New exact solution to the Einstein-Dirac equations

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A new exact solution to the Einstein-Dirac equations is presented. The solution represents neutrinos moving in a static plane-symmetric curved space-time which the neutrinos themselves generate.

I. INTRODUCTION

Neutrinos have been of great interest recently in connection with general relativity. Twenty years ago, Brill and Wheeler[1] discussed the Dirac equation in curved spacetime, and in the following decade the theory was formulated in terms of orthonormal Cartan frames by Lichnerowicz[2] and by Brill and Cohen,[3] making it possible to solve the Einstein-Dirac equations with a minimum of labor.

The Cartan formalism was used by Davis and Ray[4] to find a particular solution of the Einstein-Dirac equations for neutrinos. In this communication we present a more general solution with the same metric form but with a wave function which depends on both x and t. (The wave function of Davis and Ray depended only on x.)

We use the conventions of Jauch and Rohrlich[5] for the Dirac γ matrices. We use units in which $\hbar = c = 1$; and we use the notation of Brill and Wheeler[1] with regard to $\psi^\dagger$, ψ^*, and $\nabla_\mu\psi$, although we denote the time coordinate by 0 instead of 4.

We seek a simultaneous solution of the Einstein-Dirac equations:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1}$$

and

$$\gamma^\mu \nabla_\mu \psi + m\psi = 0\,, \tag{2}$$

with

$$T_{\mu\nu} = \tfrac{1}{4}\left[\psi^\dagger \gamma_\mu \nabla_\nu \psi + \psi^\dagger \gamma_\nu \nabla_\mu \psi - (\nabla_\mu \psi^\dagger)\gamma_\nu \psi - (\nabla_\nu \psi^\dagger)\gamma_\mu \psi\right]\,. \tag{3}$$

The derivation of Eq. (3) is discussed by Brill and Wheeler,[1] as well as by Fletcher.[6]

The metric[7] is

$$ds^2 = e^{2u}(dx^2 - dt^2) + e^{2v}(dy^2 + dz^2)\,, \tag{4}$$

in which u and v are functions only of x. This is the same form as the metric used by Davis and Ray[4] (however, our u and v are different functions of x). We use the orthonormal frame defined by

$$\omega^0 = e^u dt\,, \quad \omega^1 = e^u dx\,, \quad \omega^2 = e^v dy\,, \quad \omega^3 = e^v dz\,.$$

Hence, $\omega_0 = e^{-u}\partial/\partial t$, $\omega_1 = e^{-u}\partial/\partial x$, $\omega_2 = e^{-v}\partial/\partial y$, and $\omega_3 = e^{-v}\partial/\partial z$. We seek a solution for ψ of the form $\psi_0(x)e^{-i\omega t}$, where ψ_0 is a spinor function of x, and ω is a positive real number.

II. THE DIRAC EQUATION

In this section we solve the Dirac equation for neutrinos. Before we can write down the Dirac equation, we need to know the matrices Γ_μ which appear in the formula for the covariant derivatives: $\nabla_\mu\psi = \omega_\mu(\psi) - \Gamma_\mu\psi$, where $\Gamma_\mu = -\frac{1}{4}\gamma^\alpha{}_{\nu\mu}\gamma_\alpha\gamma^\nu$ and $\gamma^\alpha{}_{\nu\mu}$ are the Ricci rotation coefficients, while γ^ν are the usual flat-space Dirac matrices. (See Ref. 3 for definitions and further discussion.) Repeated Greek indices are summed from 0 to 3. For the metric (4), the Γ_μ are

$$\begin{aligned}
\Gamma_0 &= \tfrac{1}{2}e^{-u}u'\gamma^0\gamma^1\,,\\
\Gamma_1 &= 0\,,\\
\Gamma_2 &= \tfrac{1}{2}e^{-u}v'\gamma^1\gamma^2\,,\\
\Gamma_3 &= \tfrac{1}{2}e^{-u}v'\gamma^1\gamma^3\,.
\end{aligned} \tag{5}$$

