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Convergent polynomial expansion and scaling in diffraction scattering I. *pp* scattering

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Using Mandelstam analyticity of the s and $\cos\theta$ planes and conformal mapping, a variable χ is constructed which has the potentialities of reproducing Regge behavior and/or some known scaling variables. The role of the physical region in the mapped plane for the optimized polynomial expansion (OPE) is emphasized. Ambiguities in using the OPE in terms of Laguerre polynomials at finite energies are pointed out. However, at finite energies there exists a convergent polynomial expansion (CPE) for which the nature of polynomials and the rate of convergence vary with energy. The first term in the expansion gives a good fit to the world data on forward slopes for pp scattering for all energies with effective shapes of spectral function, but yields a good fit to the high-energy data for $s > 35$ GeV^2 with the theoretical boundaries. The possible existence of a scaling function at asymptotic energies as a series in Laguerre polynomials in the new variable χ is pointed out. Available high-energy data on the pp cross-section ratio for $p_{\mathrm{lab}} \geq 50$ GeV/c and all angles exhibit scaling in this variable. It is found that at high energies scaling occurs even for larger-$|t|$ data lying well outside the diffraction peak. The implication of this type of scaling in the data analysis at high energies using OPE is pointed out. The energy dependence of dip position at high energies is predicted to be $|t_d(s)| = 4m_\pi^2 \{\sinh[(4.35 \pm 0.05)/ 4m_\pi^2 b(s)]^{1/2}\}^2$, which is in very good agreement with the existing data.

I. INTRODUCTION

Almost six years ago Auberson, Kinoshita, and Martin[1] proved, using results of axiomatic field theory, that for spinless-particle scattering amplitudes which qualitatively saturate the Froissart-Martin bound, the amplitude ratio $T(s,t)/T(s,0)$ must approach an entire function of variable $t(\ln s)^2$ for $s \to \infty$ and $|t| < 4m_\pi^2$. As early as 1970, Singh and Roy[2] eastablished that an upper bound for the amplitude ratio of the absorptive part scales in the variable $t\sigma_{\mathrm{tot}}^2/\sigma_{\mathrm{el}}$ for small $|t|$. Thus, scaling of the diffraction peak is obtained if the unitarity bound is saturated. Evidence for this type of scaling has been investigated by Divakaran and Gangal,[3] who have noted that the approach to scaling in the experimental data slows down with increasing value of the variable. Recently, scaling in a similar variable has been hypothesized from geometrical considerations at high energy and using the group-contraction method.[4] Cornille and Martin[5] have examined the rigorous foundations of scaling properties of the cross-section ratio

$$f(s,t) = \frac{d\sigma}{dt}(s,t) \Big/ \frac{d\sigma}{dt}(s,0) \tag{1}$$

in the context of diffraction scattering. They have proved scaling in the variable $t\sigma_{\mathrm{tot}}^2/\sigma_{\mathrm{el}}$ for the case when $\sigma_{\mathrm{tot}}/(\ln s)^2 \to 0$ in the asymptotic region. It has been remarked[5] that the models which saturate the Fröissart-Martin bound automatically possess these properties if they satisfy s-channel unitarity and have a good analyticity in t. Cornille[6] has examined different conditions of scaling and shown that one of the scaling variables at asymptotic energies can be $tb(s)$, $b(s)$ being the slope parameter in the forward direction. He has defined a class of scaling functions in which are included sums of powers, exponentials, and classical orthogonal polynomials with positive coefficients. Auberson and Roy[7] have shown that under certain conditions the scaling variable at asymptotic energies can be $tb(s)$. Experimental data on $\pi^\pm p$, $K^\pm p$, and $p^\pm p$ elastic scattering appear to support the geometrical-scaling hypothesis.[8-10] In particular, pp scattering data from laboratory momentum 501 to 1500 GeV/c exhibit geometrical scaling[9,10] in the variable $t\sigma_{\mathrm{tot}}$. Recent CERN-ISR data appear to spoil[10] the simple scaling picture in the Krisch variable.[11] Krisch has modified the variable[12] in which, although scaling is exhibited by data near forward angles,[13] the energy dependence is not removed for larger values of $|t|$. One main criticism against the Krisch-type representation for the amplitude is that it violates the Cerrulus-Martin lower bound.[14] All the results on scaling derived so far can be broadly classified into two classes: exact results based upon properties of the S matrix derived from axiomatic field theory (Refs. 1–3 and 5–7), and model-dependent results based on geometrical considerations (Refs. 4, 10, and 11–13).

The optimized polynomial expansion (OPE) for scattering amplitudes[15,16] and form factors has found its successful application[17] in many areas of particle physics. Convergent expansions have been developed[18] for the virtual Compton scattering amplitude and to exhibit Bjorken scaling behavior, including scaling violations[19] as observed in the case of deep-inelastic electron-nucleon scattering. It is tempting to investigate whether scaling in diffraction scattering can be exhibited by means of OPE. Differential-scattering-cross-section data have been parametrized by this method for phase-shift analysis.[20] The polynomial expansion generally involves unknown parameters which vary with energy. Thus the differential cross section at each energy is fitted with a new set of parameters. But if it can be shown that the data exhibit scaling when plotted against a suitably chosen conformally mapped variable, the parameters, and hence the scaling function in terms of corresponding orthogonal polynomials, are known once a set of parameters are determined by fitting the data at any single energy in the scaling region. Thus, demonstration of scaling of the data in terms of a suitable conformally mapped variable has importance from considerations of the economic use of computer time. In the present work we show that if analyticity of the s plane is exploited along with that of the $\cos\theta$ plane by conformal mapping and OPE in a specific manner, it is possible for pp scattering to construct a variable $\chi(s,t)$ which has the potentialities of reproducing Regge behavior and/or known scaling variables in the appropriate kinematical region. The first term in OPE in terms of this variable gives a good fit to the data on forward slopes even at ISR energies. The unknown parameters in the variable are thus determined by fitting the slope-parameter data. At high energies and all angles the variable is $b(s)z$, where $b(s)$ is the slope parameter and z is the parabolic variable proposed earlier. For $|t| \ll t_R$, where t_R is the boundary of spectral function ρ_{st} (ρ_{su}), the variable becomes $tb(s)$. The role of the physical region along with that of the figure of convergence in OPE has been emphasized. It is pointed out that Laguerre-polynomial expansion for OPE can be used only at asymptotic energies. At finite energies one can use a sequence of polynomials for which the convergence of the series is not maximum and the nature of such polynomials varies with energy. It is argued that a Laguerre-polynomial expansion in terms of χ may lead to a scaling function only at high energies if the coefficients in the expansion do not depend upon energy. Experimental data on $f(s,t)$ for pp scattering support this view. When data on $f(s,t)$ are plotted against χ for different energies and angles, they come closer and closer to lying on a universal curve at higher energies, showing that χ is a good scaling variable. In particular, we find that all the small-angle data with $|t| \leqslant 1.2$ GeV^2 and $P_{\rm lab} \geqslant 19.2$ GeV/c lie on the same scaling curve. Available data at all angles and for $P_{\rm lab} \geqslant 50$ GeV/c fall on the same scaling curve. To examine whether scaling is exhibited by this variable for larger angles well outside the diffraction peak region, we have used recent experimental data at $P_{\rm lab} = 200$ GeV/c and $P_{\rm lab} = 1500$ GeV/c for our analysis. Surprisingly, we find that all the available data, including those near the regions of the prominent dip and secondary maximum, and for larger values of $|t|$ with $|t| \leqslant 10$ GeV^2, lie on the same scaling curve. Having thus explicitly demonstrated scaling in OPE, the energy dependence of the dip position at high energies is predicted to be

$$|t_d(s)| = 4m_\pi^2\{\sinh[\chi_0/4m_\pi^2 b(s)]^{1/2}\}^2, \tag{2}$$

with

$$\chi_0 = 4.35 \pm 0.05\,,$$

where $|t_d(s)|$ is the dip position in the $d\sigma/dt$ vs $|t|$ plot. Such a prediction is found to be in good agreement with the available experimental data. At various stages of development of this work we point out several limitations of the present approach. Scaling in the geometrical models of Krisch[11,12] and Hansen and Krisch[13] has been hypothesized and geometrical scaling for collision of extended hadrons has been assumed by Dias de Deus.[9] Scaling variables in other cases[2,3,5-7] have been proved to exist from more rigorous foundations. In the present work, scaling in the variable χ is hypothesized *a priori* from the uniqueness of OPE at asymptotic energies and the experimental data support such a hypothesis.

