline{\rho}\overline{\psi}$ (other components are zero) where τ and ρ are spinor affine connection components as discussed e.g. in Refs. 26 and 36 (they are also called "spin coefficients" in the NP formalism). With these choices for P_{AB} and $G_{AX'}$, Eqs. (3.17) and (3.18) are written in the NP formalism according to the rules in Refs. 26 and 36, and the dyad components of ϕ_{AB} are given their NP labels $\phi_0 \equiv \phi_{11}$, $\phi_1 \equiv \phi_{12}$, and $\phi_2 \equiv \phi_{22}$. (This procedure is illustrated in detail below for the simpler spin-$\frac{1}{2}$ case.) Then the dyad components of the wave operator acting on the potential, the left side of Eq. (3.17), become

$$X'R' = 1'1' : [-(\Delta - \gamma + \overline{\gamma} + \mu)(D + 2\overline{\epsilon} - \overline{\rho}) + (\delta + \overline{\alpha} + \beta - \tau)(\overline{\delta} + 2\overline{\beta} - \overline{\tau}) + \overline{\nu}\overline{\kappa} - \overline{\lambda}\overline{\sigma}]\overline{\psi} ,$$

$$1'2' : \tfrac{1}{2}[(\overline{\delta} - \alpha + \overline{\beta} + \pi + \overline{\tau})(D + 2\overline{\epsilon} - \overline{\rho}) - (D + \epsilon + \overline{\epsilon} - \rho + \overline{\rho})(\overline{\delta} + 2\overline{\beta} - \overline{\tau}) + (\Delta - \gamma - \overline{\gamma} + \mu - \overline{\mu})\overline{\kappa} - (\delta - \overline{\alpha} + \beta - \overline{\pi} - \tau)\overline{\sigma}]\overline{\psi} ,$$

$$2'2' : [+(D + \epsilon - \overline{\epsilon} - \rho)\overline{\sigma} - (\overline{\delta} - \alpha - \overline{\beta} + \pi)\overline{\kappa} - \overline{\sigma}(D + 2\overline{\epsilon} - \overline{\rho}) + \overline{\kappa}(\overline{\delta} + 2\overline{\beta} - \overline{\tau})]\overline{\psi} ,$$

while the dyad components of the gauge terms, the right side of Eq. (3.17), are

$$X'R' = 1'1' : [-(\Delta - \gamma + \overline{\gamma} + \mu)(-2\overline{\rho}) + (\delta + \overline{\alpha} + \beta - \tau)(-2\overline{\tau})]\overline{\psi} ,$$

$$1'2' : \tfrac{1}{2}[+(\overline{\delta} - \alpha + \overline{\beta} + \pi + \overline{\tau})(-2\overline{\rho}) - (D + \epsilon + \overline{\epsilon} - \rho + \overline{\rho})(-2\overline{\tau})]\overline{\psi} ,$$

$$2'2' : [-\overline{\sigma}(-2\overline{\rho}) + \overline{\kappa}(-2\overline{\tau})]\overline{\psi} .$$

Equality of the $(2'2')$ components holds identically because of the alignment of o^A along a principal direction of the Weyl tensor: a calculation in the NP formalism shows the $(2'2')$ component of the left side to be just $(2\overline{\rho}\overline{\sigma} - 2\overline{\tau}\overline{\kappa} + \overline{\Psi}_0)\overline{\psi}$. For the $(1'2')$ component of the left side, an NP calculation gives

$$[(D + \epsilon + \overline{\epsilon} - \rho)\overline{\tau} - (\overline{\delta} - \alpha + \overline{\beta} + \pi)\overline{\rho} - \overline{\sigma}(\delta + 2\overline{\alpha} + \overline{\pi}) + \overline{\kappa}(\Delta + 2\overline{\gamma} + \overline{\mu}) - 2\overline{\Psi}_1]\overline{\psi} ,$$

which again is identically equal to the right side because $\bar{\sigma}$, $\bar{\kappa}$, and $\bar{\Psi}_1$ vanish by assumption. Finally, the equality of the $(1'1')$ components, omitting quantities proportional to $\bar{\kappa}$ and $\bar{\sigma}$, yields a scalar wave equation for $\bar{\psi}$:

$$[(\Delta - \gamma + \bar{\gamma} + \mu)(D + 2\bar{\epsilon} + \bar{\rho}) - (\delta + \bar{\alpha} + \beta - \tau)(\bar{\delta} + 2\bar{\beta} + \bar{\tau})]\bar{\psi} = 0\,, \quad (3.28)$$

whose solutions $\bar{\psi}$ contain all of the information of the Maxwell field. In fact, Eq. (3.18) explicitly gives the Maxwell field spinor in terms of $\bar{\psi}$, the three components of which become

$$\begin{aligned}
\phi_0 &= -(D - \epsilon + \bar{\epsilon} - \bar{\rho})(D + 2\bar{\epsilon} + \bar{\rho})\bar{\psi}\,, \\
2\phi_1 &= -[(D + \epsilon + \bar{\epsilon} + \rho - \bar{\rho})(\bar{\delta} + 2\bar{\beta} + \bar{\tau}) \\
&\qquad + (\bar{\delta} - \alpha + \bar{\beta} - \pi - \bar{\tau})(D + 2\bar{\epsilon} + \bar{\rho})]\bar{\psi}\,, \qquad (3.29)\\
\phi_2 &= -[(\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\bar{\delta} + 2\bar{\beta} + \bar{\tau}) - \lambda(D + 2\bar{\epsilon} + \bar{\rho})]\bar{\psi}\,,
\end{aligned}$$

where terms proportional to $\bar{\kappa}$ and $\bar{\sigma}$ have been omitted from these expressions. These field components and scalar wave equation are just Eqs. (5.5) and (5.6) of Ref. 20. The vector potential, Eq. (3.24), becomes for this P_{AB} and $G_{AX'}$,

$$A^\mu = -\bar{m}^\mu(D + 2\bar{\epsilon} + \bar{\rho})\bar{\psi} + l^\mu(\bar{\delta} + 2\bar{\beta} + \bar{\tau})\bar{\psi} + \text{c.c.}\,, \quad (3.30)$$

where l^μ and $\bar{m}^\mu$ are elements of the null tetrad canonically associated[3] with the chosen spin dyad.

Similarly, the two alternate formulations of the Debye potential derived in Ref. 20 for type D spacetimes follow directly from the spinor formulas (3.17), (3.18), and (3.24), as is now shown.

For the formulation which interchanges the roles of the congruences with tangents l^μ and n^μ, the two shear-free null geodesic congruences of the type D space, we take $P_{11} = \psi$, $G_{11'} = -2\bar{\pi}\bar{\psi}$, $G_{21'} = -2\bar{\mu}\bar{\psi}$; the remaining components of the potential and guage spinor vanish. Proceeding as above, omitting quantities proportional to Ψ_3, λ, and ν which are the NP quantities whose vanishing ensures the shear-free null geodesic condition for the n^μ congruence as well as that n^μ be a repeated principal direction of the Weyl tensor, we obtain the wave equation

$$[(D + \epsilon - \bar{\epsilon} - \rho)(\Delta - 2\bar{\gamma} - \bar{\mu}) - (\bar{\delta} - \alpha - \bar{\beta} + \pi)(\delta - 2\bar{\alpha} - \bar{\pi})]\bar{\psi} = 0 \quad (3.31)$$

for the $(2'2')$ component of Eq. (3.17) and identities for the other two components. The field tensor components, Eq. (3.18), are

$$\begin{aligned}
\phi_0 &= -(\delta - \bar{\alpha} - \beta + \bar{\pi})(\delta - 2\bar{\alpha} - \bar{\pi})\bar{\psi}\,, \\
2\phi_1 &= [-(\delta - \bar{\alpha} + \beta + \bar{\pi} + \tau)(\Delta - 2\bar{\gamma} - \bar{\mu}) \\
&\qquad - (\Delta - \gamma - \bar{\gamma} - \mu + \bar{\mu})(\delta - 2\bar{\alpha} - \bar{\pi})]\bar{\psi}\,, \qquad (3.32)\\
\phi_2 &= -(\Delta + \gamma - \bar{\gamma} + \bar{\mu})(\Delta - 2\bar{\gamma} - \bar{\mu})\bar{\psi}\,,
\end{aligned}$$

and the vector potential obtained from Eq. (3.24) is

$$A^\mu = -n^\mu(\delta - 2\bar{\alpha} - \bar{\pi})\bar{\psi} + m^\mu(\Delta - 2\bar{\gamma} - \bar{\mu}) + \text{c.c.} \quad (3.33)$$

In each of these formulations, the vector potential is transverse to the special congruence, that is, $A^\mu l_\mu = 0$ in the first case and $A^\mu n_\mu = 0$ in the second. The third scheme involves a vector potential transverse to neither congruence: Choose $P_{12} = \psi$, other components zero, and $G_{12'} = -2\bar{\pi}\bar{\psi}$, $G_{22'} = -2\bar{\mu}\bar{\psi}$, $G_{11'} = -2\bar{\rho}\bar{\psi}$, $G_{21'} = -2\bar{\tau}\bar{\psi}$. Then the scalar wave equation for the potential is

$$[(\Delta - \gamma - \bar{\gamma} + \mu - \bar{\mu})D - (\delta - \bar{\alpha} + \beta - \bar{\pi} - \tau)\bar{\delta}]\bar{\psi} = 0 \quad (3.34)$$

and the field components are

$$\begin{aligned}
\phi_0 &= [(D - \epsilon + \bar{\epsilon} + \bar{\rho})\delta + (\delta - \bar{\alpha} - \beta + \bar{\pi})D]\bar{\psi}\,, \\
\phi_1 &= [(D + \epsilon + \bar{\epsilon} + \rho - \bar{\rho})\Delta + (\bar{\delta} - \alpha + \bar{\beta} - \pi - \bar{\tau})\delta]\bar{\psi}\,, \qquad (3.35)\\
\phi_2 &= [(\Delta + \gamma - \bar{\gamma} + \bar{\mu})\bar{\delta} + (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})\Delta]\bar{\psi}
\end{aligned}$$

from Eqs. (3.17) and (3.18), respectively, omitting terms proportional to κ, σ, λ, and ν. The vector potential from Eq. (3.24) is found to be

$$A^\mu = n^\mu D\bar{\psi} - m^\mu\bar{\delta}\bar{\psi} + \bar{m}^\mu\delta\bar{\psi} - l^\mu\Delta\bar{\psi}\,. \quad (3.36)$$

These constructive procedures for the Maxwell field tensor and vector potential are not restricted to vacuum spaces, nor is the first of the three schemes [Eqs. (3.28)–(3.30)] restricted to type D spacetimes; the method covers the generalized Goldberg-Sachs[2,22] class of spaces: all those algebraically special spaces—vacuum or not—which possess both a repeated principal null direction of the Weyl tensor and a shear-free congruence of null geodesics along that direction. Illustrations showing the applicability of this method of constructing Maxwell fields to astrophysical spacetimes are given in Ref. 20.

D. Summary of Sec. III

In this section it has been shown that the curved-space generalizations of Eqs. (2.12)–(2.15) are

$$\nabla^{A(X'}\nabla_{AZ'}\bar{P}^{Y')Z'} = \nabla^{A(X'}G_A{}^{Y')}\,, \quad (3.17)$$

$$A_{BY'} = \nabla_{BZ'}\bar{P}_{Y'}{}^{Z'} - G_{BY'} + \text{c.c.}\,, \quad (3.24)$$

$$\phi_{AB} = \nabla_{(AX'}A_{B)}{}^{X'}\,, \quad (3.27)$$

and

$$\phi_{AB} = \nabla_{(AX'}\nabla_{B)Y'}\bar{P}^{X'Y'} - \nabla_{(AX'}G_{B)}{}^{X'}\,. \quad (3.18)$$

In other words, if a Hertzian potential spinor $\bar{P}^{Y'Z'}$ and gauge spinor $G_{AY'}$ are related by the wave condition (3.17), then the vector potential $A_{BY'}$ of Eq. (3.24) and the field spinor ϕ_{AB} of Eqs. (3.18) and (3.27) identically solve the curved-space Maxwell equations.