The Dirac equation becomes an equation for ψ_0:

$$\frac{d\psi_0}{dx} = \left[i\omega\gamma^1\gamma^0 - (v' + \tfrac{1}{2}u')\right]\psi_0\,. \tag{6}$$

The solution of (6) is

$$\psi_0(x) = \exp\left[i\omega\gamma^1\gamma^0 x - (v + \tfrac{1}{2}u)\right]\psi_c\,, \tag{7}$$

where ψ_c is an arbitrary constant spinor. ψ may now be written as

$$\psi = e^{-(v+u/2)}(\cos\omega x + i\gamma^1\gamma^0 \sin\omega x)\,e^{-i\omega t}\,\psi_c\,. \tag{8}$$

III. THE ENERGY-MOMENTUM TENSOR

Equation (3) can be used to calculate the components of $T_{\mu\nu}$. The results are

$$T_{00} = T_{11} = \tfrac{1}{4}e^{-u}\psi^\dagger(4i\omega\gamma^0)\psi\,, \tag{9}$$

$$T_{10} = T_{01} = \tfrac{1}{4}e^{-u}\psi^\dagger(-4i\omega\gamma^1)\psi\,, \tag{10}$$

$$T_{20} = T_{02} = \tfrac{1}{4}e^{-u}\psi^\dagger\left[-2i\omega\gamma^2 + \gamma^1\gamma^2\gamma^0(v' - u')\right]\psi\,, \tag{11}$$

$$T_{30} = T_{03} = \tfrac{1}{4}e^{-u}\psi^\dagger\left[-2i\omega\gamma^3 + \gamma^1\gamma^3\gamma^0(v' - u')\right]\psi\,. \tag{12}$$

All other components are identically zero.

IV. THE EINSTEIN EQUATIONS

In this section we use the Einstein field equations (1) to find u, v, and ψ_c. From Eq. (9), $T^{\mu}{}_{\mu}=0$. The Einstein equations become

$$R_{\mu\nu}=8\pi G T_{\mu\nu}\,. \tag{13}$$

The nonvanishing components of the Ricci tensor for the metric (4) are

$$R_{00}=e^{-2u}(u''+2u'v')\,, \tag{14}$$

$$R_{11}=-e^{-2u}[u''+2v''-2u'v'+2(v')^2]\,, \tag{15}$$

$$R_{22}=R_{33}=-e^{-2u}[v''+2(v')^2]\,. \tag{16}$$

The above results for $T_{\mu\nu}$ and $R_{\mu\nu}$, together with the Einstein field equations in the form (13), tell us that

$$T_{01}=T_{02}=T_{03}=0\,, \tag{17}$$

$$R_{00}=R_{11}\,, \tag{18}$$

$$R_{22}=R_{33}=0\,. \tag{19}$$

Equation (19) has the solution

$$v=\tfrac{1}{2}\ln(ax+b)\,, \tag{20}$$

where a and b are constants of integration. Next we solve (18) simultaneously with (14), (15), and (20) to find that

$$u=-\tfrac{1}{4}\ln(ax+b)+cx+d \tag{21}$$

in which a and b are the same constants as in (20), but c and d are new integration constants.

We now set the T_{0i} [Eqs. (10)–(12)] equal to zero [see Eq. (17)], using Eq. (8) to express ψ in terms of ψ_c. This yields