In Sec. II we emphasize the correct physical region along with the correct figure of convergence in OPE. The need for using s-plane analyticity has been pointed out and a new variable has been constructed. In Sec. III we parametrize the data on forward slopes and plot the data on the cross-section ratio against the new scaling variable. The agreement of predictions of the energy dependence of dip positions with experimental data are also studied in this section. In Sec. IV we discuss the results and limitations of this approach.

II. CONFORMAL MAPPING WITH s- AND $\cos\theta$-PLANE ANALYTICITY

In this section we emphasize the correct physical region in the mapped plane for OPE. We point out the need for using s-plane analyticity and develop a new variable to describe diffraction scattering. In developing representations for differential cross sections in this section we assume that scattering near forward angles is due to its absorptive part alone. With this assumption, the contribution due to pion poles has been neglected. There is a host of papers[1,2,5-7,21] which contain such an assumption. It has been shown that the unitarity upper bound for the absorptive part of the scattering amplitude derived by Singh and Roy[2] saturates the high-energy data near forward angles[2] for πN and pp scattering.[22] Experimental measurements at high energies show that the real part of the amplitude is small near forward angles for NN and πN scattering. Theoretically, although pion pole contribution is zero in the forward direction, it is not absent away from the forward direction and the interference between the pole and the cut contributions may significantly affect the slope parameter. Although the data analysis in the previous[23,24] and present papers has been carried out by neglecting the pion-pole contribution, it is important to investigate how it modifies the fits to the slope-parameter data. However, we note that the simple picture of scaling presented subsequently in this paper will be perhaps difficult to realize if the pion-pole contribution is explicitly retained. From the point of neglecting the pole contribution, our representations do not possess the correct analyticity properties.

Using Mandelstam analyticity in the $x=\cos\theta$ plane and the techniques of OPE, a model of high-energy scattering of hadrons was proposed[23] which reproduced well-known phenomenological fits within appropriate limits. The formula proposed for the scattering amplitude in terms of a conformal-mapping variable z possesses the necessary $t \longleftrightarrow u$ symmetry but does not violate the Cerrulus-Martin[14] lower bound. At high energies and near the forward directions, scattering is almost pure imaginary. Therefore, the experimental data at forward angles can serve as a good guide in obtaining information about the imaginary part at nearby unphysical regions. With this idea, conformal mapping was developed[24] for the asymmetric cut plane of analyticity and a formula for slope parameters was developed which could successfully correlate experimental data on slopes of forward peaks with equations to boundaries of spectral functions. It was found that the formula could account for the shrinkage of forward peaks for several processes yielding constant values of slope at ISR energies. From fits to the data on slopes for various processes, effective shapes of spectral functions were computed. Experimental measurements[25] on slope parameters for high-energy pp scattering show a $\sim \ln s$ type of increase consistent with the exchange of a Pomeron of slope $\alpha'(0)=0.28\ \mathrm{GeV}^{-2}$ in the t channel. This feature is absent in the model proposed earlier.[23,24] In this section we show by using the analyticity property in the s plane, along with that of the $\cos\theta$ plane by means of OPE, that it is possible to construct a variable in a simple way which has the potentialities of saturating Regge behavior and reproducing some known scaling variables in the appropriate kinematical region.

Before proceeding further, we discuss certain aspects of earlier works[23,24] which are relevant in the present context. In Refs. 21 and 22, conformal mappings of the cut $x=\cos\theta$ plane onto the z plane were used such that the cuts were mapped onto the branches of a parabola with the focus at the origin and the scattering amplitude (equivalently the differential cross section) was expanded as a series in Laguerre polynomials with an exponential weight factor, the correct expression for which is

$$\frac{d\sigma}{dt}=\exp(-\alpha z)\sum_n a_n(s)L_n(2\alpha z). \tag{3}$$

The presence of the number α, which determines the size of the parabola in the z plane, was omitted from the above expansion in earlier works. We will see subsequently that the energy dependence of α is very crucial for the scaling of $f(s,t)$. In Ref. 24 the variable z has been constructed through a series of successive transformations of the asymmetric cut x plane

$$\begin{aligned} y&=2x+x_- - x_+,\\ y_{\max}&=2+x_- - x_+,\\ y_{\rm in}&=x_- + x_+,\\ w&=\frac{y_{\max}^2-y^2}{y_{\rm in}^2-y_{\max}^2}, \end{aligned} \tag{4}$$

and

$$z=(\sinh^{-1}\sqrt{w})^2, \tag{5}$$

where $-x_-$ (x_+) is the start of the left- (right-) hand cut in the x plane. For the symmetric cut plane of analyticity, as in the case for pp scattering, the variable (4) reduces to the one proposed in Ref. 23 with

$$w=\frac{1-x^2}{x_+^2-1}. \tag{6}$$

It is well known[26] that the correct physical region for Laguerre-polynomial expansion is the entire right half of the real axis in the z plane. In the conformal transformations discussed above, the image of the physical region in the z plane at any finite energy is only a part of the right half of the real axis, the other part being the image of the imaginary axis of the x plane. The image of the physical region spreads the entire right half of the real axis like $\sim(\ln s)^2$ as $s\to\infty$. Thus the physical region appropriate for Laguerre-polynomial expansion is achieved only at asymptotic energies, although the parabolic figure of convergence is achieved for all energies. The length of the physical region and the weight function decide the nature of the polynomial.[26] Since the length of the image of the physical region in the z plane changes with energy, the orthogonal polynomials as decided by the length of the image of the physical region in the mapped plane are not necessarily the Laguerre polynomials at finite energies. Now we ask the question as to what happens if we write the series (3) for all energies with Laguerre polynomials as has been done earlier.[23,24] In that case, a_n's are decided only if $f(s,t)$ is known on the entire right half of the real axis, a part of which is the image of the imaginary axis of the y plane on which $f(s,t)$ is not known. Thus, for finite energies the expansion (3) in terms of Laguerre polynomials is ambigous for lack of the correct physical region.

For any given finite energy one can define[27] a sequence of orthogonal polynomials $\{p_n(z)\}$ with the knowledge of the length of the physical region in the z plane and the same exponential weight function $\exp(-\alpha z)$. Since the length of the physical region varies with energy, the nature of the polynomials, by definition, is not the same for all finite energies, although the weight function has been chosen[27] to be the same. It is to be noted that the exponential weight function occurs naturally from the theory of orthogonal polynomials[26] for the OPE in terms of Laguerre polynomials with a semi-infinite physical region, but the construction[27] of orthogonal polynomials $\{p_n(z)\}$ for finite energies with the same weight function is a choice. As we will presently see, such a choice gives rise to an exponential fit to the diffraction peak near forward angles. For finite energies one can write, instead of (3),

$$\frac{d\sigma}{dt}=\exp(-\alpha z)\sum_n c_n(s)p_n(2\alpha z), \tag{7}$$

where $\{p_n(z)\}$ is a sequence of orthogonal polynomials whose nature depends upon energy. Since the nature of the polynomials $\{p_n(z)\}$ changes, the corresponding figure of convergence also changes with energy, the figures of convergence for finite energies may not necessarily coincide with the parabola, and the interior of the figure of convergence at finite energies may not contain the whole image of the cut plane. Thus the series (7) may not be maximally convergent for all energies, but only at asymptotic energies when the correct physical region is achieved $\{p_n(z)\}\to\{L_n(z)\}$, and the series (3) is maximally convergent. Since we are considering the convergence of (7) in the mapped plane, the area enclosed by a figure of convergence at any finite energy is likely to be the image of a larger portion of the x plane than the smaller area enclosed by the Lehmann ellipse in it. Therefore we suppose that for any finite energy the convergence of series (7) is faster than the convergence of the partial wave expansion in Legendre polynomials in x. For this reason we call (7) a convergent polynomial expansion (CPE). To summarize the statements made above, expansion in terms of Laguerre polynomials in z is an asymptotic expansion and there is ambiguity in using an asymptotic expansion for finite energies. The discrepancy from the correct physical region at finite energies leads to CPE (7) where the rate of convergence of the series varies with energy. The sequence of polynomials $\{p_n(z)\}$ is neither unique nor maximally convergent.

