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Improved unitarity bound on the slope parameter

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Given the total cross section, total elastic cross section, and forward slope parameter, an upper bound is derived on the nonforward slope parameter $b(s,t) = 2d\ln A(s,t)/dt$, where $A(s,t)$ is the absorptive part of the elastic scattering amplitude. The bound significantly improves an earlier bound on $b(s,t)$ and comparison with data leads to some interesting conclusions.

I. INTRODUCTION

The small-$|t|$ data on the pp elastic differential cross section $\sigma(s,t)$ at various values in a wide range of energy (from $\sqrt{s} = 6$ GeV to the present CERN ISR energies) has shown[1,2,3] that $d\ln\sigma(s,t)/dt$, which appears to change little for $0.1 \leq |t| < 0.3$ (GeV/c)^2, takes up a higher, more or less constant value for $|t| \leq 0.1$ (GeV/c)^2. In contrast to these trends, Ankenbrandt *et al.*[4] reported some time back that their pp scattering preliminary data at 200 GeV/c gives a decreasing slope parameter for $|t| \leq 0.09$ (GeV/c)^2. So far, no satisfactory theory exists for these phemomena. Since information on $d\ln\sigma(s,t)/dt$ for small t needs very accurate (and difficult) measurements of $\sigma(s,t)$ it is good to check the consistency of any set of data on $d\ln\sigma(s,t)/dt$ with unitarity. With these motivations, both upper and lower unitarity bounds were obtained[5] on $b(s,t) \equiv 2d\ln A(s,t)/dt$ in the diffraction peak region [$A(s,t)$ is the absorptive part of the elastic scattering amplitude] and compared with data at different energies. In particular, the upper bounds on $b(s,t)$ at 200 GeV/c were very close to the data on Ref. 4 and merited an effort to improve the bounds.

This short paper reports the improved upper bounds on $b(s,t)$ with σ_{elastic} as input in addition to that used in Ref. 5, namely, σ_{total}, $A'(s,0)$ [$=dA(s,0)/dt$], and unitarity (excepting its boundedness aspect, which is no longer enforced explicitly). Comparison with the data of Ref. 4 shows a significant improvement over the previous bound.

II. THE BOUNDS

We first, as in Ref. 5, obtain an upper bound on $A'(s,t)$, then suitably integrate this over t to get a lower bound on $A(s,t)$ and combine these to get an upper bound on $b(s,t)$. Given $A(s,0)$, $A'(s,0)$, σ_{el}, and $a_l \geq 0$, where

$$A(s,t) = (\sqrt{s}/k)\sum_{l=0}^{\infty}(2l+1)a_l P_l(z), \quad z = \cos\theta, \tag{2.1}$$

we can show that

$$A'(s,t) \leq A_U'(s,t) = \frac{\sqrt{s}}{2k^3}\sum_{l\in U}\eta_l P_l'(z), \tag{2.2}$$

provided $\beta > 0$. Here $P_l'(z) = dP_l(z)/dz$, $l \in U$ if

$$\frac{8\pi\beta\eta_l}{k^2(2l+1)} = g_l \equiv \frac{\sqrt{s}}{2k^3}P_l'(z) - \frac{\sqrt{s}}{-2kt}\mu - \frac{\sqrt{s}}{k^3}\lambda P_l'(1) \geq 0, \tag{2.3}$$

and the multipliers μ, λ, β are to be obtained from

$$A'(s,0) = \frac{\sqrt{s}}{2k^3}\sum_{l\in U}\eta_l P_l'(1), \tag{2.4}$$

$$A(s,0) = \frac{\sqrt{s}}{k}\sum_{l\in U}\eta_l, \tag{2.5}$$

and

$$\sigma_{el} = \frac{4\pi}{k^2}\sum_{l\in U}\eta_l^2/(2l+1). \tag{2.6}$$

Remarks: (1) Equation (2.2) is easily proved by direct subtraction. Proof is omitted here. (2) We do not expect (2.2) to go into the corresponding bound of Ref. 5 in the limit $\sigma_{el}/\sigma_t \to 1$, because $a_l \leq 1$ is not imposed in getting (2.2). In the practical case to be discussed later, however, it turns out that the boundedness is automatically satisfied.

As in Ref. 5, integration of (2.2) over t immediately gives a lower bound on $A(s,t)$. Call this $A_L(s,t)$. One then has, if $A_L > 0$, the following.