$$\psi_c^*\gamma^0\gamma^1\psi_c=0\,, \tag{22}$$

$$\psi_c^*[\gamma^0\gamma^2A(x)+i\gamma^1\gamma^2B(x)]\psi_c=0\,, \tag{23}$$

$$\psi_c^*[\gamma^0\gamma^3A(x)+i\gamma^1\gamma^3B(x)]\psi_c=0\,, \tag{24}$$

in which

$$A=2\omega\cos 2\omega x+(v'-u')\sin 2\omega x \tag{25}$$

and

$$B=2\omega\sin 2\omega x-(v'-u')\cos 2\omega x\,. \tag{26}$$

Equations (23) and (24) will certainly be satisfied[8] if

$$\psi_c^*\gamma^0\gamma^2\psi_c=0\,, \tag{27}$$

$$\psi_c^*\gamma^1\gamma^2\psi_c=0\,, \tag{28}$$

$$\psi_c^*\gamma^0\gamma^3\psi_c=0\,, \tag{29}$$

and

$$\psi_c^*\gamma^1\gamma^3\psi_c=0\,. \tag{30}$$

We want a simultaneous solution of the five equations (27) through (30) and (22). The solutions are

$$\psi_c=\begin{pmatrix}s\\s\\q\\q\end{pmatrix}e^{i\varphi}\quad\text{and}\quad\psi_c=\begin{pmatrix}s\\-s\\q\\-q\end{pmatrix}e^{i\varphi}\,, \tag{31}$$

where s, q, and φ are arbitrary real numbers. These solutions satisfy the Einstein-Dirac equations provided that the constraint $8\pi G\omega\psi_c^*\psi_c=ac$ (which comes from $R_{00}=8\pi GT_{00}$ and $R_{11}=8\pi GT_{11}$) is met.

To summarize, we have found the solutions

$$\psi=e^{-(v+u/2)}(\cos\omega x+i\gamma^1\gamma^0\sin\omega x)\times e^{-i\omega t}\begin{pmatrix}s\\\pm s\\q\\\pm q\end{pmatrix}e^{i\varphi}\,, \tag{32}$$

with

$$u=-\tfrac{1}{4}\ln(ax+b)+cx+d \tag{33}$$

and

$$v=\tfrac{1}{2}\ln(ax+b)\,. \tag{34}$$

All components of $T_{\mu\nu}$ are zero except T_{00} and T_{11}, which are equal.

V. PHYSICAL INTERPRETATION

What physical situation does our solution (32) represent, and why should T_{01} be zero? To answer these questions, ψ must be rewritten in a more enlightening form, with x dependence of the form $e^{i\omega x}$ and $e^{-i\omega x}$. The two solutions become

$$\psi=e^{-(v+u/2)}e^{-i\omega t}re^{i\varphi}\times\left[\begin{pmatrix}1\\1\\-i\\-i\end{pmatrix}e^{i\omega x}e^{i\theta}+\begin{pmatrix}1\\1\\i\\i\end{pmatrix}e^{-i\omega x}e^{-i\theta}\right] \tag{35}$$

and

$$\psi=e^{-(v+u/2)}e^{-i\omega t}re^{i\varphi}\times\left[\begin{pmatrix}1\\-1\\i\\-i\end{pmatrix}e^{i\omega x}e^{i\theta}+\begin{pmatrix}1\\-1\\-i\\i\end{pmatrix}e^{-i\omega x}e^{-i\theta}\right], \tag{36}$$

in which r, φ, and θ are real. When ψ is written in this form, Eq. (9) becomes

$$T_{00}=T_{11}=8\omega r^2 e^{-(2v+2u)}, \tag{37}$$

and the equations $R_{00}=8\pi G T_{00}$ and $R_{11}=8\pi G T_{11}$ are satisfied if

$$ac=64\pi G\omega r^2. \tag{38}$$

If we replace u and v by zero in (35) and (36) (the "flat-spacetime limit"), we can compare these wave functions with the usual plane-wave solution to the Dirac equation in flat spacetime, which may be written in the form

$$\begin{bmatrix} C \\ D \\ \left(\frac{-ip^3}{p^0+m}\right)C-\frac{(ip^1+p^2)}{p^0+m}D \\ \frac{-i(p^1+ip^2)}{p^0+m}C+\left(\frac{ip^3}{p^0+m}\right)D \end{bmatrix} e^{ip_\mu x^\mu}, \tag{39}$$

where C and D are arbitrary real or complex numbers. We will denote the "flat-spacetime limits" of (35) and (36) by (35′) and (36′).