Besides achieving accelerated convergence by mapping the cut plane into the interior of the figure of convergence, the convergence of polynomial expansions (3) and (7) is further accelerated near forward angles because for $|t|\ll t_R$ and energies such that $(4p^2+t_L-\Delta/s)\gg|(t_L-t)|$, $z\simeq -t/t_R$. In this kinematical region, only the first term in the convergent expansion is important and (3) or (7) yields an exponential fit to the diffraction scattering of the form

$$\frac{d\sigma}{dt}=Ae^{bt}.$$

Thus the exponential fit observed even at relatively lower energies in diffraction scattering is accounted for by (7). In earlier works,[21,22] convergent expansion (3) has been used[22] to account for the shrinkage of the diffraction peak for pp, $\bar{p}p$, and $K^{\pm}p$ scattering for all energy ranges yielding a constant slope for $s\to\infty$. It can be argued that the conformal transformation of the type (4), when applied to the unsymmetrical cut plane of analyticity, introduces spurious cuts in the mapped plane by folding a part of the physical region on top of the other part. For example, the double sheet structure in the y plane (or, equivalently, in the x plane) is folded together in going from the y to the w plane through the transformation defined by Eqs. (4) and (5). Such a transformation introduces three spurious branch points in the z plane at $z_1=\{\ln[(4p^2+t_L-\Delta/s)/t_R]\}^2/4$, $z_2=\infty+i\infty$, and $z_3=\infty-i\infty$, one of which lies on the physical region and two others lie at the ends of the branches of the parabola. But for the symmetrical cut x plane this transformation, which is the same as (6) does not introduce any spruious branch point. Considered as a function of x, the representations (3) or (7) do not possess any other branch points except at $x=x_+$ and $x=-x_-$ for the unsymmetrical cut plane. Thus the conformal transformation does not introduce extra branch points to affect analyticity property, but the presence of these three spruious branch points affects the convergence of the polynomial expansion in the z plane. At present we are interested in pp scattering, in which case the cut x plane is symmetric, thus the question of the spurious singularities does not arise. Taking only the first term in the expansion from (3) or (7), the expression for the forward slope is obtained as

$$b(s) = -\alpha \frac{dz}{dt}\bigg| = \frac{\alpha}{t_R}\left(1 - \frac{t_R}{4p^2 + t_R}\right), \tag{8}$$

where

$$t_R = 4m_\pi^2 + \frac{4\lambda^4}{s - 4m^2} \tag{9}$$

is the effective shape of the spectral function boundary ρ_{st} (ρ_{su}). For $\lambda = m_\pi$, Eq. (9) describes the theoretical boundary.

So far the cut structure of the x plane only has been exploited and the formula (7) yields a constant value of the slope parameter at ISR energies. The scattering amplitude is a function of two independent variables, which we choose as s and $\cos\theta$. Experimentally, the slope parameter is measured by fitting the fixed-energy and small-but-nonzero-angle data on the differential cross section and extrapolating the fit to the forward direction. The slope of the forward peak is thus given at various energies from the analysis of data at different fixed energies. The scattering of hadrons is due to the nature of hitherto unknown forces, which are represented by cuts in the planes of both the variables. Since the forward slope is determined from fits to the differential-cross-section data at various angles, it has been reasonable to exploit x-plane analyticity. But since the slope parameter is also a function of s, the singularities of the absorptive part of the amplitude in the s plane might be playing a dominant role in accounting for a part of the energy dependence. We suppose that the failure of formula (8) to account for the ISR energy data may be due to the lack of inclusion of s-plane analyticity. The most general way to use analyticity properties in two independent variables is via the double-variable dispersion relation, as suggested by the Mandelstam representation. But one of the well-known problems in this case is that the discontinuities across the cuts are not known *a priori*. In the OPE one need not know the discontinuity across the cuts.[17] It has been pointed out earlier[23,24] that in expansion (3) not only $a_n(s)$ but also α are energy-dependent parameters. The parameters a_n's do not enter into the expression for the forward slope, but $b(s) \propto \alpha$. Thus the energy dependence of α may affect the slope parameter significantly. The most general way of exploiting s-plane analyticity by OPE is to expand the function depending upon s in polynomials of a suitably chosen variable. Cutkosky[28] has suggested the use of separate conformal mappings and polynomial expansions for the individual factors in a function each of which is a function of the independent variable. Since formula (8) has accounted for the major part of the available data on slope parameters, we suppose that only a few terms in the expansion for α will be needed to account for the complete energy dependence of $b(s)$. Thus, for the sake of simplicity, we will develop convergent expansion for $\alpha(s)$. As a result we will see that almost all the energy dependence from the cross-section-ratio data at high energies is removed, so much so that one may not worry about the energy dependence of the other parameters.

In the s plane the physical region extends from $s = 4m^2$ to ∞ along the positive real axis. There is a left-hand cut from $s = 4m^2 - 2m_\pi^2$ to $-\infty$ for the absorptive part of the amplitude, where m is the nucleon mass. The conformal transformation

$$\eta(s) = \sinh^{-1}\left(\frac{s - 4m^2}{2m_\pi^2}\right)^{1/2} \tag{10a}$$

maps the cut onto the boundary of a strip of width π in the η plane, the whole plane of analyticity being squeezed into the interior of the strip. Similarly, the transformation

$$\xi(s) = \eta^2 = \left[\sinh^{-1}\left(\frac{s - 4m^2}{2m_\pi^2}\right)^{1/2}\right]^2 \tag{10b}$$

maps the cuts onto the branches of a parabola with the focus at the origin, the entire plane of analyticity being mapped onto its interior. However, in both cases the physical region in the s plane is mapped onto the right half of the real axis. As discussed earlier, the correct physical region has been achieved for Laguerre-polynomial expansion in terms of $\xi(s)$. But the correct physical region for expansion in terms of Hermite polynomials, which converge inside a strip, is from $-\infty$ to $+\infty$, and this has not been achieved by the transformation $\eta(s)$. Nevertheless, one can use the convergent expansion for $\alpha(s)$ in a Taylor series in terms of η or ξ. The unitarity restrictions on the slope parameter allow us to keep only a few terms in the Taylor series. We see that for

$$s \gg 4m^2 + t_R, \tag{11}$$

$\eta(s) \sim \ln s$ and $\xi(s) \sim (\ln s)^2$, and from relation (8) we see that $b(s) \sim \alpha(s)$ when condition (11) is satisfied. It has[7,21] been shown that the maximum growth rate of the slope parameter is $(\ln s)^2$. Thus the convergent expansion can be written as

$$\alpha(s) = \begin{cases} C_0 + C_1\xi, & (12a) \\ d_0 + d_1\eta + d_2\eta^2. & (12b) \end{cases}$$