These spinor equations are the equivalents of the corresponding differential form equations

$$\Delta P = dG + \delta W\,,$$

$$A = \delta P - G\,,$$

$$f = dA\,,$$

and

$$f = d\delta P - dG = \delta W - \delta dP$$

of Ref. 20.

The above Hertz potential formulas, valid in all spacetimes, are shown to yield a scalar Debye superpotential for the electromagnetic field in the generalized Goldberg-Sachs class of spacetimes. The general Maxwell field is explicitly constructed by solving the scalar wave equation (3.28) and differentiating its solution to yield the vector potential (3.30) or field components (3.29).

IV. THE WEYL NEUTRINO IN CURVED SPACE

The two-component or Weyl neutrino equation in curved space is just the one-index case of Eq. (2.1). In the section, Eqs. (2.2) and (2.3) for the spin-$\frac{1}{2}$ case will be generalized to curved space, as will the version incorporating gauge transformations, Eqs. (2.4) and (2.5).[21,26] A Debye one-component reduction of these equations will then be derived in the NP formalism using the spin dyad frame methods of Refs. 3, 26, and 36.

In order to solve

$$\nabla^{AX'}\phi_A = 0\,, \qquad (4.1)$$

a potential P_A of the Hertz type is introduced and assumed to obey

$$\nabla^A{}_{Z'}\nabla_{AW'}\bar{P}^{W'} = 0\,. \qquad (4.2)$$

Then if the neutrino field is given by

$$\phi_A = \nabla_{AW'}\bar{P}^{W'}\,, \qquad (4.3)$$

Eq. (4.1) is immediately verified to be satisfied by ϕ_A by virtue of the wave equation (4.2).

If gauge transformations are now included in the form suggested by Eqs. (2.4) and (2.5), the wave equation becomes

$$\nabla^A{}_{Z'}\nabla_{AW'}\bar{P}^{W'} = \nabla^A{}_{Z'}G_A \qquad (4.4)$$

and the neutrino field is given by

$$\phi_A = \nabla_{AW'}\bar{P}^{W'} - G_A\,. \qquad (4.5)$$

Again the Weyl neutrino equation is immediately seen to be solved by ϕ_A.

Equations (4.4) and (4.5), which comprise a generalization to curved space of the $s=\frac{1}{2}$ Hertz formalism with gauge transformations, are now used as a framework for a Debye-type one-component reduction of P^A in the generalized Goldberg-Sachs class of spaces. To this end a spin dyad frame is chosen with o^A oriented along the repeated principal null direction of the Weyl tensor, with the result that spin coefficients κ and σ and NP Weyl tensor components Ψ_0 and Ψ_1 are zero.

As an illustration of the method of translating from spinor to NP notation, Eq. (4.1) will be written in the latter formalism. In terms of the quantities $\zeta_a{}^A$ which connect lower-case dyad indices to upper-case spinor indices via

$$\eta_a = \zeta_a{}^A\eta_A\,, \quad \eta_A = -\zeta^a{}_A\eta_a\,,$$

and which are chosen in the NP formalism to be $\zeta_a{}^A = \delta_a{}^A$, Eq. (4.1) becomes

$$\begin{aligned} 0 &= \nabla_{AX'}\phi^A = \nabla_{AX'}\zeta_a{}^A\phi^a \\ &= \phi^a\nabla_{AX'}\zeta_a{}^A + \zeta_a{}^A\nabla_{AX'}\phi^a \quad \text{by the Leibnitz rule} \\ &= \epsilon^{ab}\epsilon^{AB}(\phi_b\nabla_{AX'}\zeta_{aB} + \zeta_{aB}\nabla_{AX'}\phi_b)\,. \end{aligned} \qquad (4.6)$$

The definition[26,36] of the dyad components of the spinor affine connection gives

$$\nabla_{AX'}\zeta_{aB} = -\zeta^e{}_A\bar{\zeta}^{x'}{}_{X'}\zeta^f{}_B\Gamma_{afex'}\,.$$

Furthermore,

$$\zeta_{aB}\nabla_{AX'}\phi_b = -\zeta^e{}_A\bar{\zeta}^{x'}{}_{X'}\zeta^f{}_B(\zeta_{af}\nabla_{ex'}\phi_b)\,,$$

so that Eq. (4.6) becomes

$$\begin{aligned} 0 &= -\epsilon^{ab}\epsilon^{AB}\zeta^e{}_A\bar{\zeta}^{x'}{}_{X'}\zeta^f{}_B(\phi_b\Gamma_{afex'} + \epsilon_{af}\nabla_{ex'}\phi_b) \\ &= -\epsilon^{ab}\epsilon^{AB}\delta^e{}_A\delta^{x'}{}_{X'}\delta^f{}_B(\phi_b\Gamma_{afex'} + \epsilon_{af}\nabla_{ex'}\phi_b)\,. \end{aligned} \qquad (4.7)$$

Writing out the $x' = 1'$ component explicitly yields

$$\begin{aligned} 0 &= -\epsilon^{12}\epsilon^{12}(\phi_2\Gamma_{1211'} + \epsilon_{12}\nabla_{11'}\phi_2) - \epsilon^{12}\epsilon^{21}(\phi_2\Gamma_{1121'}) \\ &\quad - \epsilon^{21}\epsilon^{12}(\phi_1\Gamma_{2211'}) - \epsilon^{21}\epsilon^{21}(\phi_1\Gamma_{2121'} + \epsilon_{21}\nabla_{21'}\phi_1) \\ &= -(\nabla_{11'} + \Gamma_{1211'} - \Gamma_{1121'})\phi_2 + (\nabla_{21'} + \Gamma_{2211'} - \Gamma_{2121'})\phi_1\,. \end{aligned}$$

Substituting the NP notation[26,36] gives

$$0 = -(D + \epsilon - \rho)\phi_2 + (\bar{\delta} + \pi - \alpha)\phi_1\,. \qquad (4.8)$$

Similarly the $x' = 2'$ component of Eq. (4.7) becomes

$$0 = -(\delta + \beta - \tau)\phi_2 + (\Delta + \mu - \gamma)\phi_1\,. \qquad (4.9)$$

Equations (4.8) and (4.9) are the NP spin dyad form of Eq. (4.1) and are fully general; the special properties $\kappa = \sigma = \Psi_0 = \Psi_1 = 0$ for an aligned dyad in the generalized Goldberg-Sachs class of spaces have not been used in their derivation.

The above methods are now used to derive from Eqs. (4.4) and (4.5) the spin-$\frac{1}{2}$ Debye potential formalism for the generalized Goldberg-Sachs spacetimes. The choices $P_2 = \psi$, $P_1 = 0$ for the potential and $G_1 = \bar{\rho}\bar{\psi}$, $G_2 = \bar{\tau}\bar{\psi}$ for the gauge spinor are made in order to obtain a Debye one-component potential which satisfies a decoupled scalar wave equation. With these special choices, Eq. (4.4) is now written in an aligned NP dyad frame.

The form of the right-hand side of Eq. (4.4) is just the negative of Eq. (4.6) by the dummy index rule. Thus the right-hand side of Eq. (4.4) in NP notation is given by Eqs. (4.8) and (4.9) with the above choices for G_1 and G_2:

$$\nabla^{A}{}_{1'}G_A = (D+\epsilon-\rho)\bar{\tau}\bar{\psi} - (\bar{\delta}+\pi-\alpha)\bar{\rho}\bar{\psi}, \tag{4.10a}$$

$$\nabla^{A}{}_{2'}G_A = (\delta+\beta-\tau)\bar{\tau}\bar{\psi} - (\Delta+\mu-\gamma)\bar{\rho}\bar{\psi}. \tag{4.10b}$$

Next it is noted that the factor $\nabla_{AW'}\bar{P}^{W'}$ in the left-hand side of Eq. (4.4) is just the complex conjugate of $\nabla_{BA'}P^{B}$. Since this is again the same form as Eq. (4.6), one finds from Eqs. (4.8) and (4.9) that

$$\nabla_{1W'}\bar{P}^{W'} = -(D+\bar{\epsilon}-\bar{\rho})\bar{\psi}, \tag{4.11a}$$

$$\nabla_{2W'}\bar{P}^{W'} = -(\bar{\delta}+\bar{\beta}-\bar{\tau})\bar{\psi}. \tag{4.11b}$$

Using the notation $\nabla_{AW'}\bar{P}^{W'} \equiv \chi_A$, it is seen that the left-hand side of Eq. (4.4) is $\nabla^{A}{}_{Z'}\chi_A$, which again is given by the negative of Eqs. (4.8) and (4.9), with dyad components of χ_A substituted from Eqs. (4.11):

$$\nabla^{A}{}_{1'}\nabla_{AW'}\bar{P}^{W'} = [-(D+\epsilon-\rho)(\bar{\delta}+\bar{\beta}-\bar{\tau}) + (\bar{\delta}+\pi-\alpha)(D+\bar{\epsilon}-\bar{\rho})]\bar{\psi}, \tag{4.12a}$$

$$\nabla^{A}{}_{2'}\nabla_{AW'}\bar{P}^{W'} = [-(\delta+\beta-\tau)(\bar{\delta}+\bar{\beta}-\bar{\tau}) + (\Delta+\mu-\gamma)(D+\bar{\epsilon}-\bar{\rho})]\bar{\psi}. \tag{4.12b}$$

Several of the NP-notation Ricci identities and derivative commutators are now used to show that Eq. (4.10a) and (4.12a), the two sides of Eq. (4.4), are identically equal under the assumption that $\kappa=\sigma=\Psi_0=\Psi_1=0$. The calculation begins with the NP expression in (4.12a):

$$\begin{aligned}\nabla^{A}{}_{1'}\nabla_{AW'}\bar{P}^{W'} &= [(\bar{\delta}+\pi-\alpha)(D+\bar{\epsilon}-\bar{\rho}) - (D-\rho+\epsilon)(\bar{\delta}+\bar{\beta}-\bar{\tau})]\bar{\psi}\\ &= [\bar{\delta}D - D\bar{\delta} + \bar{\delta}(\bar{\epsilon}-\rho) - D(\bar{\beta}-\bar{\tau}) + (\pi-\alpha)(D+\bar{\epsilon}-\bar{\rho}) + (\rho-\epsilon)(\bar{\delta}+\bar{\beta}-\bar{\tau})]\bar{\psi}\\ &= [(\alpha+\bar{\beta}-\pi)D + \bar{\kappa}\Delta - \bar{\sigma}\delta - (\rho+\bar{\epsilon}-\epsilon)\bar{\delta} + \bar{\delta}(\bar{\epsilon}-\bar{\rho}) - D(\bar{\beta}-\bar{\tau}) + (\pi-\alpha)(D+\bar{\epsilon}-\bar{\rho}) + (\rho-\epsilon)(\bar{\delta}-\bar{\beta}-\bar{\tau})]\bar{\psi}\end{aligned}$$

(by the commutator Eq. (4.4) of Ref. 3).

But by Eq. (4.2e) of Ref. 3,

$$D\bar{\beta} - \bar{\beta}D = \bar{\delta}\bar{\epsilon} - \bar{\epsilon}\bar{\delta} = (\bar{\alpha}+\bar{\pi})\bar{\sigma} + (\rho-\epsilon)\bar{\beta} - (\bar{\mu}+\bar{\gamma})\bar{\kappa} - (\alpha-\pi)\bar{\epsilon} + \Psi_1.$$

Substitution of this expression and subsequent cancellation yield

$$\nabla^{A}{}_{1'}\nabla_{AW'}\bar{P}^{W'} = [(D+\epsilon-\rho)\bar{\tau} - (\bar{\delta}+\pi-\alpha)\bar{\rho}]\bar{\psi} + (\text{terms proportional to } \kappa,\ \sigma,\ \text{and } \Psi_1).$$

But this is identically equal to (4.10a) as claimed. Hence Eq. (4.4) gives

$$\nabla^{A}{}_{1'}\nabla_{AW'}\bar{P}^{W'} - \nabla^{A}{}_{1'}G_A \equiv 0, \tag{4.13a}$$

$$\nabla^{A}{}_{2'}\nabla_{AW'}\bar{P}^{W'} - \nabla^{A}{}_{2'}G_A = [(\Delta+\mu-\gamma)(D+\bar{\epsilon}) - (\delta+\beta-\tau)(\bar{\delta}+\bar{\beta})]\bar{\psi} = 0. \tag{4.13b}$$

This is the desired decoupled wave equation for ψ.