If we set $m=0$, $p^0=\omega$, and $p^2=p^3=0$, (39) becomes

$$\begin{bmatrix} C \\ D \\ \frac{-ip^1}{\omega}D \\ \frac{-ip^1}{\omega}C \end{bmatrix} e^{-i\omega t+ip^1x}. \tag{40}$$

Then, setting $p^1=\omega$ and $C=D$ in (40) yields the $e^{i\omega x}$ term in (35′), while setting $p^1=-\omega$ and $C=D$ yields the $e^{-i\omega x}$ term in (35′). Similarly, we can obtain the two terms of (36′) by setting $p^1=\pm\omega$ and $C=-D$. Thus, each of our two solutions reduces, in the limit of flat spacetime, to a linear combination of two beams of neutrinos, traveling in the $+x$ and $-x$ directions.

In flat spacetime, the wave function (39) has the energy-momentum tensor

$$T_{\mu\nu}=2(|C|^2+|D|^2)\frac{p_\mu p_\nu}{p^0+m}. \tag{41}$$

[This was calculated from Eq. (3).] We cannot apply (41) directly to a wave function, such as (35′), which is a superposition of terms with different four-momenta, because (3) is bilinear—not linear—in ψ; there may be cross terms. However, it can be shown that for both (35′) and (36′), in flat spacetime, the cross terms are zero, for all components of $T_{\mu\nu}$. This means that we may apply (41) separately to the $e^{i\omega x}$ and $e^{-i\omega x}$ terms and then add the results to obtain the total $T_{\mu\nu}$. Furthermore, in both (35′) and (36′) the $e^{i\omega x}$ and $e^{-i\omega x}$ terms have equal values of $|C|^2+|D|^2$. From the form of (41), we see that the contributions to T_{01} from the two terms have opposite signs and cancel, whereas for T_{00} and T_{11} the two contributions have the *same* sign.

Another interesting quantity is the probability flux, $S^\mu=i\psi^\dagger\gamma^\mu\psi$. For the flat-spacetime wave function (39),

$$S^\mu=(|C|^2+|D|^2)\frac{2p^\mu}{p^0+m}. \tag{42}$$

For the superposition of two beams, (35′) or (36′), moving in flat spacetime with opposite momentum, the S^i are zero but S^0 is positive. Similarly, for the curved-spacetime wave function (35) or (36), all components of S^μ are zero except S^0 (which is $\psi^*\psi$),

$$S^0=8r^2e^{-(2v+u)}. \tag{43}$$

To acquire a still better understanding of our solutions, we may obtain information about the helicities of the neutrino beams, using the helicity projection operators $\frac{1}{2}(1+i\gamma_5)$ and $\frac{1}{2}(1-i\gamma_5)$. We find that in one solution the beam traveling in the x direction is entirely positive helicity and the beam traveling in the $-x$ direction is completely negative helicity; in the other solution the situation is reversed. The experimentalists have observed only one helicity, so it may be that neither of the solutions presented here is physically realizable.

VI. BOUND STATES?

Another interesting question is whether the wave function, (35) or (36), represents a bound state. How do ψ, $T^{\mu\nu}$, and S^μ behave as $x\to\pm\infty$?

The metric is

$$ds^2=(ax+b)^{-1/2}e^{2cx}e^{2d}(dx^2-dt^2)+(ax+b)(dy^2+dz^2). \tag{44}$$

Because of the square root, we must have $x>-b/a$ if $a>0$, $x<-b/a$ if $a<0$. At $x=-b/a$, the coordinate system has a singularity. However, we still do not know if the actual spacetime is singular—it may be possible to extend the spacetime beyond $x=-b/a$ by using a different coordinate system. To investigate this possibility, we calculate $R^\alpha{}_{\beta\mu\nu}$:

$$R^0{}_{101}=-u''e^{-2u}=-\tfrac{1}{4}a^2(ax+b)^{-3/2}e^{-2cx-2d}, \tag{45}$$

$$R^0{}_{202}=R^0{}_{303}=-u'v'e^{-2u}=[\tfrac{1}{8}a^2(ax+b)^{-3/2}-\tfrac{1}{2}ac(ax+b)^{-1/2}]e^{-2cx-2d}, \tag{46}$$

$$R^1{}_{212} = R^1{}_{313} = -[v'' + v'(v-u)']e^{-2u} = [\tfrac{1}{8}a^2(ax+b)^{-3/2} + \tfrac{1}{2}ac(ax+b)^{-1/2}]e^{-2cx-2d}, \tag{47}$$

$$R^2{}_{323} = -(v')^2 e^{-2u} = -\tfrac{1}{4}a^2(ax+b)^{-3/2}e^{-2cx-2d}. \tag{48}$$

All other components either are zero or can be obtained from the symmetries of the Riemann tensor.

Now consider the invariant quantity $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. This is found to be

$$e^{-4cx-4d}[\tfrac{3}{4}a^4(ax+b)^{-3} + 4a^2c^2(ax+b)^{-1}],$$

which is infinite when $ax+b=0$. Hence, the spacetime cannot be extended beyond $x=-b/a$.

The constraint $ac = 64\pi G\omega r^2$ forces a and c to have the same sign, since $\omega>0$. Now, ψ contains the exponential factor $e^{-(v+u/2)}$, while T_{00} and T_{11} contain $e^{-(2v+2u)}$ and S^0 contains $e^{-(2v+u)}$. But $u=-\frac{1}{4}\ln(ax+b)+cx+d$, $v=\frac{1}{2}\ln(ax+b)$, so

$$\psi \sim (ax+b)^{-3/8}e^{-(c/2)x},$$

$$T_{00} \text{ and } T_{11} \sim (ax+b)^{-1/2}e^{-2cx},$$

$$S^0 \sim (ax+b)^{-3/4}e^{-cx}.$$

If $a>0$, then $c>0$ and the exponential factors decrease as $x\to\infty$, while if $a<0$ then $c<0$ and the exponential factors decrease as $x\to-\infty$. Both cases are analogous to bound states.

It should also be noted that all nonzero components of $R^{\alpha}{}_{\beta\mu\nu}$ also approach zero exponentially as $|x|\to\infty$; spacetime becomes flat at large distances.

VII. SPECIAL CASES OF THE SOLUTION

Up to this point we have not discussed the possibility of a, c, ω, or r being zero. Even if some of these quantities are zero, our solutions are still valid,[9] provided that $ac = 64\pi G\omega r^2$.

Recall that ψ is proportional to r [Eqs. (35) and (36)], T_{00} and T_{11} are proportional to ωr^2 [Eq. (37)], and S^0 is proportional to r^2 [Eq. (43)].

If $c=0$ but $a\neq 0$, the metric is

$$(ax+b)^{-1/2}e^{2d}(dx^2-dt^2)+(ax+b)(dy^2+dz^2). \tag{49}$$

According to Eqs. (45) through (48), spacetime is still curved. [By a simple coordinate transformation,

$$x=e^{-d}x''+\frac{1-b}{a}, \quad y=y'',$$

$$t=e^{-d}t'', \quad z=z'',$$

the metric (49) becomes the metric of Davis and Ray,[4]

$$(kx''+1)^{-1/2}(dx''^2-dt''^2)+(kx''+1)(dy''^2+dz''^2),$$

after we set $k=ae^{-d}$.]