With the energy dependence of $\alpha(s)$ specified by (12), we see that the image of the cuts in the $\chi(s,t)$ plane, with

$$\chi(s,t) = \alpha(s)z \tag{13}$$

for fixed but physical values of s, is a parabola whose size depends upon s. The apex of the parabola is on the negative real axis at $\mathrm{Re}\chi = -\frac{1}{4}\pi^2\alpha(s)$, and the *latus rectum* of the parabola is also proportional to $\alpha(s)$. As $s \to \infty$, the physical region spreads the entire right half of the real axis like $\sim(\ln s)^4$ and the images of the singularities which lie on the boundary are pushed away to infinity like $\sim(\ln s)^2$. One of the main reasons[23] for developing parabolic transformation was to enlarge the domain of convergence at high energies as compared to the small Lehmann ellipse. In the present case we see that for fixed but large energies the size of the parabola is enormously increased and at infinite energies the domain of convergence is the entire χ plane minus the points at infinity. We can rewrite the expansion (3) for $f(s,t)$ as

$$f(s,t)=e^{-\chi}\sum_{n=0} e_n L_n(2\chi)\ , \tag{14}$$

where the coefficients e_n's are related to the a_n's of (3) by the relation

$$e_n=\frac{a_n}{\sum_{n=0}a_n L_n(0)}$$

and a similar relation can be written in terms of c_n and $p_n(0)$ for finite energies.[27] The coefficients e_n's are, in general, energy-dependent unknown parameters. To take into account the energy dependence of the partial-wave amplitudes $a_n(s)$ of Eq. (3) or $c_n(s)$ of Eq. (7), the analyticity of the s plane of partial-wave amplitudes may be exploited by conformal mapping and CPE. Although in the present paper we confine our analysis to pp scattering, partial-wave amplitudes do not possess the same analytic structure as the total amplitude for unequal-mass scattering. Further complication may arise because of the introduction of a set of free parameters, because of the convergent series expansion, for each amplitude. Instead of using the expansions (3) or (7), if one uses the expansion (14), we show in the present work that one need not worry about the energy dependence of e_n's. We will show in the next section that the energy dependence of α is sufficient to account for the energy dependence of the high-energy pp cross-section ratio at all available angles and thus e_n's will turn out to be energy independent. With the expansion (14), the expression for the slope parameter, which is defined as

$$b(s)=\frac{d}{dt}\ln f(s,t)\bigg|_{t=0}=-\frac{d\chi(s,t)}{dt}\bigg|_{t=0}\ , \tag{15}$$

is valid if we retain only the first term in the convergent expansion (14) as a good approximation to the data near forward angles. Expression (15) yields the same formula as (8), with $\alpha(s)$ given by (12). We list below some properties of the variables $\chi(s,t)$ and the series expansion (14):

(i) It is clear from (8) that for large energies, when inequality (11) is satisfied,

$$b(s)\simeq\frac{\alpha(s)}{4m_\pi^2}=\frac{\chi(s,t)}{4m_\pi^2 z}\ .$$

Thus

$$\chi(s,t)=4m_\pi^2 b(s)z \tag{16}$$

for large energies and all angles.

(ii) For large energies but small angles, such that

$$|t|\ll t_R,$$

$$\chi(s,t)\sim tb(s), \tag{17}$$

which is the scaling variable proposed by others.[5-7]

(iii) If we retain the second (third) term in (12a) [(12b)],

$$\chi(s,t)\sim t(\ln s)^2 \tag{18}$$

for large energies and small angles, which is the scaling variable of Auberson, Kinoshita, and Martin.[1]

(iv) If we use expression (12b) and retain only up to the second term

$$\chi(s,t)\sim t\ln s \tag{19}$$

for large energies and small angles. In this case (14) saturates the Regge behavior whether the e_n's are energy dependent or not, and d_1 is related to the Pomeron slope.

In view of the properties discussed above, it is worth examining scaling properties of pp data in the variable χ. From the point of view of the length of the physical region as discussed earlier, the polynomials in (14) are uniquely the Laguerre polynomials and the series expansion coverages at the fastest rate at asymptotic energies. But at finite energies the nature of the polynomials, and hence the function (14), changes with energy.[27] Thus, if (14) defines a unique scaling function at all, it must be at large energies and with e_n's independent of s. In the next section we will show that the experimental data on the pp-scattering cross-section ratio support this view and χ is a good scaling variable. In the next section we first determine the unknown parameters in (12) and hence determine χ by fitting the forward slopes.

III. SLOPE PARAMETERS, SCALING, AND ENERGY DEPENDENCE OF DIP POSITIONS

In this section we first examine how the inclusion of the s-plane analyticity, as discussed in the previous section, improves the fit to the slope-parameter data. When this is done, the unknown parameters in $\chi(s,t)$ are determined and a plot of the data on $f(s,t)$ against $\chi(s,t)$ shows that there is scaling. From this graph and our definition of the variables as defined in the previous section we predict the energy dependence of the position of dips which is in excellent agreement with the data.

A. Energy dependence of forward slopes

To examine whether conformal mapping of the s plane improves fit to the world data on pp forward slopes, we use formulas (8)–(10a), and (12b) and rewrite the slope parameter as

$$b(s)=[d_0+d_1\eta(s)+d_2\eta^2(s)] \times \frac{1}{t_R(s)}\left(1-\frac{t_R(s)}{4_p^2+t_R(s)}\right) \qquad (20)$$

to fit the data.[25,29] We have used expression (12b) because this choice has the capability of saturating the maximum growth rate of the slope parameter, including Regge behavior, depending upon whether both d_1 and d_2 or any of them is nonzero. The choice $d_1=0$ is equivalent to chosing expression (12a) for $\xi(s)$. It has been already pointed out that the expression (20) can be derived from the series (7) which has different types of polynomials $\{p_n(z\}$ at different energies. At asymptotic energies the series (7) converges at the fastest rate and the polynomials are uniquely the Laguerre polynomials, but at finite energies the rate of convergence of the series (7) changes with energy. Thus, at the risk of having used a nonunique set of polynomials and a nonunique rate of convergence with respect to finite energies, we use formula (20) to fit the slope-parameter data for all available energies. By now there are available[25,29] extensive data on $b(s)$ for different s and small-$|t|$ values. To avoid complication in choosing and fitting the data Hansen and Krish[13] have computed average equivalent data from all the available data points for small-$|t|$ values. We thus use 16 data points of Hansen and Krisch which represent all the available forward-slope data for small $|t|$ and fit them with formula (20) by taking at first $d_2=0$. Thus, a three-parameter fit to the data with

$$d_0=0.659$$
$$d_1=0.050 \qquad (21)$$
$$\lambda=0.424 \text{ GeV}$$

is obtained[30] with $\chi^2/\text{NDF}=1.59$. This fit has been shown as the solid curve in Fig. 1. The dotted curve in Fig. 1 is the best fit curve of Ref. 24. To see how much nearer are the average data of Hansen and Krisch[13] and our fit to the actual data, we have also plotted the actual data in Fig. 1. We find that the present fit is significantly improved over the previous fit[24] especially at ISR energies. From the value of d_1 we calculate the coefficient of the $t\ln s$ term in the exponent to be 0.32, which yields the slope of the Pomeron trajectory as $\alpha'(0)=0.16$ GeV^{-2}. We next included d_2 as a free parameter. Its value was found to be consistent with zero without any improvement on the fit. We then tried formula (20) with $d_1=0$ and d_2 as a free parameter, which is equivalent to using formula (12a) for $\propto$. With this choice the total χ^2 was nearly the same as the fit (21). Thus according to present analysis, the data on $b(s)$ are consistent with $\sim\ln s$ or $\sim(\ln s)^2$ type of asymptotic behavior. Comparing the value of λ with the previous result,[24] we find that the present analysis yields a domain of analyticity smaller than the previous analysis, but larger than the theoretical result, for the absorptive part of the amplitude.