The dyad components of Eq. (4.5) are now written in the NP notation. They may immediately be found from Eqs. (4.11) and the assumed form of G_A and are given by

$$\phi_1 = -(D+\bar{\epsilon})\bar{\psi}, \tag{4.14a}$$

$$\phi_2 = -(\bar{\delta}+\bar{\beta})\bar{\psi}. \tag{4.14b}$$

Equations (4.13b) and (4.14) are the Debye potential scheme for the Weyl neutrino field; they may be written in coordinates in specific spacetimes of astrophysical interest.

An alternative Debye potential scheme valid in type D spaces with a shear-free congruence of null geodesics along each of the repeated principal directions of the Weyl tensor may be obtained by taking $P_1=\psi$, $P_2=0$ and suitable gauge terms, or alternatively by applying the $l \leftrightarrow n$, $m \leftrightarrow \bar{m}$ transformation of the NP formalism to Eqs. (4.13b) and (4.14).

V. CURVED-SPACE TREATMENT OF GRAVITATIONAL PERTURBATIONS

A. Metric and Weyl tensor perturbations of spacetimes

The computation of gravitational perturbations of algebraically special vacuum spacetimes is reduced to solving a linear wave equation for a complex scalar superpotential. The procedure by which this is accomplished is a generalization to curved spaces of the spin-2 Hertz formalizm of Sec. II above[21,26] and gives the proofs for our previously published results.[20,21]

One consequence of the results of this section may be regarded as an extension of a theorem of Wald[37] concerning the determination of a gravitational perturbation of a spacetime by a scalar quantity. His work shows that knowledge of a single gauge-invariant tetrad component of the perturbed

Weyl tensor of a type D vacuum spacetime, obtained by solving Teukolsky's equation,[13] in principle determines all aspects of the perturbed spacetime. (This statement in turn generalizes the analogous result of Fackerell and Ipser[12] for electromagnetic perturbations of type D vacuum spaces.) The present Debye potential procedure extends the result in two respects, in that (1) it applies to the wider class of algebraically special vacuum spaces, and (2) the perturbed space is uniquely fixed not only in principle but by direct construction. Thus, it is shown below that for a vacuum space, the existence of a single perturbed curvature component which is both tetrad- and identification-gauge-invariant in the sense of Stewart and Walker[38] is sufficient to ensure that the full perturbation is explicitly determined by the information in one complex scalar (the Debye potential).

There is an added degree of complication in the field equations for gravitational perturbations of spacetimes beyond that which is encountered in extending the lower-spin zero-rest-mass field treatments to curved space. For spins $\frac{1}{2}$ and 1, one argues that the correct formulation for weak zero-rest-mass fields on a curved background is just the corresponding test-field equation, since both neutrino and electromagnetic fields produce contributions to the stress-energy tensor which are quadratic in the field strength and which hence may be neglected in a consistent linearization of the Einstein-Maxwell or Einstein-Weyl neutrino system. In the gravitational case, on the other hand, it is the Bianchi identities[36] involving the Weyl tensor,

$$\nabla^{AX'}\psi_{ABCD}=\nabla_{(B}{}^{Z'}\Phi_{CD)}{}^{X'}{}_{Z'} \tag{5.1}$$

(with $\Phi_{ABX'Y'}$ the trace-free Ricci spinor), which bear the formal resemblance to the spin-2 test-field equation treated above in linearized theory (perturbations of flat space). The spin-2 test-field equation is in general

$$\nabla^{AX'}\psi_{ABCD}=0 \tag{5.2}$$

in a fixed background; this is *not* the perturbation version of Eq. (5.1), not even under the assumption that the Ricci spinor vanishes in the perturbed space, since the covariant derivative operator will also acquire a *perturbed part of the same order as the perturbation in* ψ_{ABCD} *itself.*

Fortunately, by working with small perturbations of the metric tensor, it is possible to obtain a second-order equation representing a linearization of the Einstein vacuum field equations about a given exact background space which overcomes this difficulty: Its dependence on the perturbed space is only through explicit occurrence of the perturbed metric; all differentiations are purely background operations. This equation is just Eq. (2.20),

$$2R_{\alpha\beta}=h_{;\alpha\beta}+h_{\alpha\beta;\rho}{}^{\rho}-h^{\rho}{}_{\alpha;\beta\rho}-h^{\rho}{}_{\beta;\alpha\rho}=0\,, \tag{5.3}$$

sometimes referred to as the Palatini identity.[39] For these reasons, of the two proofs in Sec. IIC for the gravitational perturbations of flat space, the one which concentrates on the metric perturbations and shows that Eq. (5.3) is satisfied is the one which will be generalized here to curved space.

What will be shown in this section is that a metric perturbation given by

$$h_{CD}{}^{M'N'}=\nabla_{(CP'}\nabla_{D)Q'}\overline{P}^{M'N'P'Q'}-\nabla_{(CP'}G_{D)}{}^{M'N'P'}+\text{c.c.} \tag{5.4}$$

obeys Eq. (5.3), provided that the totally symmetric Hertz spinor $\overline{P}^{M'N'P'Q'}$ and the gauge spinor $G_A{}^{M'N'P'}=G_A{}^{(M'N'P')}$ obeys a generalized version of the wave equation (2.24) given by

$$\nabla^{A(M'}\nabla_{AX'}\overline{P}^{N'P'Q')X'}+3\overline{\Psi}_{X'Y'}{}^{(M'N'}\overline{P}^{P'Q')X'Y'}=\nabla^{A(M'}G_A{}^{N'P'Q')}. \tag{5.5}$$

Then the perturbed Weyl spinor, which follows from Eq. (2.19), is just

$$\begin{aligned}\psi_{ABCD}&=\nabla_{(AW'}\nabla_{BX'}h_{CD)}{}^{W'X'}\\&=\nabla_{(AW'}\nabla_{BX'}\nabla_{CY'}\nabla_{D)Z'}\overline{P}^{W'X'Y'Z'}\\&\quad-\nabla_{(AW'}\nabla_{BX'}\nabla_{CY'}G_{D)}{}^{W'X'Y'}\\&\quad+\nabla_{(AW'}\nabla_{BX'}\nabla_E{}^{(W'}\nabla_F{}^{X')}P_{CD)}{}^{EF}\\&\quad-\nabla_{(AW'}\nabla_{BX'}\nabla_E{}^{(W'}G^{X')}{}_{CD)}{}^{E}.\end{aligned} \tag{5.6}$$

In fact, the results for the gravitational case are slightly more special than the lower spins in the following respect: for spins $\frac{1}{2}$ and 1, the equations analogous to (5.5) and (5.6) are established for all spacetimes and are only subsequently specialized to the generalized Goldberg-Sachs class when a one-component Debye potential is chosen in a special aligned null frame. In the gravitational case, we proceed by specializing Eqs. (5.4) and (5.5) to a one-component Debye potential in the vacuum algebraically special spaces in the course of the proof.

The proof that Eq. (5.4) is the solution to Eq. (5.3) by virtue of the wave equation (5.5) involves somewhat lengthy spinor manipulations which will be sketched here. As in the curved-space spin-1 treatment of Sec. III, the strategy of the calculation is to substitute the expression (5.4) into Eq. (5.3) and to group the resulting terms into those which cancel one another by the wave condition (5.5), and the remainder which vanish by the Ricci identities.

On substitution of the metric (5.4) into the field operator (5.3), the first term $h_{;\alpha\beta}$ vanishes by the traceless property of the expression (5.4) for the metric (symmetric pairs of spinor indices being trace-free). The remaining terms of Eq. (5.3) become

$$h_{\alpha\beta;\rho}{}^{\rho} \longleftrightarrow \nabla^{AX'}\nabla_{AX'}(\nabla_{(CP'}\nabla_{D)Q'}\bar{P}^{M'N'P'Q'} - \nabla_{(CP'}G_{D)}{}^{M'N'P'}),$$

$$h^{\rho}{}_{\alpha;\beta\rho} \longleftrightarrow \nabla_{AX'}\nabla_D{}^{N'}(\nabla^{(A}{}_{P'}\nabla_{C)Q'}\bar{P}^{X'M'P'Q'} - \nabla^{(A}{}_{P'}G_{C)}{}^{X'M'P'}),$$

$$h^{\rho}{}_{\beta;\alpha\rho} \longleftrightarrow \nabla_{AX'}\nabla_C{}^{M'}(\nabla^{(A}{}_{P'}\nabla_{D)Q'}\bar{P}^{X'N'P'Q'} - \nabla^{(A}{}_{P'}G_{D)}{}^{X'N'P'}),$$

so

$$\begin{aligned}2R_{CD}{}^{M'N'} = \nabla^{AX'}\nabla_{AX'} &\times(\nabla_{(CP'}\nabla_{D)Q'}\bar{P}^{M'N'P'Q'} - \nabla_{(CP'}G_{D)}{}^{M'N'P'})\\ &-\nabla_{AX'}\nabla_D{}^{N'}(\nabla^{(A}{}_{P'}\nabla_{C)Q'}\bar{P}^{X'M'P'Q'} - \nabla^{(A}{}_{P'}G_{C)}{}^{X'M'P'})\\ &-\nabla_{AX'}\nabla_C{}^{M'}(\nabla^{(A}{}_{P'}\nabla_{D)Q'}\bar{P}^{X'N'P'Q'} - \nabla^{(A}{}_{P'}G_{D)}{}^{X'N'P'}).\end{aligned} \tag{5.7}$$

First the terms proportional to the gauge spinor are combined. It is convenient at this stage to note that the first term of Eq. (5.7), $\Box\nabla_{(CP'}\nabla_{D)Q'}\bar{P}^{M'N'P'Q'}$, may be rewritten as $\nabla_{(CP'}\nabla_{D)Q'}\Box\bar{P}^{M'N'P'Q'}$ plus commutators, and to note further that $\Box\bar{P}^{M'N'P'Q'} = 2\nabla^{A(M'}\nabla_{AX'}\bar{P}^{N'P'Q')X'}$ plus commutators [see Eq. (A18)]. Hence the wave equation (5.5) may be used to express $\Box\nabla_{(CP'}\nabla_{D)Q'}\bar{P}^{M'N'P'Q'}$ as $2\nabla_{(CP'}\nabla_{D)Q'}\nabla^{A(M'}G_A{}^{N'P'Q')}$ plus terms proportional to $\bar{P}^{M'N'P'Q'}$ (which will be treated in detail below). Therefore the gauge spinor terms in Eq. (5.7), denoted for convenience by R_G, are

$$R_G = 2\nabla_{(CP'}\nabla_{D)Q'}\nabla^{A(M'}G_A{}^{N'P'Q')} - \nabla^{AX'}\nabla_{AX'}\nabla_{(CP'}G_{D)}{}^{M'N'P'} + \nabla_{AX'}\nabla_D{}^{N'}\nabla^{(A}{}_{P'}G_{C)}{}^{X'M'P'} + \nabla_{AX'}\nabla_C{}^{M'}\nabla^{(A}{}_{P'}G_{D)}{}^{X'N'P'}. \tag{5.8}$$

By suitable ordering of the derivative operators, these gauge terms may be written as the same gauge terms appearing in the flat-space proof of Sec. II C (2) above (which cancel one another), plus commutator terms which may be expressed via the Ricci identities in terms of the background curvature spinors. Toward this end, R_G is written as

$$\begin{aligned}R_G = {}&\tfrac{1}{2}\nabla_{CP'}\nabla_{DQ'}\nabla^{AM'}G_A{}^{N'P'Q'} + \tfrac{1}{2}\nabla_{CP'}\nabla_{DQ'}\nabla^{AN'}G_A{}^{M'P'Q'} + \tfrac{1}{2}\nabla_{CP'}\nabla_{DQ'}\nabla^{AP'}G_A{}^{M'N'Q'}\\ &+\tfrac{1}{2}\nabla_{CP'}\nabla_{DQ'}\nabla^{AQ'}G_A{}^{M'N'P'} - \tfrac{1}{2}\Box\nabla_{CP'}G_D{}^{M'N'P'} - \tfrac{1}{2}\Box\nabla_{DP'}G_C{}^{M'N'P'}\\ &+\tfrac{1}{2}\nabla_{AX'}\nabla_D{}^{N'}\nabla_D{}^{N'}\nabla^A{}_{P'}G_C{}^{X'M'P'} + \tfrac{1}{2}\nabla_{AX'}\nabla_D{}^{N'}\nabla_{CP'}G_A{}^{X'M'P'} + \tfrac{1}{2}\nabla_{AX'}\nabla_C{}^{M'}\nabla^A{}_{P'}G_D{}^{X'N'P'}\\ &+\tfrac{1}{2}\nabla_{AX'}\nabla_C{}^{M'}\nabla_{DP'}G^{AX'N'P'} + (\nabla_{DQ'}\nabla_{CP'} - \nabla_{CP'}\nabla_{DQ'})\nabla^{A(M'}G_A{}^{N'P'Q')}.\end{aligned} \tag{5.9}$$