Upper bound on $b(s,t)$.

$$b(s,t) \leq b_U(s,t) = \frac{2A_U'(s,t)}{A_L(s,t)}. \tag{2.7}$$

Let us evaluate these bounds in the diffraction-peak domain. Here we use $P_l'(z) \to \tau J_1(\tau)/(-t/k^2)$, where $\tau^2 = (-t/k^2)l^2$, and replace the summations over l by integrations over τ. If a large number of l's contribute to the sums, the error committed in this approximation is negligible for k large and $|t|$ small. Condition (2.3) now reads

$$g(\tau) \equiv \tau J_1(\tau) - \mu - \lambda\tau^2 \geq 0. \tag{2.8}$$

Let $\nu_{1,m}$ denote the mth zero of $J_1(x)$ and $\rho_{1,m}$ the value of x at the mth maximum of $J_1(x)$. Now if

(i) $\tau_2 < \nu_{1,1}$,

(ii) $\mu/\nu_{1,2} + \lambda\nu_{1,2} \geq J_1(\rho_{1,2})$, (2.9)

then (2.8) will be satisfied only for $0 \leq \tau \leq \tau_2$ when $\mu < 0$ and only for $0 < \tau_1 \leq \tau \leq \tau_2$ when $\mu > 0$, and $\nu_{1,2} > \sqrt{\mu/\lambda}$ where the unknowns μ, λ, β, τ_2 (and τ_1) are to be obtained from the constraints $A'(s,0)$, $A(s,0)$, σ_{el}, and

$$g(\tau_i)=0, \quad i=1,2 \text{ if } \mu>0 \ (i=2 \text{ if } \mu<0). \tag{2.10}$$

It is of course assumed that $\lambda > 0$—this is true at least for small t. ($\lambda = \frac{1}{2}\delta A_U'(s,t)/\delta A'(s,0) \rightarrow \frac{1}{2}$ as $t \rightarrow 0$). The conditions (2.9) essentially restrict the value of t over which this evaluation holds. We define

$$A_m = \int_\Gamma d\tau\, \tau^m g(\tau), \quad m=1,2,3\ldots, \tag{2.11}$$

$$B = \int_\Gamma d\tau\, \tau[g(\tau)]^2, \tag{2.12}$$

and

$$I = \int_\Gamma d\tau\, \tau^2 J_1(\tau) g(\tau), \tag{2.13}$$

where $\Gamma=[0,\tau_2]$ if $\mu<0$ and $\Gamma=[\tau_1,\tau_2]$ if $\mu>0$. Introducing dimensionless parameters R, r and a variable y^2,

$$R \equiv 36\pi\left(\frac{\sigma_{el}}{\sigma_t^2}\right)\frac{A'(s,0)}{A(s,0)}, \tag{2.14}$$

$$r \equiv \frac{\sigma_{el}}{\sigma_t}, \tag{2.15}$$

and

$$y^2 \equiv \frac{4A'(s,0)}{A(s,0)}(-t), \tag{2.16}$$

We rewrite the constraints as follows:

$$R = \frac{9}{8}\frac{A_3 B}{A_1^3}, \tag{2.17}$$

$$y^2 = \frac{A_3}{A_1} \tag{2.18}$$

[and $r=(1/\bar{\beta})B/A_1$, $\bar{\beta}=16\pi\beta(-t)/(k\sqrt{s})$]. Equations (2.17) and (2.18) can be solved together with Eq. (2.10) to get values of μ, λ, τ_2 (and τ_1). We use them to calculate the bound given by

$$\tilde{A}_U'(s,t) \equiv \frac{A_U'(s,t)}{A'(s,0)} = \frac{2I}{A_3}, \tag{2.19}$$

and the subsequent lower bound on $A(s,t)$ given by

$$A_L(s,t) = A(s,0) - \int_t^0 dt\, A_U'(s,t) \tag{2.20}$$

(as in Ref. 5). If the bound (2.20) is positive, then the upper bound on $b(s,t)$ is

$$b(s,t) \leq b_U(s,t) = \frac{2A_U'}{A_L}. \tag{2.21}$$

The bounds now are functions of R, r, and y^2. For comparison with our earlier results,[5] we reintroduce $\alpha = 32\pi A'(s,0)/A(s,0)\sigma_t$ and $x^2 = -t\sigma_t/8\pi$. Since $R = 9\alpha r/8$ and $y^2 = \alpha x^2$, the bounds become functions of α, r, and x^2. Numerically, the integrated lower bound $A_L(s,t)$ coincides with the one directly obtained by Auberson and Roy.[6]

Lower bound on $b(s,0)$. As in Ref. 5, we can obtain a lower bound on $b(s,0)$ in terms of data at a nonforward value of t. Let $\tilde{A}_U'(s,t_1) \equiv f(x_1,\alpha,r)$, $x_1^2 = -t_1\sigma_t/8\pi$, and $\tilde{A}(s,t_1) = A(s,t_1)/A(s,0)$, where t_1 is any value of t where $A(s,t)$ and $b(s,t)$ are known. One can show that

$$1 \leq \frac{\alpha f(x_1,\alpha,r)\sigma_t}{16\pi b(s,t_1)\tilde{A}(s,t_1)}. \tag{2.22}$$

This gives an implicit lower bound on α [$=\alpha_0(t_1)$] and hence on $b(s,0)$. These can be easily computed.