Two limitations of the present representation have been pointed out in Sec. II. First, the representation does not conform to the correct analytic properties in so far as the pion poles have been neglected. Second, at finite energies the sequence of polynomials $\{p_n(z)\}$ is neither unique, nor the convergence of polynomial expansion maximum. Before proceeding further it is necessary at this stage to point out yet another limitation which is apparent from the value of λ obtained by using formula (20) to analyze the slope-parameter data for all energies. We see that although asymptotes to the effective shape of the spectral function are the same as the theoretical ones, $\lambda\simeq 3m_\pi$, which implies that the two-pion cut is weak at lower energies. Such an implication of a high value of λ was pointed out in Ref. 24. But according to the general notions of S-matrix theory, the nearest singularity has the maximum influence and perhaps there is no reason to believe that the contribution from the nearby region of the spectral function is small. Therefore the effective shape of the spectral function deduced in this case from the experimental data may not be correct. But such an effective shape of the spectral-function boundary is not needed if we confine our attention to the high-energy data for $s>35$ GeV^2. To see that in fact, one can fit the high-energy data with the theoretical elastic boundary ($\lambda=m_\pi$). We have shown such a fit with the values of d_0 and d_1 given by (21). This fit has been shown in Fig. 1 and extrapolated to lower energies by the dot-dashed curve, which coincides with the solid curve for $s>35$ GeV^2. At high energies the two fits are the same because in this region the distinction between the elastic and effective boundary does not exist. As we will see in this section, scaling of the cross-section-ratio data in the variable χ will be exhibited for small-angle data with $P_{\text{lab}}\geqslant 19.2$ GeV/c and for larger-angle data with $P_{\text{lab}}\geqslant 50$ GeV/c. Therefore, fitting the slope-parameter data with the elastic boundary alone does not affect our results on scaling in any manner.

In the literature, besides the present one, the only fit in which χ^2/NDF has been reported in all the available energy ranges is due to Hansen and

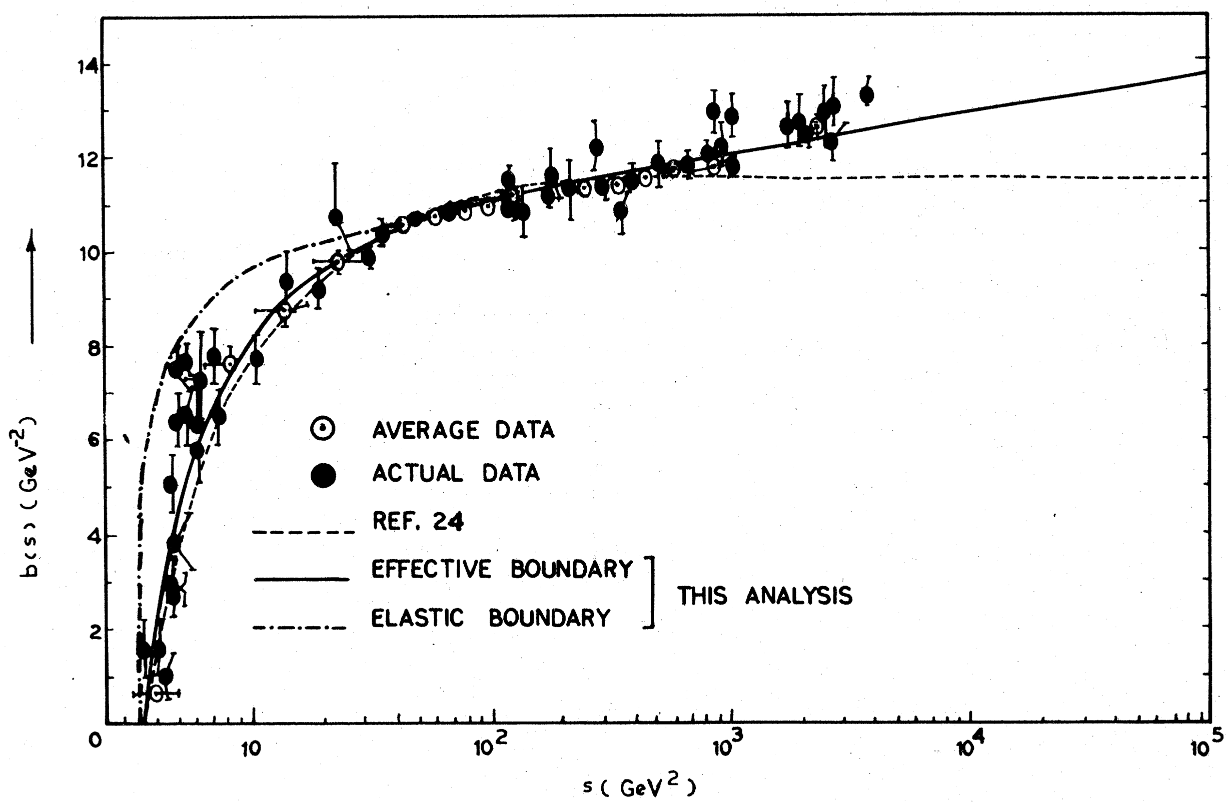

FIG. 1. Slope parameter of pp scattering as a function of s. The solid curve is the fit by the present formula with the effective boundary of spectral functions, and the dot-dashed line is the fit with the theoretical boundary. The dotted line denotes the fit of Ref. 24. The open circles are the average data points of Hansen and Krisch (Ref. 13). The closed circles are the actual data points taken from Ref. 25.

Krisch.[13] Their model gives a much better value of χ^2/NDF than the present one. The main objection against their model is the violation of the Cerrulus-Martin lower bound[14] of the scattering amplitude. The present model, with several limitations already pointed out, illustrates the success of the theory of analytic approximation by conformal mapping and convergent polynomial expansion in continuation of earlier works.[24] More important is the result that the fit has determined unknown parameters in $\chi(s,t)$, which will be shown to be a good scaling variable.

B. Scaling of cross-section-ratio data

The unknown parameters in χ have been determined in the first part of this section from the fit to the slope-parameter data. Thus, knowing the variable $\chi(s,t)$ as a function of s and t, we can now plot the data[29] on $f(s,t)$ against χ. In plotting the data we have separated the smaller- and larger-$|t|$ regions. Plot (a) of Fig. 2 shows small-$|t|$ data for different values of laboratory momenta starting from $P_{\rm lab}=3$ to 1500 GeV/c. It is clear that the data come closer and closer to lying on a scaling curve as energy increases. In particular, all the small-$|t|$ data for $P_{\rm lab}\geq 19.2$ GeV/c lie on the same scaling curve. Plot (b) of Fig. 2 shows the same data points for higher values of energy lying in the range $50\leq P_{\rm lab}\leq 1500$ GeV/c. It is very clear that all the small-$|t|$ data in this plot lie on the same scaling curve. We then examined the efficiency of χ as a scaling variable for large-angle data. Plot (a) in Fig. 3 shows larger-$|t|$ data, including those in the dip region for different values of energies. Plot (b) of Fig. 3 shows the same data points for $50\leq P_{\rm lab}\leq 1500$ GeV/c. It is clear that even the data for larger values of $|t|$ in this range of energies lie on a single curve. From Fig. 3(b) this conclusion is seen to be surprisingly true where the 200 GeV/c data of Akerlof *et al.*[29] and Hartmann *et al.*[31] and the 1500 GeV/c data of De Kerret *et al.*[29] are shown to lie on the same scaling curve.