Because $c=0$, either r or ω must be zero. If $r=0$, then ψ is zero, so we have a vacuum solution of the Einstein field equations. If $r\neq 0$, but $\omega=0$, then by Eq. (32),

$$\psi=(ax+b)^{-3/8}\begin{pmatrix} s\\ \pm s\\ q\\ \pm q\end{pmatrix}e^{i\varphi}e^{-d/2}. \tag{50}$$

Neither the wave function ψ nor the neutrino probability density S^0 is zero, but $T_{\mu\nu}$ is zero: We have "ghost neutrinos" in curved spacetime, very similar to the "ghost neutrinos" discussed by Davis and Ray.[4] But we understand why these "ghost neutrinos" make no contribution to $T_{\mu\nu}$—they have zero energy.

If instead $a=0$, then spacetime is flat, since by Eqs. (45) through (48), the $R^{\alpha}{}_{\beta\mu\nu}$ are all zero. The metric is

$$b^{-1/2}e^{2cx}e^{2d}(dx^2-dt^2)+b(dy^2+dz^2). \tag{51}$$

If $r=0$, then we have the uninteresting case of empty flat spacetime in a non-Cartesian coordinate system. If $\omega=0$ but $r\neq 0$, then we again use Eq. (32) to find

$$\psi=b^{-3/8}e^{-d/2}e^{-(c/2)x}e^{i\varphi}\begin{pmatrix} s\\ \pm s\\ q\\ \pm q\end{pmatrix}. \tag{52}$$

The wave function (52) in the metric (51) is an example of "ghost neutrinos" in flat spacetime. It should be possible to perform a transformation to Cartesian coordinates [to determine whether (52) represents the constant spinor solution to the Dirac equation in Cartesian coordinates, or something more interesting]. The following coordinate transformation accomplishes this:

$$t'=b^{-1/4}e^{d}c^{-1}e^{cx}\sinh ct, \tag{53}$$

$$x'=b^{-1/4}e^{d}c^{-1}e^{cx}\cosh ct, \tag{54}$$

$$y'=b^{1/2}y, \tag{55}$$

$$z'=b^{1/2}z. \tag{56}$$

Here, the primed coordinates are Cartesian coordinates.

Next, we transform the wave function ψ, given by Eq. (52) in the unprimed coordinate system. To find the necessary spacetime-dependent Lorentz transformation of the basis vectors, we apply the orthonormal basis vectors ω_μ to the primed coordinates written as functions of the unprimed coordinates in Eqs. (53) through (56). Thus,

$$\omega_0(t')=b^{1/4}e^{-cx}e^{-d}\frac{\partial}{\partial t}(b^{-1/4}e^{d}c^{-1}e^{cx}\sinh ct)$$
$$=\cosh ct\,,$$
$$\omega_1(t')=\sinh ct\,,$$
$$\omega_0(x')=\sinh ct\,,$$
$$\omega_1(x')=\cosh ct\,,$$
$$\omega_2(y')=1\,,$$
$$\omega_3(z')=1\,,$$

and all others are zero. The matrix of the $\omega_\mu(x'^\nu)$ represents the Lorentz transformation of the basis vectors. The spinor transformation corresponding to this Lorentz transformation[10] is

$$\psi'=(\cosh\tfrac{1}{2}ct-\gamma^0\gamma^1\sinh\tfrac{1}{2}ct)\psi\,, \tag{57}$$

where ψ is given by (52). The result is

$$\psi'=b^{-3/8}e^{-d/2}e^{i\varphi}e^{-(c/2)x}\begin{bmatrix} s\cosh\frac{1}{2}ct\pm iq\sinh\frac{1}{2}ct\\ \pm s\cosh\frac{1}{2}ct+iq\sinh\frac{1}{2}ct\\ \mp is\sinh\frac{1}{2}ct+q\cosh\frac{1}{2}ct\\ -is\sinh\frac{1}{2}ct\pm q\cosh\frac{1}{2}ct\end{bmatrix}. \tag{58}$$