III. COMPARISON WITH DATA

We now compare our bounds with the data of Ref. 4. It should be mentioned that these data are preliminary and we use them only to illustrate the power of the method.

As usual we assume (1) spin independence and (2) negligible real parts. This leads to $\tilde{A}(s,t) \simeq [16\pi\sigma(s,t)]^{1/2}/\sigma_t$ and $b(s,t) \simeq d\ln\sigma(s,t)/dt$. Now a quadratic fit to the data of Ref. 4 gives $b(s,0) = 10.1\ (\text{GeV}/c)^{-2}$. With $\sigma_t = 38.9$ mb this gives $\alpha = 5.08$. Taking[7] $\sigma_{el}/\sigma_t = 0.183$, we evaluate the upper bound on $b(s,t)$, Eq. (2.21). This is shown in Fig. 1 where the results of Ref. 5 are reproduced for comparison.[8] The improvement incurred by inclusion of σ_{el} as input is very clear. Moreover, it can be seen that the bound with $\alpha = 5.08$ is not satisfied by the small-t data points, thus ruling out $b(s,0) = 10.1\ (\text{GeV}/c)^{-2}$. In fact if the data can be fitted with $b(s,0) = 10.73\ (\text{GeV}/c)^{-2}$ ($\alpha = 5.4$) which is the value quoted by Akerlof *et al.*,[7] the data on $\tilde{b}(s,t)$ reduce by a factor of 0.94 and the second data point (at center) just lies on the bound with $\alpha = 5.4$.

Consider now Eq. (2.22). Taking $t_1 = -0.032$ $(\text{GeV}/c)^2$ and $b(s,t_1)$ (central point) and $A(s,t_1)$ from Ref. 4, we evaluate (2.22) to find $b(s,0)$

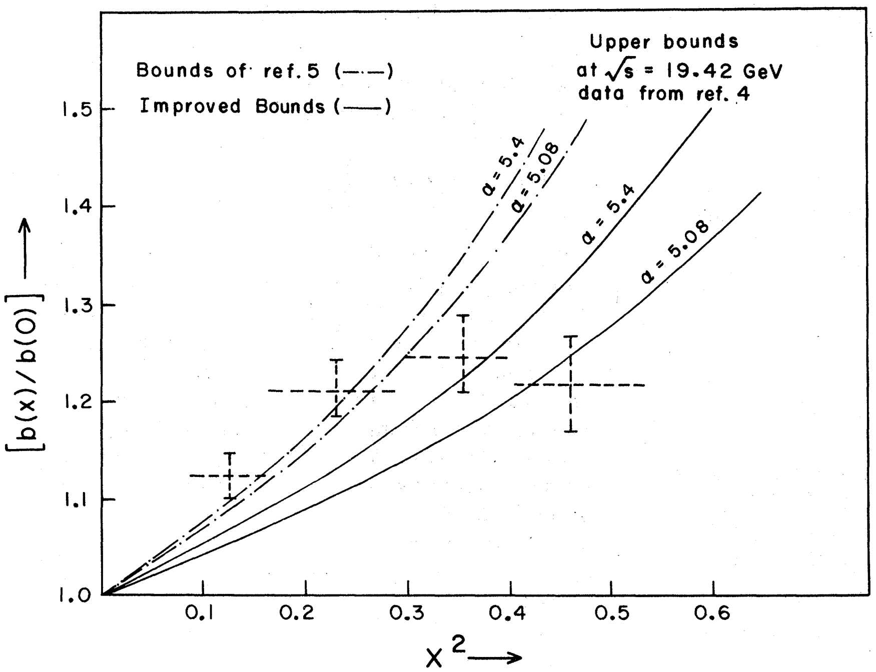

FIG. 1. Upper bounds of Ref. 5 (broken curves) on $b(s,t)/b(s,0)$ at 200 GeV/c for $\alpha = 5.08$ [$b(0) = 10.1$] and $\alpha = 5.4$ [$b(0) = 10.7$]. The solid lines show the improved bounds with $\sigma_{el}/\sigma_t = 0.183$ for the same values of α.

≥ 10.54 (GeV/c)$^{-2}$. [Note that the McDowell-Martin bound gives $b(s,0) \geq 9.66$ (GeV/c)$^{-2}$.] The bound of Ref. 5 had given $b(s,0) \geq 10.4$ (GeV/c)$^{-2}$.

The present paper has thus shown that unitarity restrictions can give very useful guidelines on possible extrapolations to $t = 0$ of the small-t data.

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[8]This paper defines $b(s,t)$ to be twice the one defined in Ref. 5. This of course does not affect $b(s,t)/b(s,o)$ shown in Fig. 1.