Of these eleven terms, the second and eighth may be combined to give

$$\tfrac{1}{2}\nabla_{AZ'}\nabla_D{}^{Z'}\nabla_{CR'}G^{AM'N'R'} + \tfrac{1}{2}\nabla_{CP'}(\nabla_{DQ'}\nabla^{AN'} - \nabla^{AN'}\nabla_{DQ'})G_A{}^{M'P'Q'} + \tfrac{1}{2}(\nabla_{CP'}\nabla^{AN'} - \nabla^{AN'}\nabla_{CP'})\nabla_{DQ'}G_A{}^{M'P'Q'} + \tfrac{1}{2}\nabla^{AN'}(\nabla_{CP'}\nabla_{DQ'} - \nabla_{DQ'}\nabla_{CP'})G_A{}^{M'P'Q'};$$

the first and tenth similarly give

$$\tfrac{1}{2}\nabla_{AZ'}\nabla_C{}^{Z'}\nabla_{DR'}G^{AM'N'R'} + \tfrac{1}{2}\nabla_{CP'}(\nabla_{DQ'}\nabla^{AM'} - \nabla^{AM'}\nabla_{DQ'})G_A{}^{N'P'Q'} + \tfrac{1}{2}(\nabla_{CP'}\nabla^{AM'} - \nabla^{AM'}\nabla_{CP'})\nabla_{DQ'}G_A{}^{N'P'Q'};$$

the third is rewritten as

$$\tfrac{1}{2}\nabla_{CZ'}\nabla^{AZ'}\nabla_{DR'}G_A{}^{M'N'R'} + \tfrac{1}{2}\nabla_{CP'}(\nabla_{DQ'}\nabla^{AP'} - \nabla^{AP'}\nabla_{DQ'})G_A{}^{M'N'Q'};$$

the fourth as

$$\tfrac{1}{2}\nabla_{DZ'}\nabla^{AZ'}\nabla_{CR'}G_A{}^{M'N'R'} + \tfrac{1}{2}(\nabla_{CP'}\nabla_{DQ'} - \nabla_{DQ'}\nabla_{CP'})\nabla^{AQ'}G_A{}^{M'N'P'} + \tfrac{1}{2}\nabla_{DQ'}(\nabla_{CP'}\nabla^{AQ'} - \nabla^{AQ'}\nabla_{CP'})G_A{}^{M'N'P'};$$

the seventh as

$$\tfrac{1}{2}\nabla_D{}^{N'}\nabla_{A(X'}\nabla^A{}_{P')}G_C{}^{X'M'P'} + \tfrac{1}{2}(\nabla_{AX'}\nabla_D{}^{N'} - \nabla_D{}^{N'}\nabla_{AX'})\nabla^A{}_{P'}G_C{}^{X'M'P'};$$

and the ninth as

$$\tfrac{1}{2}\nabla_C{}^{M'}\nabla_{A(X'}\nabla^A{}_{P')}G_D{}^{X'N'P'} + \tfrac{1}{2}(\nabla_{AX'}\nabla_C{}^{M'} - \nabla_C{}^{M'}\nabla_{AX'})\nabla^A{}_{P'}G_D{}^{X'N'P'}.$$

In all terms thus generated which contain once-contracted second covariant derivatives, the identity (A9), the curved-space generalization of Eq. (A8), is applied. When expression (5.9) for R_G is rewritten in this way, all terms containing d'Alembertian operators cancel one another, as do several of the commutators, leaving only commutator terms. These in turn are expressed as symmetrized once-contracted second derivatives by application of Eq. (A12), with the intention of applying the Ricci identities (A19) and (A20) to express all of R_G in terms of curvature quantities. The remaining terms in R_G obtained in this manner are

$$\begin{aligned} R_G = {} & \tfrac{3}{2}\nabla_{CP'}\nabla_{(DR'}\nabla^{A)R'}G_A{}^{N'P'M'} + \nabla_{(CR'}\nabla^{A)R'}\nabla_{DQ'}G_A{}^{N'M'Q'} + \nabla_{H(P'}\nabla^{HN')}\nabla_{DQ'}G_C{}^{M'P'Q'} \\ & + \nabla_{H(P'}\nabla^{HM')}\nabla_{(DQ'}G_{C)}{}^{N'P'Q'} + \tfrac{1}{2}\nabla_{CP'}\nabla_{H(Q'}\nabla^{HM')}G_D{}^{N'P'Q'} + \tfrac{1}{2}\nabla_{CP'}\nabla_{H(Q'}\nabla^{HN')}G_D{}^{M'P'Q'} \\ & + \tfrac{1}{2}\nabla_{CP'}\nabla_{H(Q'}\nabla^{HP')}G_D{}^{M'N'Q'} + \tfrac{1}{2}\epsilon_{CD}\nabla_{H(P'}\nabla^{H}{}_{Q')}\nabla^{AQ'}G_A{}^{M'N'P'} + \epsilon_{DC}\nabla_{H(Q'}\nabla^{H}{}_{P')}\nabla^{A(M'}G_A{}^{N'P'Q')} \\ & + \tfrac{1}{2}\epsilon_{CD}\nabla^{AN'}\nabla_{H(P'}\nabla^{H}{}_{Q')}G_A{}^{M'P'Q'} - \tfrac{1}{2}\nabla_{(CR'}\nabla_{D)}{}^{R'}\nabla^{A}{}_{Z'}G_A{}^{M'N'Z'} + \tfrac{1}{2}\nabla_{DQ'}\nabla_{H(P'}\nabla^{HQ')}G_C{}^{M'N'P'} \\ & + \tfrac{1}{2}\nabla_{DQ'}\nabla_{(CR'}\nabla^{A)R'}G_A{}^{M'N'Q'} + \tfrac{1}{2}\nabla_D{}^{N'}\nabla_{A(X'}\nabla^{A}{}_{P')}G_C{}^{X'M'P'} + \tfrac{1}{2}\nabla_C{}^{M'}\nabla_{A(X'}\nabla^{A}{}_{P')}G_D{}^{X'N'P'} \\ & + \tfrac{1}{2}\nabla_{(AR'}\nabla_{D)}{}^{R'}\nabla^{A}{}_{P'}G_C{}^{M'N'P'} + \tfrac{1}{2}\nabla_{(AR'}\nabla_{C)}{}^{R'}\nabla^{A}{}_{P'}G_D{}^{M'N'P'} \, . \end{aligned} \tag{5.10}$$

The next step is to apply the Ricci identities (A19) and (A20) to these terms, with the Ricci spinor $\Phi_{ABW'X'}$ and the curvature scalar Λ set equal to zero (vacuum spacetimes). The terms of Eq. (5.10) then yield, respectively,

$$\begin{aligned} R_G = {} & 0 - \Psi_{CADF}\nabla^{F}{}_{Q'}G^{AN'M'Q'} + \overline{\Psi}_{P'}{}^{N'M'}{}_{R'}\nabla_{DQ'}G_C{}^{R'P'Q'} + \overline{\Psi}_{P'}{}^{N'M'}{}_{R'}\nabla_{(DQ'}G_{C)}{}^{R'P'Q'} \\ & + \tfrac{1}{2}\nabla_{CP'}(\overline{\Psi}_{Q'}{}^{M'N'}{}_{R'}G_D{}^{R'P'Q'} + \overline{\Psi}_{Q'}{}^{M'P'}{}_{R'}G_D{}^{N'R'Q'}) + \tfrac{1}{2}\nabla_{CP'}(\overline{\Psi}_{Q'}{}^{N'M'}{}_{R'}G_D{}^{R'P'Q'} + \overline{\Psi}_{Q'}{}^{N'P'}{}_{R'}G_D{}^{M'R'Q'}) \\ & + \tfrac{1}{2}\nabla_{CP'}(\overline{\Psi}_{Q'}{}^{P'M'}{}_{R'}G_D{}^{R'N'Q'} + \overline{\Psi}_{Q'}{}^{N'P'}{}_{R'}G_D{}^{M'R'Q'}) + \tfrac{1}{2}\epsilon_{CD}(\overline{\Psi}_{P'Q'}{}^{M'}{}_{R'}\nabla^{AQ'}G_A{}^{R'N'P'} + \overline{\Psi}_{P'Q'}{}^{N'}{}_{R'}\nabla^{AQ'}G_A{}^{M'R'P'}) \\ & + \epsilon_{DC}(\overline{\Psi}_{P'Q'}{}^{M'}{}_{R'}\nabla^{A(R'}G_A{}^{N'P'Q')} + \overline{\Psi}_{P'Q'}{}^{N'}{}_{R'}\nabla^{A(M'}G_A{}^{R'P'Q')}) \\ & + \tfrac{1}{2}\epsilon_{CD}\nabla^{AN'}\overline{\Psi}_{P'Q'}{}^{M'}{}_{R'}G_A{}^{R'P'Q'} + 0 + \tfrac{1}{2}\nabla_{DQ'}(\overline{\Psi}_{P'}{}^{Q'M'}{}_{R'}G_C{}^{R'N'P'} + \overline{\Psi}_{P'}{}^{Q'N'}{}_{R'}G_C{}^{M'R'P'}) \\ & + 0 + \tfrac{1}{2}\nabla_D{}^{N'}\overline{\Psi}_{X'P'}{}^{M'}{}_{R'}G_C{}^{X'R'P'} + \tfrac{1}{2}\nabla_C{}^{M'}\overline{\Psi}_{X'P'}{}^{N'}{}_{R'}G_D{}^{X'R'P'} + \tfrac{1}{2}\Psi_{ADCE}\nabla^{A}{}_{P'}G^{EM'N'P'} + \tfrac{1}{2}\Psi_{CDGH}\nabla^{G}{}_{X'}G^{HM'N'X'} \, , \end{aligned} \tag{5.11}$$

where Ψ_{ABCD} is the Weyl curvature spinor of the unperturbed spacetime. It is seen that the three terms containing the unconjugated Weyl spinor cancel one another. The remaining terms proportional to the conjugate Weyl spinor of the background may then be combined, although it is convenient first to fix the gauge spinor $G_A{}^{M'N'P'}$ by demanding that the wave equation (5.5) yield a decoupled scalar wave equation for a one-component potential in an aligned dyad.

Thus the rest of the proof consists of (1) combining the terms in Eq. (5.7) which are proportional to the potential spinor (to be denoted collectively by R_P) and expressing these in terms of the background Weyl spinor in a manner analogous to the procedure leading to Eq. (5.11) for R_G, (2) making special choices for P_{ABCD} and $G_A{}^{M'N'P'}$ in an aligned dyad which lead via Eq. (5.5) to a decoupled scalar wave equation for the potential, and (3) inserting these choices into R_G and R_P and showing that they imply $R_G + R_P = 0$, i.e., that the perturbed vacuum Einstein field equations are satisfied.

