Corrections to the sixth-order anomalous magnetic moment of the muon

Morten Laursen and Mark A. Samuel
Quantum Theoretical Research Group, Department of Physics, Oklahoma State University, Stillwater, Oklahoma 74074
(Received 30 October 1978)

The contribution to the sixth-order muon anomaly from fourth-order electron vacuum polarization is determined analytically to order m_e/m_μ. The result including the contribution from graphs containing two second-order lepton vacuum polarization subgraphs is $(\alpha/\pi)^3\{(2/9)\ln^2(m_\mu/m_e)+(403/108-4\pi^2/9)\times\ln(m_\mu/m_e)+\zeta(3)/2+2\pi^2/27+5/27+[383\pi^2/135-16\pi^2\ln(2)/9-13\pi^3/18]m_e/m_\mu\}$.

In sixth order, the difference between the muon and electron magnetic moments can be expressed, for $m_\mu/m_e \gg 1$, as

$$a_\mu^{(6)}-a_e^{(6)}=\left(\frac{\alpha}{\pi}\right)^3[A\ln^2(m_\mu/m_e)+B\ln(m_\mu/m_e)+C+D(m_e/m_\mu)+O((m_e/m_\mu)^2\ln^2(m_\mu/m_e))]\,. \tag{1}$$

A and B are completely known analytically[1,2]:

$$A=\tfrac{2}{9}\,,$$
$$B=\tfrac{31}{27}+7\pi^2/9-\frac{2\pi^2}{3}\ln 2+\zeta(3)\,. \tag{2}$$

All contributions to C, except the light-by-light contribution $C^{(\gamma\gamma)}$, are also known analytically[3] ($C^{(\gamma\gamma)}$ is known numerically[4,5]):

$$C=\tfrac{1075}{216}-\tfrac{25}{18}\pi^2+\tfrac{5}{3}\pi^2\ln 2-3\zeta(3)+3C_4+C^{(\gamma\gamma)}\,,$$

where

$$C_4=\tfrac{11}{648}\pi^4-\tfrac{2}{27}\pi^2\ln^2 2-\tfrac{1}{27}\ln^4 2-\tfrac{8}{9}a_4 \tag{3}$$

and

$$a_4=\sum_1^\infty\frac{1}{2^n n^4}\,.$$

The only contribution to D which is known analytically is the double-bubble contribution[6] [diagram (d) of Fig. 1]

$$D^{(d)}=-4\pi^2/45\,. \tag{4}$$

The contribution to D due to the other diagrams of Fig. 1 is known numerically:

$$D^{(a+b+c)}=-5.6776256\,,$$
$$D^{(e+f)}=0. \tag{5}$$

In this paper we present an analytic calculation of $D_{(a+b+c)}$, the contribution to D from fourth-order electron vacuum polarization [the proper diagrams (a), (b), and (c) of Fig. 1].

This quantity is given by the following expression[7,8]:

$$D^{(a+b+c)}=\frac{\pi}{2}-2\pi\int_0^1\frac{dx\,x}{(1-x^2)^{3/2}}\left[\frac{1}{\pi}\mathrm{Im}\,\pi^{*(4)}(x)-\frac{1}{\pi}\mathrm{Im}\,\pi^{*(4)}(1)\right]\Big/\left(\frac{\alpha}{\pi}\right)^2, \tag{6}$$

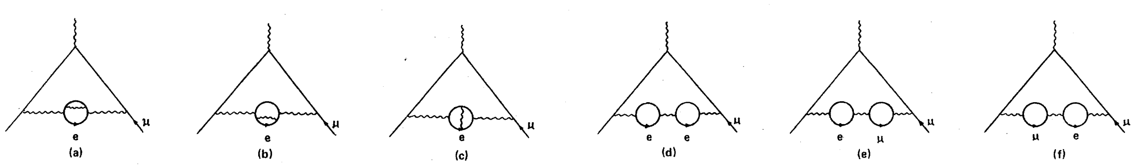

FIG. 1. Feynman diagrams representing the fourth-order vacuum polarization contribution to the sixth-order muon anomaly.