The geometrical-scaling variable[9] $t\sigma_{\rm tot}$ has

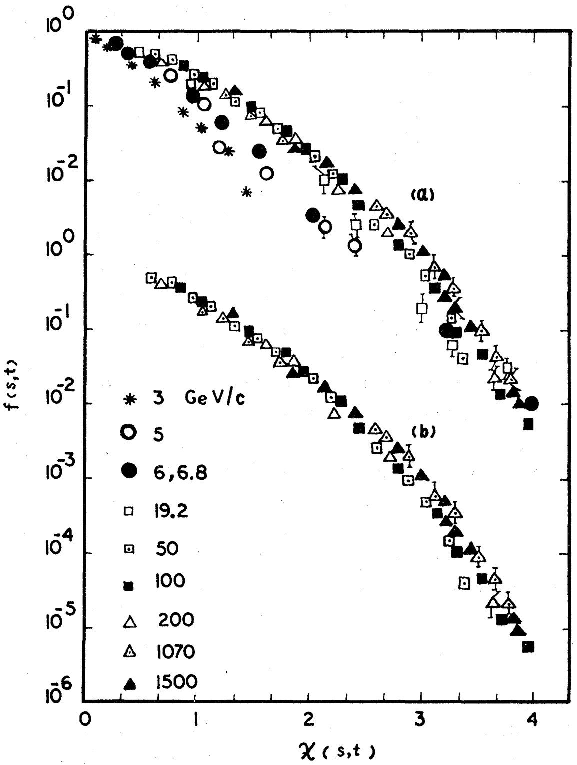

FIG. 2. (a) Scaling and approach to scaling in small-angle pp-scattering data as a function of the scaling variable χ starting from $P_{lab}=3$ GeV/c. (b) Scaling and approach to scaling in small-angle pp-scattering data starting from $P_{lab}=50$ GeV/c.

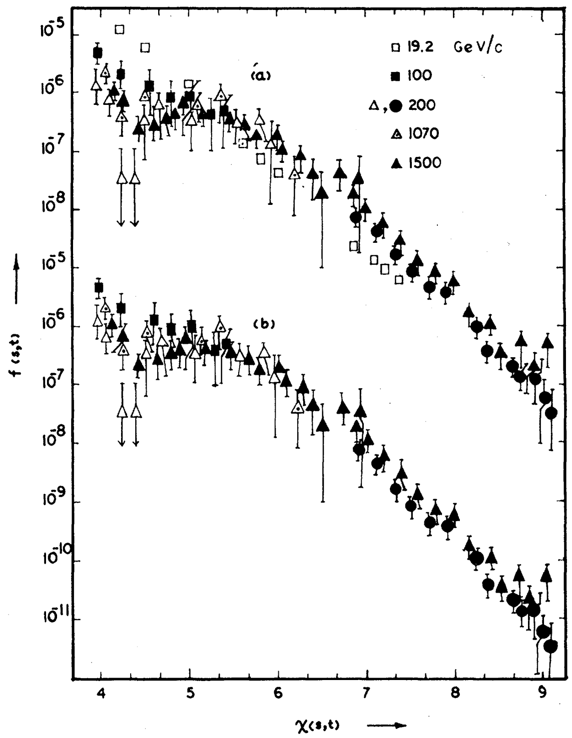

FIG. 3. (a) Scaling and approach to scaling in larger-angle pp-scattering data starting from $P_{lab}=19.2$ GeV/c. (b) Scaling and approach to scaling in larger-angle pp-scattering data at high energies starting from $P_{lab}=100$ GeV/c. The solid circles denote large-$|t|$ data for $P_{lab}=200$ GeV/c from Ref. 31.

been tested to be a bad scaling variable by Hansen and Krisch[13] even near forward angles. These authors[13] propose the scaling variable $\beta^2 P_\perp{}^2 \sigma_{tot}/38.3$ using the Lorentz-contracted geometrical model of Krisch,[12] which proves to be very good near forward angles. It has been shown by Giacomelli and pointed out by a Hansen and Krisch[13] that the energy dependence is not removed for larger-angle data. But in our variable, scaling of the data for $P_{lab} \geqslant 50$ GeV/c has been demonstrated even for much larger value of $|t|$ than in Ref. 13. Besides these variables,[9,13] the only work in which the scaling of the data has been tested for both smaller and larger angles is due to Divakaran and Gangal,[3] who have observed that the scaling for the larger-angle data becomes worse for larger values of the variable $|t|\sigma_{tot}{}^2/\sigma_{el}$. The scaling of the data in our variable is much better than the scaling in the variables $t\sigma_{tot}$ and $t\sigma_{tot}{}^2/\sigma_{el}$. Particularly, Fig. 3(b) demonstrates the spectacular success of χ for high-energy and large-angle data as compared to other variables. Theoretical justification of the assumption on scaling in the geometrical models[9] has been made from general physical grounds, but scaling in the model of Krisch[11,12] and Hansen and Krisch[13] has been hypothesized. The scaling of the upper bound of the amplitude ratio in the variable $t\sigma_{tot}{}^2/\sigma_{el}$ has been proved[2] at asymptotic energies based on rigorous constraints of axiomatic analyticity and unitarity. In the present approach, scaling is not proved *a priori* from rigorous theoretical basis, but the possibility of scaling in χ is hypothesized from the uniqueness of OPE at asymptotic energies. Experimental data on the cross-section ratio support such a hypothesis.

For the sake of completeness we restate our arguments here in favor of scaling. It has been already remarked that in the expansion (14) the polynomials are uniquely the Laguerre polynomials only at asymptotic energies. Whereas the domain of convergence of polynomials decided by the length of the image of the physical region at finite energies may not be the whole interior of the parabola, the figure of convergence at asymptotic energies coincides with the parabola. Thus, from the point of view of the correct physical region, series (14) converges at the fastest rate at asymptotic energies. Then it may be just possible that, if e_n's are independent of energy, the same number of terms in (14) with the same coefficients describe $f(s,t)$ at all energies. Now the fact that the data exhibit scaling in χ verifies such an assertion and the fact that e_n's are independent of s for large s. In the present case the scaling function in (14) can be known, once e_n's are determined by fitting the data at any single high energy, but we have not attempted to find such a function by data fitting at present. Although several scaling variables have been suggested, no unique scaling function has been proposed yet in the literature. Recently Cornille[6] has defined a class of scaling functions in which are included sums of powers, exponentials, and classical orthogonal polynomials with positive coefficients. Our representation suggests the scaling function to be a series in Laguerre polynomials with exponential weight function.

Let us examine the angular range of validity of the present representations. In our conformal mapping variable z, only the structure of the nearest branch points in the x plane has been included. By the conformal mapping, the images of the start of the left- and the right-hand cuts are brought closest to the image of the forward direction in the mapped plane. By a suitable normalization, the variable z has been so constructed that the representations yield forward peak structure. Further, only the analyticity property of the absorptive part of the amplitude has been expolited and the absorptive part is known to be having dominant contribution only near forward angles. For scattering at larger angles, short-range forces represented by more distant branch points and the real part of the amplitude may be important and these have not been directly included[31] in the formula. Therefore it can be argued that the present model is developed for the near forward direction and need not work for larger-angle regions in the representation of the data. But we have already seen that the variable χ serves as a good scaling variable for high-energy larger-$|t|$ data well outside the forward diffraction peak region. At present there is no convincing explanation of the scaling of the data outside the diffraction peak as observed in the present case and also in the variable of Singh and Roy.[2,3] Here we put forward some plausible heuristic reasons in favor of the scaling of the data for larger values of $|t|$. In the present work only the domain of analyticity of the absorptive part of the amplitude, which is dominant near the forward direction, has been taken for conformal mapping and representation in terms of CPE. At larger angles the real part contributes significantly and the real part can be represented by CPE in terms of a similar parabolic variable as defined by (6), where the start of the cut $x_+(-x_-)$ is decided by the domain of analyticity of the real part. This type of representation for the real and imaginary parts has been used by Chao,[20] using conformal mapping on different ellipses. If we ignore the presence of the poles, the domains of analyticity for the real and the imaginary parts are the same for high energies and hence the parabolic variable becomes the same for the real part also. Thus, at high energies a part of the representation (14) may be due to the absorptive part and the other part may be due to the real part. Another possible reasoning may be that the real part is much less as compared to the absorptive part even for larger $|t|$. Concerning the short-range forces which might be influencing scattering at larger angles, it may be argued that their effects have been indirectly taken into account by conformal mapping, although explicit structure of the distant branch points[32] is absent in the representation (14). By means of conformal mapping, the start of more distant cuts is brought closer to the physical region in the mapped plane than in the x plane. It is to be emphasized that these arguments in favor of scaling observed at high energies and larger angles are only plausibility arguments, without having rigorous basis. From our point of view, violations of scaling at larger angles would mean the explicit presence of more distant branch-point structures[31] in the analytic representation.