The potential spinor terms R_P in Eq. (5.7) are seen to be

$$-\nabla_{AX'}\nabla_D{}^{N'}\nabla^{(A}{}_{P'}\nabla_{C)Q'}\overline{P}^{X'M'P'Q'} - \nabla_{AX'}\nabla_C{}^{M'}\nabla^{(A}{}_{P'}\nabla_{D)Q'}\overline{P}^{X'N'P'Q'} \tag{5.12}$$

plus the commutator terms referred to above which arise from expressing $\Box\nabla_{(CP'}\nabla_{D)Q'}\overline{P}^{M'N'P'Q'}$ in terms of the gauge spinor. These commutator terms are obtained by noting that

$$\begin{aligned} \Box\nabla_{(CP'}\nabla_{D)Q'}\overline{P}^{M'N'P'Q'} = {} & \nabla_{(CP'}\nabla_{D)Q'}\Box\overline{P}^{M'N'P'Q'} + \nabla^{AX'}(\nabla_{AX'}\nabla_{(CP'} - \nabla_{(CP'}\nabla_{AX'})\nabla_{D)Q'}\overline{P}^{M'N'P'Q'} \\ & + \nabla^{AX'}\nabla_{(CP'}(\nabla_{AX'}\nabla_{D)Q'} - \nabla_{D)Q'}\nabla_{AX'})\overline{P}^{M'N'P'Q'} + (\nabla^{AX'}\nabla_{(CP'} - \nabla_{(CP'}\nabla^{AX'})\nabla_{D)Q'}\nabla_{AX'}\overline{P}^{M'N'P'Q'} \\ & + \nabla_{(CP'}(\nabla^{AX'}\nabla_{D)Q'} - \nabla_{D)Q'}\nabla^{AX'})\nabla_{AX'}\overline{P}^{M'N'P'Q'} \, , \end{aligned} \tag{5.13}$$

where symmetrizations are over indices C and D. In the first of these terms, the d'Alembertian may be rewritten according to Eq. (A18) as

$$\Box\overline{P}^{M'N'P'Q'} = 2\nabla^{A(M'}\nabla_{AX'}\overline{P}^{N'P'Q')X'} + (\nabla^{AX'}\nabla_A{}^{(M'} - \nabla_A{}^{(M'}\nabla^{AX'})\overline{P}^{N'P'Q')}{}_{X'} \, , \tag{5.14}$$

where the symmetrization includes just M', N', P', Q'. The first term of Eq. (5.14) is replaced, as stated above, by $2\nabla^{A(M'}G_A{}^{N'P'Q')} - 6\overline{\Psi}_{X'Y'}{}^{(M'N'}\overline{P}^{P'Q')X'Y'}$ via the wave equation (5.5); the gauge term is accounted for above in R_G. When the terms thus given by Eqs. (5.13) and (5.14) are included, we find that

$$\begin{aligned} R_P = {} & -\nabla_{AX'}\nabla_D{}^{N'}\nabla^{(A}{}_{P'}\nabla_{C)Q'}\overline{P}^{X'M'P'Q'} - \nabla_{AX'}\nabla_C{}^{M'}\nabla^{(A}{}_{P'}\nabla_{D)Q'}\overline{P}^{X'N'P'Q'} + \nabla_{(CP'}\nabla_{D)Q'}(\nabla^{AX'}\nabla_A{}^{(M'} - \nabla_A{}^{(M'}\nabla^{AX'})\overline{P}^{N'P'Q')}{}_{X'} \\ & + \nabla^{AX'}(\nabla_{AX'}\nabla_{(CP'} - \nabla_{(CP'}\nabla_{AX'})\nabla_{D)Q'}\overline{P}^{M'N'P'Q'} + \nabla^{AX'}\nabla_{(CP'}(\nabla_{AX'}\nabla_{D)Q'} - \nabla_{D)Q'}\nabla_{AX'})\overline{P}^{M'N'P'Q'} \\ & + (\nabla^{AX'}\nabla_{(CP'} - \nabla_{(CP'}\nabla^{AX'})\nabla_{D)Q'}\nabla_{AX'}\overline{P}^{M'N'P'Q'} + \nabla_{(CP'}(\nabla^{AX'}\nabla_{D)Q'} - \nabla_{D)Q'}\nabla^{AX'})\nabla_{AX'}\overline{P}^{M'N'P'Q'} \\ & - 6\nabla_{(CP'}\nabla_{D)Q'}\overline{\Psi}_{X'Y'}{}^{(M'N'}\overline{P}^{P'Q')X'Y'} \, . \end{aligned} \tag{5.15}$$

The first two terms are put into a form suitable for application of the Ricci identities by noting that

$$
\begin{aligned}
-\nabla_{AX'}\nabla_D{}^{N'}\nabla^{(A}{}_{P'}\nabla_{C)Q'}\overline{P}^{X'M'P'Q'} = {} & -\nabla_D{}^{N'}\nabla_{A(X'}\nabla^A{}_{P')}\nabla_{CQ'}\overline{P}^{X'M'P'Q'} \\
& - (\nabla_{AX'}\nabla_D{}^{N'} - \nabla_D{}^{N'}\nabla_{AX'})\nabla^{(A}{}_{P'}\nabla_{C)Q'}\overline{P}^{X'M'P'Q'} - \tfrac{1}{2}\nabla_D{}^{N'}\nabla_{AX'}(\nabla_{CP'}\nabla^A{}_{Q'} - \nabla^A{}_{Q'}\nabla_{CP'})\overline{P}^{X'M'P'Q'}
\end{aligned}
\tag{5.16}
$$

and similarly that

$$
\begin{aligned}
-\nabla_{AX'}\nabla_C{}^{M'}\nabla^{(A}{}_{P'}\nabla_{D)Q'}\overline{P}^{X'N'P'Q'} = {} & -\nabla_C{}^{M'}\nabla_{A(X'}\nabla^A{}_{P')}\nabla_{DQ'}\overline{P}^{X'N'P'Q'} \\
& -(\nabla_{AX'}\nabla_C{}^{M'} - \nabla_C{}^{M'}\nabla_{AX'})\nabla^{(A}{}_{P'}\nabla_{D)Q'}\overline{P}^{X'N'P'Q'} \\
& - \tfrac{1}{2}\nabla_C{}^{M'}\nabla_{AX'}(\nabla_{DP'}\nabla^A{}_{Q'} - \nabla^A{}_{Q'}\nabla_{DP'})\overline{P}^{X'N'P'Q'}\,.
\end{aligned}
\tag{5.17}
$$

After substitution of Eqs. (5.16) and (5.17) into R_P, all commutators are replaced by symmetrized once-contracted second derivatives, just as above for R_G, by application of the identity (A12). The resulting expression for R_P analogous to Eq. (5.10) is

$$
\begin{aligned}
R_P = {} & \tfrac{1}{2}\nabla_{(CP'}\nabla_{D)Q'}\nabla_{H(X'}\nabla^{HM')}\overline{P}^{N'P'Q'X'} + \tfrac{1}{2}\nabla_{(CP'}\nabla_{D)Q'}\nabla_{H(X'}\nabla^{HN')}\overline{P}^{M'P'Q'X'} \\
& + \tfrac{1}{2}\nabla_{(CP'}\nabla_{D)Q'}\nabla_{H(X'}\nabla^{HP')}\overline{P}^{M'N'Q'X'} + \tfrac{1}{2}\nabla_{(CP'}\nabla_{D)Q'}\nabla_{H(X'}\nabla^{HQ')}\overline{P}^{M'N'P'X'} \\
& + \tfrac{1}{2}(\nabla_C{}^{X'}\nabla_{H(X'}\nabla^H{}_{P')} + \nabla^A{}_{P'}\nabla_{(AR'}\nabla_{C)}{}^{R'})\nabla_{DQ'}\overline{P}^{M'N'P'Q'} + \tfrac{1}{2}(\nabla_D{}^{X'}\nabla_{H(X'}\nabla^H{}_{P')} + \nabla^A{}_{P'}\nabla_{(AR'}\nabla_{D)}{}^{R'})\nabla_{CQ'}\overline{P}^{M'N'P'Q'} \\
& + \tfrac{1}{2}(\nabla_D{}^{X'}\nabla_{CP'}\nabla_{H(X'}\nabla^H{}_{Q')} + \nabla^A{}_{Q'}\nabla_{CP'}\nabla_{(AR'}\nabla_{D)}{}^{R'})\overline{P}^{M'N'P'Q'} + \tfrac{1}{2}(\nabla_C{}^{X'}\nabla_{DP'}\nabla_{H(X'}\nabla^H{}_{Q')} + \nabla^A{}_{Q'}\nabla_{DP'}\nabla_{(AR'}\nabla_{C)}{}^{R'})\overline{P}^{M'N'P'Q'} \\
& - \tfrac{1}{2}(\nabla_H{}^{(X'}\nabla^H{}_{P')}\nabla_{DQ'}\nabla_{CX'} + \nabla^{(A}{}_{R'}\nabla_{C)}{}^{R'}\nabla_{DQ'}\nabla_{AP'})\overline{P}^{M'N'P'Q'} - \tfrac{1}{2}(\nabla_H{}^{(X'}\nabla^H{}_{P')}\nabla_{CQ'}\nabla_{DX'} + \nabla^{(A}{}_{R'}\nabla_{D)}{}^{R'}\nabla_{CQ'}\nabla_{AP'})\overline{P}^{M'N'P'Q'} \\
& - \tfrac{1}{2}\nabla_{CP'}(\nabla_H{}^{(X'}\nabla^H{}_{Q')}\nabla_{DX'} + \nabla^{(A}{}_{R'}\nabla_{D)}{}^{R'}\nabla_{AQ'})\overline{P}^{M'N'P'Q'} - \tfrac{1}{2}\nabla_{DP'}(\nabla_H{}^{(X'}\nabla^H{}_{Q')}\nabla_{CX'} + \nabla^{(A}{}_{R'}\nabla_{C)}{}^{R'}\nabla_{AQ'})\overline{P}^{M'N'P'Q'} \\
& - \nabla_D{}^{N'}\nabla_{A(X'}\nabla^A{}_{P')}\nabla_{CQ'}\overline{P}^{X'M'P'Q'} - \nabla_{H(X'}\nabla^{HN')}\nabla_{(DP'}\nabla_{C)Q'}\overline{P}^{X'M'P'Q'} \\
& - \nabla_{(AR'}\nabla_{D)}{}^{R'}\nabla^{(A}{}_{P'}\nabla_{C)Q'}\overline{P}^{M'N'P'Q'} - \tfrac{1}{2}\nabla_D{}^{N'}\nabla_{CX'}\nabla_{H(P'}\nabla^H{}_{Q')}\overline{P}^{X'M'P'Q'} \\
& - \nabla_C{}^{M'}\nabla_{A(X'}\nabla^A{}_{P')}\nabla_{DQ'}\overline{P}^{X'N'P'Q'} - \nabla_{H(X'}\nabla^{HM')}\nabla_{(CP'}\nabla_{D)Q'}\overline{P}^{X'N'P'Q'} \\
& - \nabla_{(AR'}\nabla_{C)}{}^{R'}\nabla^{(A}{}_{P'}\nabla_{D)Q'}\overline{P}^{M'N'P'Q'} - \tfrac{1}{2}\nabla_C{}^{M'}\nabla_{DX'}\nabla_{H(P'}\nabla^H{}_{Q')}\overline{P}^{X'N'P'Q'} - 6\nabla_{(CP'}\nabla_{D)Q'}\overline{\Psi}_{X'Y'}{}^{(M'N'}\overline{P}^{P'Q')X'Y'}\,.
\end{aligned}
\tag{5.18}
$$

To this expression for R_P the Ricci identities (A19) and (A20) are applied, yielding an expression analogous to Eq. (5.11), with each term proportional to the background Weyl spinor. This procedure gives, after some simplifications,

$$
\begin{aligned}
R_P = {} & \nabla_{(CP'}\nabla_{D)Q'}\overline{\Psi}_{X'}{}^{M'N'}{}_{R'}\overline{P}^{R'P'Q'X'} - 2\overline{\Psi}_{X'}{}^{M'N'}{}_{R'}\nabla_{(DP'}\nabla_{C)Q'}\overline{P}^{X'R'P'Q'} \\
& - \overline{\Psi}_{Z'}{}^{P'Y'}{}_{R'}\nabla_{(CP'}\nabla_{D)Y'}\overline{P}^{M'N'R'Z'} - 2\overline{\Psi}^{X'}{}_{P'R'}{}^{(M'}\nabla_{(CX'}\nabla_{D)Q'}\overline{P}^{N')R'P'Q'} \\
& - 2\overline{\Psi}^{X'}{}_{P'R'}{}^{(M'}\nabla_{(CQ'}\nabla_{D)X'}\overline{P}^{N')R'P'Q'} - \nabla_D{}^{N'}\overline{\Psi}_{X'P'}{}^{M'}{}_{R'}\nabla_{CQ'}\overline{P}^{X'R'P'Q'} \\
& - \tfrac{1}{2}\nabla_D{}^{N'}\nabla_{CX'}\overline{\Psi}_{P'Q'}{}^{M'}{}_{R'}\overline{P}^{X'R'P'Q'} - \tfrac{1}{2}\nabla_D{}^{N'}\nabla_{CX'}\overline{\Psi}_{P'Q'}{}^{X'}{}_{R'}\overline{P}^{M'R'P'Q'} \\
& - \nabla_C{}^{M'}\overline{\Psi}_{X'P'}{}^{N'}{}_{R'}\nabla_{DQ'}\overline{P}^{X'R'P'Q'} - \tfrac{1}{2}\nabla_C{}^{M'}\nabla_{DX'}\overline{\Psi}_{P'Q'}{}^{X'}{}_{R'}\overline{P}^{N'R'P'Q'} \\
& - \tfrac{1}{2}\nabla_C{}^{M'}\nabla_{DX'}\overline{\Psi}_{P'Q'}{}^{N'}{}_{R'}\overline{P}^{X'R'P'Q'} - 6\nabla_{(CP'}\nabla_{D)Q'}\overline{\Psi}_{X'Y'}{}^{(M'N'}\overline{P}^{P'Q')X'Y'}\,,
\end{aligned}
\tag{5.19}
$$

where terms containing the unconjugated Weyl spinor are omitted since they cancel one another as in R_G. [Certain of the manipulations used in obtaining Eq. (5.19) involve application of the vacuum Bianchi identity Eq. (A21) to commute the Weyl spinor and covariant derivative operator.]