Although Eq. (58) should be written in terms of the primed coordinates, it is already evident that our "ghost neutrino" solution in flat spacetime is not merely a constant spinor in Cartesian coordinates. By inverting the equations (53) through (56), we obtain the following (which is valid for $x'^2-t'^2>0$, $x'>0$, $t'>0$, $c>0$):

$$\psi'(t',x',y',z')=(2bc)^{-1/2}e^{i\varphi}\left[\frac{s}{(x'^2-t'^2)^{1/2}}\begin{pmatrix}[x'+(x'^2-t'^2)^{1/2}]^{1/2}\\ \pm[x'+(x'^2-t'^2)^{1/2}]^{1/2}\\ \mp i[x'-(x'^2-t'^2)^{1/2}]^{1/2}\\ -i[x'-(x'^2-t'^2)^{1/2}]^{1/2}\end{pmatrix}+\frac{q}{(x'^2-t'^2)^{1/2}}\begin{pmatrix}\pm i[x'-(x'^2-t'^2)^{1/2}]^{1/2}\\ i[x'-(x'^2-t'^2)^{1/2}]^{1/2}\\ [x'+(x'^2-t'^2)^{1/2}]^{1/2}\\ \pm[x'+(x'^2-t'^2)^{1/2}]^{1/2}\end{pmatrix}\right]. \tag{59}$$

VIII. GHOST NEUTRINOS

Davis and Ray[4] found that with the metric form (4) and $\psi=\psi(x)$, the only possible solutions (with $\psi\neq 0$) were "ghost neutrino" solutions: $T_{\mu\nu}=0$. We find that if $\psi=\psi_0(x)e^{-i\omega t}$ and $\omega>0$, we *never* have "ghost" solutions (even if we do not require that the Einstein equations be satisfied), since by Eq. (9), $T_{00}=\omega e^{-u}\psi^*\psi$.

If we consider a slightly more general wave function (but with the same metric form), $\psi=\psi_0(x,y,z)e^{-i\omega t}$, an identical calculation shows that T_{00} is still $\omega e^{-u}\psi^*\psi$. As long as $\omega>0$, we never have "ghost neutrinos."

We repeat that in Sec. VII, we have given a solution with $\omega=0$ (zero-energy neutrinos) and $c=0$, with a nonvanishing "ghost neutrino" wave function having the same dependence on x as the "ghost neutrino" wave function of Davis and Ray. Similarly, the current vector S^μ is not zero, and our metric is the same. Furthermore, in the zero-energy neutrino case we have given an example of "ghost neutrinos" even in flat space.

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[8]That Eqs. (27) through (30) are necessary as well as sufficient for (23) and (24) to be satisfied can be seen from the following argument (note, we are *excluding* the case $\omega = 0$):
(a) There is no (real) value of x for which A and B are both zero. This is because $A = 0$ when $\cot 2\omega x = -(v' - u')/2\omega$, while $B = 0$ when $\cot 2\omega x = 2\omega/(v' - u')$.
(b) There are an infinite number of values of x for which $A = 0$, and an infinite number for which $B = 0$, in any infinite range of x (such as $x > -b/a$ or $x < -b/a$). This is most easily shown by solving $A = 0$ and $B = 0$ graphically: we plot $y = \cot 2\omega x$ and $y = (1/2\omega)[\frac{3}{4}a/(ax+b) - c]$ on the same graph, and the intersections are the values of x for which $A(x) = 0$. A similar argument applies to B.
(c) Thus, in any infinite range of x, there are values of x for which $B = 0$ but $A \neq 0$. Substituting such a value into (23) yields (27) and (29). By interchanging A and B in this argument, we obtain (28) and (30).

[9]However, if $\omega = 0$ we have not found all possible solutions. In particular, we do not obtain the solution of Davis and Ray as a special case. This is because there are fewer constraints on the solutions when $\omega = 0$; for instance, T_{01} is automatically zero [see Eq. (9)] and does not have to be *set* equal to zero.

[10]See, for example, the discussion of the Lorentz covariance of the Dirac equation in flat spacetime by J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1967), Sec. 3-4. However, Sakurai uses a different convention for the γ matrices, and he uses ict for the time coordinate.