C. Energy dependence of dip positions

If $\chi(s,t)$ is a good scaling variable, the dip positions should fall at the same point in the $f(s,t)$ vs χ plot. This should be the case for scaling in other variables also. The dip position for pp scattering at 200 GeV/c has been measured very accurately by Akerlof *et al.*[29] In Fig. 3 this dip position corresponds to

$$\chi_0 = 4.35 \pm 0.05 \ . \tag{22}$$

Now, using the definition of $\chi(s,t)$ for high energies, we obtain the energy dependence of the dip position to be

$$|t_d(s)| = 4m_\pi^2\{\sinh[\chi_0/4m_\pi^2 b(s)]^{1/2}\}, \tag{23}$$

where $|t_d(s)|$ is the magnitude of the dip position in the $d\sigma/dt$ vs $|t|$ plot. The dip positions have been measured[10] at high energies up to 1500 GeV/c. In Fig. 4 we show the agreement of our prediction (23) with experimental data.[10] For computation we have used our fit (21) for the slope parameter. The fit (23) involves only one parameter χ_0 which has been determined from the scaling graph. It yields a χ^2/NDF $= 0.46$ for six data points[10,29] showing very good agreement. The curve has been extrapolated to higher energies.

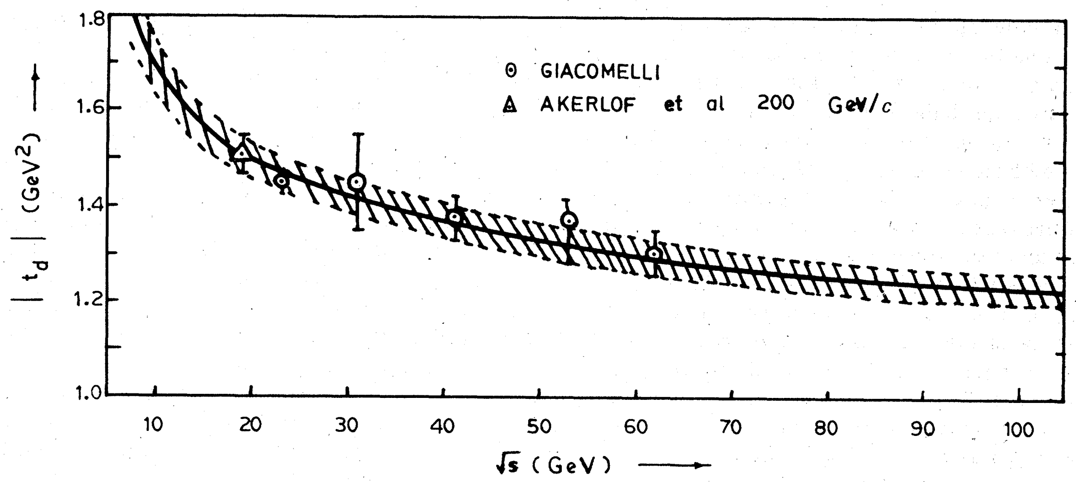

FIG. 4. Prediction of dip position as described in the text for high energies. The data points have been taken from Ref. 10 and Akerlof *et al.* (Ref. 29).

Future experiments at such energies will test our predictions which will also verify the possibility of χ being a good scaling variable for larger-angle data.

IV. RESULTS AND DISCUSSION

Using the Mandelstam analyticity of the $\cos\theta$ plane by conformal mapping and OPE, the data on the slope parameter at ISR energies could not be adequately described. That the slope-parameter data are available for different s values makes necessary the use of s-plane analyticity also. The only other parameter of the theory whose s dependence may affect slope parameter significantly is α. A conformal mapping is developed for mapping the left-hand cut of the absorptive part of the amplitude in the s plane onto the boundary of a strip (parabola). The unitarity restriction allows only the first few terms in the expansion in the Taylor series in the mapped variable for $\alpha(s)$. A good account of the forward-slope data at all available energy ranges is obtained by retaining only the first term in the expansion in the variable χ with an effective shape of spectral function, but a good description of the slope-parameter data for $s > 35$ GeV^2 is possible with the theoretical elastic boundary.

The variable χ has the potentialities of saturating Regge behavior and reproducing known scaling variables[1,5] for small $|t|$ and large s. For high energies and all angles, $\chi \sim b(s)z$, which becomes $\sim tb(s)$, the well-known scaling variable,[5-7] for small $|t|$. In view of this attractive feature of the variable and other[17-19] successes of OPE in describing strong and electromagnetic interactions, we thought it worth investigating whether the data on the cross-section ratio exhibit scaling in this variable. It is argued, in view of the requirement of the correct physical region for expansion in the mapped plane, that the polynomials in the expansion are Laguerre polynomials only at asymptotic energies. The figure of convergence of polynomials as determined by the physical region may not coincide with the parabola at finite energies; thus, the convergence of the polynomial expansion may not be maximum. The polynomial expansion, however, converges at the fastest rate at asymptotic energies. Thus, if possible, the expansion may define a unique scaling function only at asymptotic energies if the coefficients of Laguerre-polynomial expansion are independent of s. When the world data on the pp cross-section ratio are plotted against χ, all the high-energy data come closer and closer to lying on a single curve as the energy is increased. In particular, the data at all angles from 50 to 1500 GeV/c exhibit scaling in the variable χ. Thus we have demonstrated scaling in diffraction scattering by means of OPE. From the scaling graphs we find that the scaling in the variable χ is better than many others, in the sense that it removes energy dependence even from large-angle data at high energies. In particular, it is found from Fig. 3 that even all the larger-$|t|$ data for $P_{\text{lab}} = 200$ GeV/c and 1500 GeV/c lie on the same scaling curve.

Auberson, Kinoshita, and Martin[1] proved their scaling function to be an entire function in the scaling variable $\tau = t(\ln s)^2$ plane. Cornille[6] has shown that one of the scaling variables may be $tb(s)$ and that the scaling function is a series in orthogonal polynomials, including Laguerre but excluding

Hermite polynomials. In the present case the scaling function is a series in Laguerre polynomials in χ. For $s\to\infty$, the physical region spreads the entire right half of the real axis like $\sim b(s)(\ln s)^2$, and the images of cuts of the x plane which form the boundary of the parabola are pushed to infinity like $\sim b(s)$. As has been pointed out in Sec. II, the present representations do not posses the correct analytic structure for nonforward angles in the sense that the contribution due to the poles has not been included. By conformal mapping, the images of the poles fall on the negative real axis of the z plane inside the parabolic figure of convergence. But at asymptotic energies in the χ plane, these images are also pushed away to infinity like $\sim b(s)$ as $s\to\infty$. Thus, for any physical value of energy in the asymptotic energy region, the domain of analyticity for the scaling function is the entire χ plane minus the points at infinity.

From the position of the accurately measured dip[29] position at 200 GeV/c, we predict the energy dependence of dips for high-energy pp scattering. Our predictions are found to be in excellent agreement with the measured dip positions at other energies. Future high-energy measurement will test our predictions for higher energies and thus the efficiency of χ as a scaling variable.