The next step in the calculation is to make the assumption that the Weyl spinor of the background space is algebraically special, to align a spinor dyad frame so that one dyad leg coincides with the repeated principal spinor of the Weyl spinor, to choose a special potential P_{ABCD} with only one nonvanishing component in this dyad, and finally, to make a choice of the components of the gauge spinor $G_A{}^{M'N'P'}$ in this dyad which reduces Eq. (5.5) to a decoupled scalar wave equation. Thus a dyad o^A, ι^A is chosen with o^A oriented along the repeated principal spinor of the Weyl spinor, and the potential is fixed to be $\overline{P}^{1'1'1'1'} = \overline{\psi}$ in this dyad; other components vanish. Direct computation shows that if the choice $G_1{}^{1'1'1'} = 4\bar{\rho}\overline{\psi}$, $G_2{}^{1'1'1'} = 4\bar{\tau}\overline{\psi}$ is made (other components vanish), then the five dyad components of Eq. (5.5) become

$2'2'2'2'$: Both sides zero; identically satisfied.

$1'2'2'2'$: Both sides zero; identically satisfied.

$1'1'2'2'$: Left side $=[4(\bar\rho\bar\sigma-\bar\tau\bar\kappa)+\bar\Psi_0]\bar\psi$, Right side $=4(\bar\rho\bar\sigma-\bar\tau\bar\kappa)\bar\psi$. These are identically equal since alignment of the dyad implies $\Psi_0=0$.

$1'1'1'2'$: Left side $=[(D+\epsilon+3\bar\epsilon-\rho+3\bar\rho)(\bar\delta+4\bar\beta-\bar\tau)-(\bar\delta-\alpha+3\bar\beta+\pi+3\bar\tau)(D+4\bar\epsilon-\bar\rho)]\bar\psi$ $+$terms proportional to κ,σ.

Right side $=4[-(D+\epsilon+3\bar\epsilon-\rho+3\bar\rho)\bar\tau+(\bar\delta-\alpha+3\bar\beta+\pi+3\bar\tau)\bar\rho]\bar\psi$. These are seen to be identically equal by application of several NP equations and by use of $\kappa=\sigma=\Psi_1=0$.

$1'1'1'1'$: Imposing equality of the two sides yields the scalar equation[21]

$$[(\delta+3\bar\alpha+\beta-\tau)(\bar\delta+4\bar\beta+3\bar\tau)-(\Delta-\gamma+3\bar\gamma+\mu)(D+4\bar\epsilon+3\bar\rho)+3\bar\Psi_2]\bar\psi=0\,. \tag{5.20}$$

The final step of the calculation, which consists of substituting the above choices

$$\bar P^{1'1'1'1'}=\bar\psi,\quad G_1{}^{1'1'1'}=4\bar\rho\bar\psi,\quad G_2{}^{1'1'1'}=4\bar\tau\bar\psi \tag{5.21}$$

into Eqs. (5.11) and (5.19) for R_P and R_G, shows that $R_P+R_G=0$, i.e., the vacuum perturbation field equations are satisfied: Of the seventeen terms in Eq. (5.11) for R_G, the three containing the unconjugated Weyl spinor cancel, as observed above; also the tenth, fourteenth, and fifteenth terms vanish because the triple contraction of $\bar\Psi_{X'P'R'}{}^{M'}$ with $G_C{}^{X'R'P'}$ vanishes by the choice (5.21) of $G_C{}^{X'R'P'}$ and the algebraically special assumption for $\bar\Psi_{X'P'R'}{}^{M'}$. The eight remaining nonzero terms in R_G then give, after they are combined, the following dyad components in NP notation:

$$\begin{aligned}
(R_G)_{11}{}^{1'1'}&=-24(D-\epsilon+3\bar\epsilon-2\bar\rho)\bar\rho\bar\Psi_2\bar\psi\,,\\
(R_G)_{(12)}{}^{1'1'}&=-12(D+\epsilon+3\bar\epsilon+\rho-2\bar\rho)\bar\tau\bar\Psi_2\bar\psi-12(\bar\delta-\alpha+3\bar\beta-\pi-2\bar\tau)\bar\rho\bar\Psi_2\bar\psi\,,\\
(R_G)_{22}{}^{1'1'}&=24\lambda\bar\rho\bar\Psi_2\bar\psi-24(\bar\delta+\alpha+3\bar\beta-2\bar\tau)\bar\tau\bar\Psi_2\bar\psi\,,
\end{aligned} \tag{5.22}$$

where the Weyl tensor has been commuted to the right through the differential operators by means of the NP Bianchi identities,

$$\begin{aligned}
\bar\Psi_2 D&=(D-3\bar\rho)\bar\Psi_2\,,\\
\bar\Psi_2\bar\delta&=(\bar\delta-3\bar\tau)\bar\Psi_2\,.
\end{aligned} \tag{5.23}$$

When the 12 terms of R_P in Eq. (5.19) are treated similarly, it is seen that the sixth through eleventh vanish because of triple contractions of $\bar\Psi_{X'P'R'}{}^{M'}$ with $\bar P^{N'X'P'R'}$ or with $\nabla_{CQ'}\bar P^{X'P'R'Q'}$. The fifth term equals the fourth, as seen by commuting the derivatives in the fifth and observing that the resulting commutator is proportional to the Ricci tensor of the background (which is assumed zero). When the differential operators in the third term are moved to the left by the vacuum Bianchi identities (A21) and the first and third terms are translated into NP notation, they are seen to cancel. Of the remaining contributions to R_P, the sum of the second and twice the fourth terms gives dyad components

$$\begin{aligned}
{}_{11}{}^{1'1'}&=[-6(D-\epsilon+3\bar\epsilon-\bar\rho)(D+4\bar\epsilon-5\bar\rho)-24\bar\rho^2]\bar\Psi_2\bar\psi\,,\\
{}_{(12)}{}^{1'1'}&=[-3(D+\epsilon+3\bar\epsilon+\rho-\bar\rho)(\bar\delta+4\bar\beta-5\bar\tau)-3(\bar\delta-\alpha+3\bar\beta-\pi-\bar\tau)(D+4\bar\epsilon-5\bar\rho)-24\bar\rho\bar\tau]\bar\Psi_2\bar\psi\,,\\
{}_{22}{}^{1'1'}&=[6\lambda(D+4\bar\epsilon-5\bar\rho)-6(\bar\delta+\alpha+3\bar\beta-\bar\tau)(\bar\delta+4\bar\beta-5\bar\tau)-24\bar\tau^2]\bar\Psi_2\bar\psi\,,
\end{aligned} \tag{5.24}$$

where again the Bianchi identies (5.23) have been used in moving the Weyl tensor factors to the right, while the twelfth term, $-6\nabla_{(CP'}\nabla_{D)Q'}\bar\Psi_{X'Y'}{}^{(M'N'}\bar P^{P'Q')X'Y'}$, has NP dyad components

$$\begin{aligned}
{}_{11}{}^{1'1'}&=+6(D-\epsilon+3\bar\epsilon-\bar\rho)(D+4\bar\epsilon-\bar\rho)\bar\Psi_2\bar\psi\,,\\
{}_{(12)}{}^{1'1'}&=3(D+\epsilon+3\bar\epsilon+\rho-\bar\rho)(\bar\delta+4\bar\beta-\bar\tau)\bar\Psi_2\bar\psi+3(\bar\delta-\alpha+3\bar\beta-\pi-\bar\tau)(D+4\bar\epsilon-\bar\rho)\bar\Psi_2\bar\psi\,,\\
{}_{22}{}^{1'1'}&=-6\lambda(D+4\bar\epsilon-\bar\rho)\bar\Psi_2\bar\psi+6(\bar\delta+\alpha+3\bar\beta-\bar\tau)(\bar\delta+4\bar\beta-\bar\tau)\bar\Psi_2\bar\psi\,.
\end{aligned} \tag{5.25}$$

Hence the dyad components of R_P, given by Eq. (5.24) plus Eq. (5.25), are

$$\begin{aligned}
(R_P)_{11}{}^{1'1'}&=[-6(D-\epsilon+3\bar\epsilon-\bar\rho)(D+4\bar\epsilon-5\bar\rho)+6(D-\epsilon+3\bar\epsilon-\bar\rho)(D+4\bar\epsilon-\bar\rho)-24\bar\rho^2]\bar\Psi_2\bar\psi\,,\\
(R_P)_{(12)}{}^{1'1'}&=[-3(D+\epsilon+3\bar\epsilon+\rho-\bar\rho)(\bar\delta+4\bar\beta-5\tau)+3(D+\epsilon+3\bar\epsilon+\rho-\bar\rho)(\bar\delta+4\bar\beta-\bar\tau)\\
&\quad-3(\bar\delta-\alpha+3\bar\beta-\pi-\bar\tau)(D+4\bar\epsilon-5\bar\rho)+3(\bar\delta-\alpha+3\bar\beta-\pi-\bar\tau)(D+4\bar\epsilon-\bar\rho)-24\bar\rho\bar\tau]\bar\Psi_2\bar\psi\,,\\
(R_P)_{22}{}^{1'1'}&=[6\lambda(D+4\bar\epsilon-5\bar\rho)-6\lambda(D+4\bar\epsilon-\bar\rho)-6(\bar\delta+\alpha+3\bar\beta-\bar\tau)(\bar\delta+4\bar\beta-5\bar\tau)\\
&\quad+6(\bar\delta+\alpha+3\bar\beta-\bar\tau)(\bar\delta+4\bar\beta-\bar\tau)-24\bar\tau^2]\bar\Psi_2\bar\psi\,.
\end{aligned} \tag{5.26}$$

It is observed that these are just the negative of the components of R_G as given by Eq. (5.22), so that $R_P + R_G = 0$, which completes the proof. [The complex-conjugate term in the perturbed metric (5.4) may similarly be shown to result in the vanishing of expression (5.3) for the perturbed Ricci tensor upon substitution into Eq. (5.3) and application of the wave equation (5.5) for the potential and gauge spinors.]