where

$$\frac{1}{\pi}\,\mathrm{Im}\pi^{*(4)}(x)=\left(\frac{\alpha}{\pi}\right)^2\left\{(-\tfrac{5}{8}x-\tfrac{3}{8}x^3)+x(-\tfrac{1}{2}+\tfrac{1}{6}x^2)\ln\left(\frac{64x^4}{(1-x^2)^3}\right)+(\tfrac{11}{16}+\tfrac{11}{24}x^2-\tfrac{7}{48}x^4)\ln\left(\frac{1+x}{1-x}\right)\right.$$
$$+(\tfrac{1}{2}+\tfrac{1}{3}x^2-\tfrac{1}{6}x^4)\ln\left(\frac{(1+x)^3}{8x^2}\right)\ln\left(\frac{1+x}{1-x}\right)$$
$$\left.-(\tfrac{1}{2}+\tfrac{1}{3}x^2-\tfrac{1}{6}x^4)\left[4\Phi\left(-\frac{1-x}{1+x}\right)+2\Phi\left(\frac{1-x}{1+x}\right)+\frac{\pi^2}{2}\right]\right\} \tag{7}$$

and

$$\frac{1}{\pi}\,\mathrm{Im}\pi^{*(4)}(1)=\tfrac{1}{4}(\alpha/\pi)^2\,. \tag{8}$$

So we can write

$$D^{(a+b+c)}=\frac{\pi}{2}-2\pi\left[R_1+R_2+R_3+R_4+R_5-\tfrac{1}{4}\int_0^1\frac{dx\,x}{(1-x^2)^{3/2}}\right], \tag{9}$$

where the five R_i correspond to the five terms in Eq. (7). It is easy to see that R_2+R_3, R_4, and R_5 are finite and the combination

$$R_1'=R_1-\tfrac{1}{4}\int_0^1\frac{x}{(1-x^2)^{3/2}} \tag{10}$$

is also finite.

We now evaluate the integrals. Our results are as follows:

$$R_1'=\frac{1}{4}-\frac{\pi}{32}, \tag{11}$$

$$R_2+R_3=\pi\ln 2-\tfrac{211}{288}\pi\,, \tag{12}$$

and

$$R_4+R_5=\tfrac{13}{36}\pi^2-\frac{\pi}{9}\ln 2-\tfrac{15}{216}\pi\,. \tag{13}$$

Adding the terms in Eqs. (11), (12), and (13) and substituting into Eq. (9), we obtain our result

$$D^{(a+b+c)}=-\tfrac{13}{18}\pi^3-\tfrac{16}{9}\pi^2\ln 2+\tfrac{79}{27}\pi^2$$
$$=-5.6776257\,. \tag{14}$$

This is in excellent agreement with the numerical value in Eq. (5). It is interesting that, although the term proportional to π cancels out, there remains a π^3 term. This is the first time an odd power of π occurs in a $g-2$ contribution.

Using Eqs. (4), (5), and (14), the contribution to D from all the graphs of Fig. 1 can be written:

$$D^{(a+b+c)}+D^{(d)}+D^{(e+f)}=-\tfrac{13}{18}\pi^3-\tfrac{16}{9}\pi^2\ln 2+\tfrac{383}{135}\pi^2\,. \tag{15}$$

This work was supported by the U. S. Department of Energy.

[1]T. Kinoshita, Nuovo Cimento 51B, 140 (1967).
[2]B. E. Lautrup and M. A. Samuel, Phys. Lett. 72B, 114 (1977).
[3]R. Barbieri and E. Remiddi, Nucl. Phys. B90, 233 (1975).
[4]M. A. Samuel and C. Chlouber, Phys. Rev. Lett. 36, 442 (1976).
[5]C. Chlouber and M. A. Samuel, Phys. Rev. D 16, 3596 (1977).
[6]M. A. Samuel, Nucl. Phys. B70, 351 (1974).
[7]C. Chlouber and M. A. Samuel, Phys. Rev. D 17, 2817 (1978).
[8]B. E. Lautrup and E. de Rafael, Phys. Rev. 174, 1835 (1968).