The demonstration of scaling of the high-energy data in the variable χ is further important in the context of OPE for scattering amplitudes. OPE has been successfully empolyed to fit differential-cross-section data at various energies for phase-shift analysis[20] and for obtaining information on coupling constants.[20] In such cases, differential-cross-section data at every energy are fitted, involving a lot of computation. As the scattering at high energy involves a large number of parameters, the repetition of such a procedure to fit the high-energy data would be tedious. On the other hand, the optimized expansion (14), which exhibits scaling in the variable χ, will make matters simpler at high energies, saving a lot of computation. Thus, once the parameters e_n's in the expansion (14) are determined by fitting the data on $f(s,t)$ at various angles for any single high energy, the scaling function, and hence the fits for other energies, are known in the scaling region. Determination of the parameters e_n's and the scaling function by fitting high-energy pp data is deferred to a future work.

As has been pointed out in Secs. I and III, scaling in geometrical models has been either hypothesized[11-13] or assumed.[9] But scaling variables based upon the results of axiomatic field theory are supported by more physical and mathematical reasonings.[2,3,5-7] In the present work, also, scaling in the variable χ is not proved *a priori* theoretically, but hypothesized from the uniqueness of OPE at asymptotic energies. Experimental data verify such a hypothesis.

Here it is necessary to summarize limitations of the present approach to diffraction scattering which have been pointed out at various stages of this paper. First, in the spirit of earlier works,[21,22] only the cut contributions of the absorptive part have been used to parametrize the forward-slope data and demonstrate the scaling of the cross-section-ratio data. It has been demonstrated by model-independent results[2,21,22] that the upper bound on the absorptive part saturates the experimental data on the differential cross-section ratio for diffraction scattering near the forward angles. But the pion-pole contribution may be significant away from the forward direction, which may substantially affect the slope parameter. Moreover, in a typical strong absorption model, the interference between the Pomeron (cut) and the pion is essential. Although in many works[1,2,5-7,21] only the absorptive part has been taken to be responsible for scattering in the diffraction peak region, it will be interesting to investigate how pole contributions affect the fit to the slope-parameter data. But it is apparent that the simple picture of scaling as it has been discussed here will be perhaps impossible to realize if pole contributions are included explicitly. Second, although it has been pointed out[27] that expansion in orthogonal polynomials in the mapped variable z is possible for all energies with a common exponential weight function, the polynomials and hence their domains of convergence in the mapped plane are not the same for different energies. In particular, there exists the ambiguity and danger in using the OPE in terms of Laguerre polynomials at finite energies in the sense that the coefficients in the expansion cannot be determined for lack of the correct physical region in the mapped plane. However, because of the presence of the common weight function in the polynomial expansions for all energies, the same expression for the forward-slope parameter is possible for different energies. Third, the formula developed for the slope parameter requires an effective shape of the spectral function ρ_{st} (ρ_{su}) to fit the data at all energies. The effective boundary retreats too far from the theoretical elastic boundary for lower energies, which is not correct. But the difference between elastic and effective boundary disappears and the same fit is obtained for higher energies with $s>35$ GeV^2, even with the elastic boundary. Since the scaling in χ is observed only for large energies, our conclusions on the scaling of the data are true whether one uses the elastic or effective boundary. Fourth, it can be argued that the present model has been developed for scattering near forward angles and the representation need not work for large-angle data.

There is no convincing reason yet as to why scaling is observed well outside the diffraction peak region at high energies, but some plausibility arguments can be put forward in favor of scaling in this region. Although use has been made of the analyticity properties of the absorptive part of the amplitude, which is dominant near forward angles, one can use a separate CPE in terms of the parabolic variable exploiting analyticity properties of the real part as has been done by Chao,[20] using elliptic conformal mapping. But at high energies the analyticity domains for both the real and the absorptive parts tend to be the same if one ignores the presence of poles. Thus the two parabolic variables used for the real and the absorptive parts will be the same for high energies. Near forward angles the real part is known to be small, but for larger- $|t|$ values it may have significant contribution. At high energies the real part can be described by the same parabolic variables as the imaginary part. Hence it may be possible that only at high energies the data at larger- $|t|$ values may scale in χ. The alternative possibility may be that the real part is small even for larger $|t|$. The other plausible argument is that, by conformal mapping, the start of distant cuts in the x plane that represent short-range forces, which possibly influence large-angle scattering, has been brought closer to the physical region in the mapped plane than in the x plane. Although, explicitly, branch-point structures of distant cuts have not been included[32] in the mapped variable, their influence has been taken into account indirectly in this way in a crude manner. It is to be remembered that these are only plausibility arguments put forward heuristically which may or may not hold.

The fifth limitation of a more serious nature causes a problem only when the transformations (4) and (5) are used to develop CPE by the conformal mapping of the asymmetric cut x plane. Looked at as a function of x, the conformal transformation z, and hence the polynomial expansion for the amplitude, do not possess any other singularities except the dynamical branch points allowed by analyticity. But as has been discussed in Sec. II, the square transformation involved in going from the y to the w plane defined by (4) develops spurious branch points by folding a part of the physical region in the y plane on top of the other part. The images of these singularities appear as three spurious branch points at

$$z_1=\{\ln[(4p^2+t_L-\Delta/s)/t_R]\}^2/4,$$
$$z_2=\infty+i\infty,$$
$$z_3=\infty-i\infty,$$

in the z plane, out of which z_1 lies on the image of the physical region but z_2 and z_3 lie at the two extremities of the branches of the parabola. These branch points give rise to two spurious cuts lying in the interior of the parabola and thus affect the convergence of the series in Laguerre polynomials, although the representation is analytic. Recently[33] a convergent polynomial expansion has been developed by conformal mapping that achieves the correct physical region for Laguerre-polynomial expansion for all energies, but the boundary of the figure of convergence in such a case changes its shape with changing energy, approaching the limiting parabola as $s\to\infty$. In the simplest case of mappings proposed in Ref. 33, two spurious branch points at $z=0$ and $z=\infty$ appear, giving rise to a spurious cut overlapping the image of the physical region completely in the mapped plane. Ciulli[16] has discussed the convergence of polynomial expansion in terms of a mapped variable which introduces an "artificial" cut explicitly along the physical region. In his work,[16] results on the convergence of polynomial expansion have been taken to hold in the presence of an artifical spurious cut along the physical region, although in the final representation the spurious cut disappears. Thus we suppose that the spurious cut causes no problem for convergence as long as it lies on the physical region about which the polynomial expansion is done. The convergence of the polynomial expansion for the physical values of the mapped variable lying above or below the spurious cut may perhaps be taken to hold with $\mathrm{Re}z\pm i\epsilon$ prescription. Since CPE of Ref. 33 converges for all energies, it is reasonable to suppose that scaling may be achieved earlier in the energy scale if the present approach is adopted with the variable of Ref. 33. In a subsequent work[34] such a conjecture has been shown to be true for pp, $\bar{p}p$, $K^{\pm}p$, and $\pi^{\pm}p$ scattering.

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[27]Suppose that for any given s the length of the physical region is $\Gamma(s)$, which varies with s in the present case. Then, provided that we know α, the orthogonal polynomial $\{p_n(x)\}$ with $x = 2\alpha z$ can be constructed by using the relation

$$\frac{1}{2}\int_0^{\Gamma(s)} e^{-x} p_n(x) p_m(x)\, dx = \delta_{nm}. \tag{i}$$

Thus for finite energies the convergent expansion, which is not necessarily an optimized one can be written as

$$\frac{d\sigma}{dt} = e^{-\alpha z} \sum_{n=0} C_n(s) P_n(2\alpha z) \tag{ii}$$

instead of (3). Although (ii) is different from (3), both the series lead to the same form of expression near the forward directions. Because of the common weight factor in both (ii) and (3), the same formulas as (8) and (20) are valid for all energies. But we should bear in mind that for finite energies the formula is derived from a nonoptimal expansion.

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