The perturbed metric (5.4), when expressed in terms of a scalar solution $\bar{\psi}$ to the wave equation (5.20), has coordinate (or frame) components[21]

$$\begin{aligned} h_{\mu\nu} = -\{&l_\mu l_\nu[(\bar\delta+\alpha+3\bar\beta-\bar\tau)(\bar\delta+4\bar\beta+3\bar\tau)-\lambda(D+4\bar\epsilon+3\bar\rho)]+\bar m_\mu \bar m_\nu(D-\epsilon+3\bar\epsilon-\bar\rho)(D+4\bar\epsilon+3\bar\rho)\\ &-l_{(\mu}\bar m_{\nu)}[(D+\epsilon+3\bar\epsilon+\rho-\bar\rho)(\bar\delta+4\bar\beta+3\bar\tau)+(\bar\delta-\alpha+3\bar\beta-\pi-\bar\tau)(D+4\bar\epsilon+3\bar\rho)]\}\bar\psi+\text{c.c.}, \end{aligned} \tag{5.27}$$

where l_μ, m_μ, and $\bar m_\mu$ are the coordinate (or frame) components of elements of a null tetrad obtained in the canonical way from the aligned dyad (o^A, ι^A) (i.e.,

$$l^\mu \leftrightarrow o^A \bar o^{X'},\quad m^\mu \leftrightarrow o^A \bar\iota^{X'},\quad n^\mu \leftrightarrow \iota^A \bar\iota^{X'}).$$

The final result of this section is the NP expression for the perturbed Weyl spinor as given by differentiating the scalar $\bar\psi$. This expression is obtained by substituting Eq. (5.27) for $h_{\mu\nu}$ into Eq. (2.19) followed by projecting NP components, or alternatively by substituting Eq. (5.27) into Eq. (5.6). The results are given here for the case $\lambda = \nu = 0$, that is, for type D spacetimes. Their generalization to the algebraically special case may be obtained by the above procedure but involves somewhat lengthy additional terms proportional to λ which are omitted here. The perturbed Weyl tensor is[21]

$$\begin{aligned}
\psi_0 &= (D-3\epsilon+\bar\epsilon-\bar\rho)(D-2\epsilon+2\bar\epsilon-\bar\rho)(D-\epsilon+3\bar\epsilon-\bar\rho)(D+4\bar\epsilon+3\bar\rho)\bar\psi,\\
4\psi_1 &= [(D-\epsilon+\bar\epsilon+\rho-\bar\rho)(D+2\bar\epsilon+\rho-\bar\rho)(D+\epsilon+3\bar\epsilon+\rho-\bar\rho)(\bar\delta+4\bar\beta+3\bar\tau)\\
&\quad+(D-\epsilon+\bar\epsilon+\rho-\bar\rho)(D+2\bar\epsilon+\rho-\bar\rho)(\bar\delta-\alpha+3\bar\beta-\pi-\bar\tau)(D+4\bar\epsilon+3\bar\rho)\\
&\quad+(D-\epsilon+\bar\epsilon+\rho-\bar\rho)(\bar\delta-2\alpha+2\bar\beta-2\pi-\bar\tau)(D-\epsilon+3\bar\epsilon-\bar\rho)(D+4\bar\epsilon+3\bar\rho)\\
&\quad+(\bar\delta-3\alpha+\bar\beta-3\pi-\bar\tau)(D-2\epsilon+2\bar\epsilon-\bar\rho)(D+\epsilon+3\bar\epsilon-\bar\rho)(D+4\bar\epsilon+3\bar\rho)]\bar\psi,\\
6\psi_2 &= [(D+\epsilon+\bar\epsilon+2\rho-\bar\rho)(D+2\epsilon+2\bar\epsilon+2\rho-\bar\rho)(\bar\delta+\alpha+3\bar\beta-\bar\tau)(\bar\delta+4\bar\beta+3\bar\tau)\\
&\quad+(D+\epsilon+\bar\epsilon+2\rho-\bar\rho)(\bar\delta+2\bar\beta-\pi-\bar\tau)(D+\epsilon+3\bar\epsilon+\rho-\bar\rho)(\bar\delta+4\bar\beta+3\bar\tau)\\
&\quad+(D+\epsilon+\bar\epsilon+2\rho-\bar\rho)(\bar\delta+2\bar\beta-\pi-\bar\tau)(\bar\delta-\alpha+3\bar\beta-\pi-\bar\tau)(D+4\bar\epsilon+3\bar\rho)\\
&\quad+(\bar\delta-\alpha+\bar\beta-2\pi-\bar\tau)(\bar\delta-2\alpha+2\bar\beta-2\pi-\bar\tau)(D-\epsilon+3\bar\epsilon-\bar\rho)(D+4\bar\epsilon+3\bar\rho)\\
&\quad+(\bar\delta-\alpha+\bar\beta-2\pi-\bar\tau)(D+2\bar\epsilon+\rho-\bar\rho)(\bar\delta-\alpha+3\bar\beta-\pi-\bar\tau)(D+4\bar\epsilon+3\bar\rho)\\
&\quad+(\bar\delta-\alpha+\bar\beta-2\pi-\bar\tau)(D+2\bar\epsilon+\rho-\bar\rho)(D+\epsilon+3\bar\epsilon+\rho-\bar\rho)(\bar\delta+4\bar\beta+3\bar\tau)]\bar\psi,\\
4\psi_3 &= [(D+3\epsilon+\bar\epsilon+3\rho-\bar\rho)(\bar\delta+2\alpha+2\bar\beta-\bar\tau)(\bar\delta+\alpha+3\bar\beta-\bar\tau)(\bar\delta+4\bar\beta+3\bar\tau)\\
&\quad+(\bar\delta+\alpha+\bar\beta-\pi-\bar\tau)(D+2\epsilon+2\bar\epsilon+2\rho-\bar\rho)(\bar\delta+\alpha+3\bar\beta-\bar\tau)(\bar\delta+4\bar\beta+3\bar\tau)\\
&\quad+(\bar\delta+\alpha+\bar\beta-\pi-\bar\tau)(\bar\delta+2\bar\beta-\pi-\bar\tau)(D+\epsilon+3\bar\epsilon+\rho-\bar\rho)(\bar\delta+4\bar\beta+3\bar\tau)\\
&\quad+(\bar\delta+\alpha+\bar\beta-\pi-\bar\tau)(\bar\delta+2\bar\beta-\pi-\bar\tau)(\bar\delta-\alpha+3\bar\beta-\pi-\bar\tau)(D+4\bar\epsilon+3\bar\rho)]\bar\psi,\\
\psi_4 &= (\bar\delta+3\alpha+\bar\beta-\bar\tau)(\bar\delta+2\alpha+2\bar\beta-\bar\tau)(\bar\delta+\alpha+3\bar\beta-\bar\tau)(\bar\delta+4\bar\beta+3\bar\tau)\bar\psi\\
&\quad+3\Psi_2[\tau(\bar\delta+4\alpha)-\rho(\Delta+4\gamma)-\mu(D+4\epsilon)+\pi(\delta+4\beta)+2\Psi_2]\psi.
\end{aligned} \tag{5.28}$$

The terms generated by the c.c. term in Eq. (5.27) make no contribution here, except in the ψ_4 component as indicated.

The terms proportional to the unconjugated potential ψ were omitted from the formulas of Ref. 21, although they follow directly from the conjugate terms of Eq. (5.4) when substituted into Eq. (5.6) (see also Ref. 40). All such terms are in fact proportional to the background Weyl spinor Ψ_2. This is most easily seen by noting the requirement that unconjugated ψ contributions vanish in Minkowski space according to Eq. (2.5) [which for spin 2 reduces to Eq. (2.26)]. Alternatively, this may be shown explicitly by commuting the derivative operators and substituting the term $3\Psi_2\psi$ via the (conjugate) wave equation (5.20) whenever the appropriate combination of operators appears.

As remarked above, Eq. (5.28) establishes by direct construction that the perturbed Weyl tensor is derivable from a metric perturbation, namely Eq. (5.27).

B. Summary of this section

It has been proved that for algebraically special vacuum spacetimes, a scalar Debye superpotential for gravitational perturbations may be obtained from the wave equation

$$\nabla^{A(M'}\nabla_{AX'}\bar{P}^{N'P'Q')X'}+3\bar{\Psi}_{X'Y'}{}^{(M'N'}\bar{P}^{P'Q')X'Y'}=\nabla^{A(M'}G_A{}^{N'P'Q')}\,,\qquad(5.5)$$

where $\bar{\Psi}_{X'Y'}{}^{M'N'}$ is the unperturbed Weyl tensor and $\bar{P}^{N'P'Q'X'}$ and $G_A{}^{N'P'Q'}$ are a Hertz potential and gauge spinor, respectively. In terms of these spinors the perturbed metric is given by

$$h_{CD}{}^{M'N'}=\nabla_{(CP'}\nabla_{D)Q'}\bar{P}^{M'N'P'Q'}-\nabla_{(CP'}G_{D)}{}^{M'N'P'}+\text{c.c.}\qquad(5.4)$$

and the perturbed Weyl tensor by

$$\psi_{ABCD}=\nabla_{(AW'}\nabla_{BX'}h_{CD)}{}^{W'X'}\,.\qquad(5.6)$$

The corresponding scalar superpotential formulas obtained from the special aligned values [Eq. (5.21)] for $\bar{P}^{N'P'Q'X'}$ and $G_A{}^{N'P'Q'}$ are given in NP notation by Eqs. (5.20), (5.27), and (5.28), respectively.

VI. SUMMARY

This paper presents a covariant spinor framework for Hertz and Debye potentials for zero-rest-mass perturbations of certain algebraically special spacetimes. The covariant potential wave equations and the fields generated by differentiating the potentials are

$$\nabla^{AM'}\nabla_{AW'}\bar{P}^{W'}=\nabla^{AM'}G_A\qquad(4.4)$$

and

$$\phi_A=\nabla_{AW'}\bar{P}^{W'}-G_A\qquad(4.5)$$

for the two-component neutrino case,

$$\nabla^{A(M'}\nabla_{AW'}\bar{P}^{N')W'}=\nabla^{A(M'}G_A{}^{N')}\,,\qquad(3.17)$$

$$A_{CM'}=\nabla_{CP'}\bar{P}_{M'}{}^{P'}-G_{CM'}+\text{c.c.}\,,\qquad(3.24)$$

and

$$\phi_{AB}=\nabla_{(AW'}\nabla_{B)X'}\bar{P}^{W'X'}-\nabla_{(AW'}G_{B)}{}^{W'}\qquad(3.18)$$

for the case of Maxwell fields, and

$$\nabla^{A(M'}\nabla_{AW'}\bar{P}^{N'P'Q')W'}+3\Psi_{X'Y'}{}^{(M'N'}\bar{P}^{P'Q')X'Y'}=\nabla^{A(M'}G_A{}^{N'P'Q')}\,,\qquad(5.5)$$

$$h_{CD}{}^{M'N'}=\nabla_{(CP'}\nabla_{D)Q'}\bar{P}^{M'N'P'Q'}-\nabla_{(CP'}G_{D)}{}^{M'N'P'}+\text{c.c.}\,,\qquad(5.4)$$

and

$$\psi_{ABCD}=\nabla_{(AW'}\nabla_{BX'}h_{CD)}{}^{W'X'}\qquad(5.6)$$

for gravitational perturbations.

The practical consequence of these results is a computational scheme for perturbations of this class of spacetimes (including many of astrophysical importance—black holes and various cosmological models, for example[23,24]) requiring only the solution of a decoupled scalar wave equation. This equation may be summarized for the neutrino, electromagnetic, and gravitational perturbation cases by the NP equation[21]

$$\{(\Delta-[2s+1]\gamma-\bar{\gamma}+\bar{\mu})(D-2s\epsilon-[2s+1]\rho)-(\bar{\delta}-[2s+1]\alpha+\bar{\beta}-\bar{\tau})(\delta-2s\beta-[2s+1]\tau)-(s+1)(2s+1)\Psi_2\}\psi^{(s)}=0\,,\qquad(6.1)$$

where $\psi^{(s)}$ is the corresponding scalar Debye potential for $s=-\frac{1}{2}$, -1, or -2, respectively. Equation (6.1) summarizes Eqs. (3.28), (4.13b), and (5.20) of the text. The tensor or spinor components of the physical perturbation field are then given (in the NP frame) by prescribed differentiation of $\psi^{(s)}$ according to Eqs. (3.29), (4.14), and (5.27) or (5.28), respectively.

The differential operators of the exterior form Hertz potential formalism[20] have led to further developments consisting of generalizations of the notion of harmonic operator and applications to analysis of the structure of the vacuum Einstein field equations.[41,42]

The linear scalar wave equation (6.1) and that of Teukolsky[13] have led to an extensive literature on perturbations of the Kerr spacetime, including approaches[40,43,44,45] which combine both Teukolsky's derivation and the notion of a scalar superpotential. The work of Chrzanowski[43] and Chandrasekhar[44] emphasizes the "intermediate" potentials, the vector potential[44] for spin 1 and the metric perturbations for spin 2 (for a treatment of these potentials in the present context see Secs. II C, III B, and V A above; see also Ref. 20 for vector potentials). In addition the approach of Chandrasekhar has yielded a demonstration of the separability of the spin-$\frac{1}{2}$ case for nonzero mass, i.e., the Dirac equation, in a Kerr background.[46] Calculations of astrophysical processes involving perturbations of black holes and utilizing decoupled scalar wave equations also comprise a considerable literature (see e.g., Refs. 43–51).

Recent work by Wald[40] has produced very simple and elegant proofs of the Debye potential formulas for vacuum algebraically special spacetimes. The spinor proofs of the present paper (and the treatment of Ref. 20) show that the results are in fact more general in that the nonvacuum spaces of the generalized Goldberg-Sachs class are covered for spins $\frac{1}{2}$ and 1. Thus, for example, electromagnetic and neutrino perturbations of the locally rotationally symmetric perfect-fluid cosmological models may be computed by these methods.

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APPENDIX

1. Exterior form Hertz potential formalism for electromagnetism

If a 2-form Hertz potential P obeys

$$\Delta P = dG + \delta W\,, \tag{A1}$$

where G is an arbitrary 1-form and W is an arbitrary 3-form, then the vector potential 1-form A is given by

$$A = \delta P - G \tag{A2}$$

(note that A is not in the Lorentz gauge) and the Maxwell field 2-form by

$$f = dA\,, \tag{A3}$$

i.e.,

$$f = d\delta P - dG = \delta W - \delta dP\,. \tag{A4}$$

In this formalism, the proof that f is a source-free Maxwell field (i.e., that $df = \delta f = 0$) follows immediately from the identity $d^2 \equiv \delta^2 \equiv 0$; this situation is in contrast to the relatively tedious proofs given above for the equivalent formulas (3.17), (3.24), (3.27), and (3.18) of spinor analysis. The formulas (A1)–(A4) are given in Ref. 20 where they are shown to give decoupled wave equations for scalar Debye potentials in the (in general nonvacuum) generalized Goldberg-Sachs[2,22] class of spacetimes.

2. Spinor formalism

In this section those formulas of spinor analysis used in the text are given; proofs are given here for those identities not proved in Ref. 36. For more detailed presentations of spinor formalism, Refs. 26, 35, 36, and 52 are suggested.

The formulas are

$$\eta_A\eta^A = -\eta^A\eta_A \text{ (“dummy index rule,” p. 309 of Ref. 36);} \tag{A5}$$

$$\chi_{AC} - \chi_{CA} = \epsilon_{AC}\chi_D{}^D \text{ [Eq. (3.16) of Ref. 36];} \tag{A6}$$

$$\epsilon^{AB}\epsilon_{BC} = -\delta^A{}_C \text{ (p. 308 of Ref. 36);} \tag{A7}$$

$$\nabla^{AX'}\nabla_{AM'} = \tfrac{1}{2}\delta^{X'}{}_{M'}\Box \text{ in Minkowski space (where } \Box \equiv \nabla^{AX'}\nabla_{AX'}). \tag{A8}$$

Proof.

$$\nabla^A{}_{X'}\nabla_{AM'} - \nabla^A{}_{M'}\nabla_{AX'} = \epsilon_{X'M'}\nabla^A{}_{Z'}\nabla_A{}^{Z'} \text{ [by Eq. (A6)]},$$

$$\nabla^{AX'}\nabla_{AM'} - \nabla^A{}_{M'}\nabla_A{}^{X'} = -\delta^{X'}{}_{M'}\nabla^A{}_{Z'}\nabla_A{}^{Z'} \text{ [by Eq. (A7)]}$$

$$\nabla^{AX'}\nabla_{AM'} - \nabla_A{}^{X'}\nabla^A{}_{M'} = \delta^{X'}{}_{M'}\nabla^{AZ'}\nabla_{AZ'}$$

[using Eq. (A5) and the commuting of covariant derivatives]. Next Eq. (A5) is applied to the left side whose two terms are seen to be equal, giving the result claimed;

$$\nabla_{BR'}\nabla^{AR'} = \tfrac{1}{2}\delta_B{}^A\Box + \tfrac{1}{2}(\nabla_{BR'}\nabla^{AR'} - \nabla^{AR'}\nabla_{BR'}) \tag{A9}$$

[generalization of Eq. (A8) to curved space].

Proof.

$$\begin{aligned}\nabla_{BR'}\nabla^{AR'} &= \tfrac{1}{2}(\nabla_{BR'}\nabla^{AR'} - \nabla^{AR'}\nabla_{BR'}) + \tfrac{1}{2}\nabla_{BR'}\nabla^{AR'} + \tfrac{1}{2}\nabla^{AR'}\nabla_{BR'} \text{ (identically)}\\ &= \tfrac{1}{2}(\nabla_{BR'}\nabla^{AR'} - \nabla^{AR'}\nabla_{BR'}) + \tfrac{1}{2}(\nabla_{BR'}\nabla^{AR'} - \nabla^A{}_{R'}\nabla_B{}^{R'}) \text{ [by Eq. (A5)]}\\ &= \tfrac{1}{2}(\nabla_{BR'}\nabla^{AR'} - \nabla^{AR'}\nabla_{BR'}) + \tfrac{1}{2}\delta_B{}^A\Box \text{ [by Eqs. (A6) and (A7)];}\end{aligned}$$

$$F_{AW'BX'} = \epsilon_{AB}\bar{\phi}_{W'X'} + \epsilon_{W'X'}\phi_{AB}\,, \tag{A10}$$

for $F_{AW'BX'} = F_{\mu\nu}$ a real skew tensor and $\phi_{AB} \equiv \tfrac{1}{2}F_{AR'B}{}^{R'}$ [Eq. (3.26) of Ref. 36];

$$K^{BY'} = \xi^B\bar{\xi}^{Y'} \tag{A11}$$

for $K^{BY'} = K^\mu$ a real null vector [Eq. (3.24) of Ref. 36];

$$\nabla_{AW'}\nabla_{BX'} - \nabla_{BX'}\nabla_{AW'} = \epsilon_{AB}\nabla_{H(W'}\nabla^H{}_{X')} + \epsilon_{W'X'}\nabla_{(AP'}\nabla_{B)}{}^{P'} \text{ (p. 327 of Ref. 36);} \tag{A12}$$

$$\nabla_{AW'}G_{BX'} - \nabla_{BX'}G_{AW'} = \epsilon_{AB}\nabla_{H(W'}G^H{}_{X')} + \epsilon_{W'X'}\nabla_{(AP'}G_{B)}{}^{P'} \text{ [proof analogous to that of Eq. (A12)];} \tag{A13}$$

$$\begin{aligned}R_{AW'BX'CY'DZ'} = {}& \psi_{ABCD}\epsilon_{W'X'}\epsilon_{Y'Z'} + \epsilon_{AB}\epsilon_{CD}\bar{\psi}_{W'X'Y'Z'} + \epsilon_{AB}\epsilon_{Y'Z'}\Phi_{CDW'X'}\\ &+ \epsilon_{CD}\epsilon_{W'X'}\Phi_{ABY'Z'} + 2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{W'X'}\epsilon_{Y'Z'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{W'Z'}\epsilon_{X'Y'})\,,\end{aligned} \tag{A14}$$

where $R_{AW'BX'CY'DZ'}$ is the Riemann tensor, $\Phi_{ABY'Z'}=\Phi_{(AB)(Y'Z')}=\overline{\Phi}_{ABY'Z'}$ is the trace-free Ricci spinor, and $\Lambda=\frac{1}{24}R$ with R the curvature scalar [Eq. (3.52) of Ref. 36];

$$\epsilon_{AW'DZ'}{}^{BX'CY'}=i(\delta_A{}^B\delta_D{}^C\delta_{W'}{}^{Y'}\delta_{Z'}{}^{X'}-\delta_A{}^C\delta_D{}^B\delta_{W'}{}^{X'}\delta_{Z'}{}^{Y'}),$$

where $\epsilon_{AW'DZ'}{}^{BX'CY'}=\epsilon_{\alpha\delta}{}^{\beta\gamma}$ is the alternating symbol [Eq. (3.34) of Ref. 36], which implies

$$*F_{\mu\nu}=i(\epsilon_{AB}\overline{\phi}_{W'X'}-\epsilon_{W'X'}\phi_{AB})\ \text{[Eq. (3.35) of Ref. 36]}, \tag{A15}$$

if $F_{\mu\nu}$ is the real skew tensor corresponding to the spinor of Eq. (A10) and the duality operation is $*F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta}$;

$$\Box\eta_{AB}=\nabla_{(AR'}\nabla^{CR'}\eta_{B)C}+\nabla^{CR'}\nabla_{(BR'}\eta_{A)C},\quad \text{for } \eta_{AB}=\eta_{(AB)}. \tag{A16}$$

Proof. The right-hand side is

$$\tfrac{1}{2}(\nabla_{AR'}\nabla^{CR'}\eta_{BC}+\nabla_{BR'}\nabla^{CR'}\eta_{AC}+\nabla^{CR'}\nabla_{BR'}\eta_{AC}+\nabla^{CR'}\nabla_{AR'}\eta_{BC}).$$

But $\nabla_{AR'}\nabla^{CR'}-\nabla^C{}_{R'}\nabla_A{}^{R'}=\delta_A{}^C\Box$ by Eqs. (A6) and (A7), so this becomes

$$\tfrac{1}{2}(\delta_A{}^C\Box\eta_{BC}+\delta_B{}^C\Box\eta_{AC})=\Box\eta_{AB}\ \text{as claimed.}$$

Similarly,

$$\Box\overline{P}^{M'N'P'Q'}=\nabla_A{}^{(M'}\nabla^{AX'}\overline{P}^{N'P'Q')}{}_{X'}+\nabla^{AX'}\nabla_A{}^{(M'}\overline{P}^{N'P'Q')}{}_{X'},\ \text{for } \overline{P}^{M'N'P'Q'}=\overline{P}^{(M'N'P'Q')}. \tag{A17}$$

The proof is analogous to that for Eq. (A16);

$$\Box\overline{P}^{M'N'P'Q'}=2\nabla^{A(M'}\nabla_{AX'}\overline{P}^{N'P'Q')X'}+(\nabla^{AX'}\nabla_A{}^{(M'}-\nabla_A{}^{(M'}\nabla^{AX'})\overline{P}^{N'P'Q')}{}_{X'},\quad \text{for } \overline{P}^{M'N'P'Q'}=\overline{P}^{(M'N'P'Q')}. \tag{A18}$$

Proof.

$$\begin{aligned}\Box\overline{P}^{M'N'P'Q'}&=\nabla_A{}^{(M'}\nabla^{AX'}\overline{P}^{N'P'Q'}{}_{X'}+\nabla^{AX'}\nabla_A{}^{(M'}\overline{P}^{N'P'Q')}{}_{X'}\ \text{[by Eq. (A17)]}\\&=2\nabla^{A(M'}\nabla_{AX'}\overline{P}^{N'P'Q')X'}+\nabla^{AX'}\nabla_A{}^{(M'}\overline{P}^{N'P'Q')}{}_{X'}-\nabla^{A(M'}\nabla_{AX'}\overline{P}^{N'P'Q')X'}\end{aligned}$$

[by addition and subtraction of a term and by use of Eq. (A5)], which is the result claimed. Equations (A19) and (A20) are the Ricci identities, Eqs. (3.55) and (3.56) of Ref. 36:

$$\nabla_{(A}{}^{P'}\nabla_{B)P'}\xi_C=-\psi_{ABCD}\xi^D+2\Lambda\xi_{(A}\epsilon_{B)C}, \tag{A19}$$

$$\nabla_{H(W'}\nabla^H{}_{X')}\xi_D=\Phi_{DEW'X'}\xi^E. \tag{A20}$$

Equation (A21) is the vacuum Bianchi identity, Eq. (3.61) of Ref. 36:

$$\nabla^{AX'}\psi_{ABCD}=0. \tag{A21}$$

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