PHYSICAL REVIEW D VOLUME 19, NUMBER 4 15 FEBRUARY 1979

Eikonal approximation in x-ray transition radiation theory

A. I. Alikhanian and V. A. Chechin

P. N. Lebedev Physical Institute, Academy of Science of the USSR, Moscow, USSR

(Received 17 July 1978)

In connection with the calculation of x-ray transition radiation (TR) generated in a complicated configuration radiator, a new method for the calculation of the transition electromagnetic field is suggested which is analogous to the eikonal approximation in quantum mechanics. The method is based on the fact that, in the x-ray TR frequency region $\hbar\omega \gtrsim 0.1$ keV, the dielectric constant of the usual substances differs very slightly from unity: $|\epsilon(\omega) - 1| \ll 1$. The use of this condition allows one not only to obtain the well-known results of the x-ray TR theory in a simple way, but also to solve those problems for which the existing exact methods are either very complicated or they are invalid. The eikonal approximation is applicable for the calculation of the work performed by a charged relativistic particle passing through a plate or an arbitrary medium.

I. INTRODUCTION

X-ray transition radiation (TR) has recently attracted more and more attention. This is mainly due to the development of transition-radiation detectors capable of measuring the energy of relativistic charged particles in the range of Lorentz factors $\gamma \gtrsim 10^3$ where other instruments (e.g., Cherenkov counters) are ineffective.[1,2] Calculations of the x-ray TR necessary for a correct interpretation of the experiments are unwieldly and even unreliable sometimes because the TR theory based on the exact solution of the macroscopic Maxwell equations gives compact formulas for the simplest geometry only: one or several interfaces.[3-5] At the same time the operating and designed experimental arrangements for x-ray TR observations often have such a complicated geometric configuration of radiators that an exact calculation of the radiation becomes, in general, impossible. We mean, for example, radiators in the form of plastic foam,[6] deuterium foam,[7] or a colloid system.[8]

In this paper an approximate method of x ray TR intensity calculation is suggested which is applicable in the most interesting x-ray frequency region for high-energy physics ($\hbar\omega \sim 0.1$ keV). The method (similar to the eikonal approximation in quantum mechanics) is based on approximate solution of the macroscopic Maxwell equations with due regard for small parameters appearing in this frequency region (Sec. II). The eikonal approximation gives in the simplest cases the results which coincide with the exact ones in the frequency region of interest and permits the determination of the x-ray TR and transition electromagnetic field itself in radiators with complicated geometric configurations. In particular, the energy lost by a relativistic charged particle traversing a plate (Sec. III) and a random medium (Sec. IV) is calculated in this paper. The expressions obtained have a simple analytic structure which permits us to take into account the influence of the imaginary part of the dielectric constant ϵ on the transition effects more rigorously and to clarify their physical meaning.

II. APPROXIMATE METHODS IN X-RAY TRANSITION RADIATION THEORY

To calculate the TR intensity it is necessary to determine the electromagnetic field caused by a charged particle moving in an inhomogeneous medium and described by the macroscopic Maxwell equations with the dielectric constant $\epsilon(\vec{r})$.

Let us consider the approximate methods of solving the equations for the electric field which follow from Maxwell's equations:

$$(\Delta + \omega^2 \epsilon - \operatorname{grad} \operatorname{div})\vec{E}(\vec{r}) = -4\pi i \vec{j}(\vec{r}), \tag{1}$$

where $\vec{E}(\vec{r})$ and

$$\vec{j}(\vec{r}) = (2\pi)^{-1} e \int \vec{v}\delta(\vec{r} - \vec{v}t)\exp(i\omega t)dt$$

are the Fourier transform of the electric field and density of the current produced by a particle of charge e (for simplicity the argument ω will be omitted everywhere), $\vec{v}$ is the particle velocity assumed to be constant, the velocity of light C and Planck's constant $\hbar$ are taken hereafter to be equal to unity.

Let us formulate the conditions which are fulfilled in the frequency region $\omega \gtrsim 0.1$ keV and allow one to solve Eq. (1) approximately. Firstly, $\epsilon(\vec{r})$ differs slightly from some homogeneous value $\epsilon_0 \approx 1$ in this frequency region.[9] So

$$|\epsilon_1(\vec{r})| = |\epsilon(\vec{r}) - \epsilon_0| \ll |\epsilon_0|. \tag{2}$$

At any rate the condition (2) is fulfilled at $\omega \gg \max\{\omega_i\}$ (ω_i is the ionization potential of the ith electron shell of substance atoms) because in this case $\epsilon \simeq 1-\sigma/\omega^2$, where $\sqrt{\sigma}$ is the plasma frequency of the substance. For condensed media $\sqrt{\sigma} \sim 10$ eV and, hence, $|\epsilon_1| \sim 10^{-2}$. Moreover, the condition (2) is usually fulfilled even, when some of the ionization potentials lie in the frequency region $\omega \gtrsim 0.1$ keV.

Secondly, for macroscopic inhomogeneities with a characteristic dimension a, we have

$$a\omega \gg 1\,. \tag{3}$$

The TR spectrum of relativistic particles is confined to frequencies below $\omega_b \sim \sqrt{\sigma}\gamma_0$, where $\gamma_0 = \min\{\gamma, a\sqrt{\sigma}/2\}$; that is why in order that the condition (3) might be fulfilled in all the essential frequency range the following requirement is needed,

$$a\sqrt{\sigma} \gg 1\,,$$

i.e., (3a)

$$a \gg 2.10^{-6}\ \text{cm (for } \sqrt{\sigma} \sim 10 \text{ eV)}\,.$$

Thirdly, let us assume that

$$|\operatorname{grad}\epsilon_1(\vec{r})| \ll \omega|\epsilon_1(\vec{r})|\,. \tag{4}$$

With due regard for (3) the condition (4) can be broken only near sharp boundaries of inhomogeneities where $\epsilon_1(\vec{r})$ changes abruptly. But for physical reasons the boundary surface can always be considered diffused at a thickness $s \gg \omega^{-1}$ (but $s \ll a$), this ensures the fulfillment of the condition (4) everywhere.[10]

Using (2)–(4) we obtain the condition for the Born approximation to be applicable to Eq. (1), taking it in the equivalent integral form,

$$\begin{aligned}\vec{E}(\vec{r}) &= \vec{E}_0(\vec{r}) + \vec{E}_1(\vec{r})\\ &= \vec{E}_0(\vec{r}) - \omega^2 \int \hat{G}_0(\vec{r}-\vec{r}\,')\epsilon_1(\vec{r}\,')\vec{E}(\vec{r}\,')d\vec{r}\,'\\ &= \vec{E}_0(\vec{r}) - \omega^2 \int \hat{G}(\vec{r},\vec{r}\,')\epsilon_1(\vec{r}\,')\vec{E}_0(\vec{r}\,')d\vec{r}\,'\,,\end{aligned} \tag{5}$$

where

$$\vec{E}_0(\vec{r}) = \int \vec{E}_0(\vec{q})\exp(i\vec{q}\cdot\vec{r})d\vec{q}\,,$$

$$\vec{E}_0(\vec{q}) = ie\,\delta(\omega - \vec{q}\vec{v})(\omega\vec{v} - \vec{q}/\epsilon_0)(2\pi^2\Lambda_0)^{-1}\,,$$

$$\Lambda_0 = \vec{q}^{\,2} - \epsilon_0\omega^2\,,$$

$$\hat{G}_0(\vec{r}-\vec{r}\,') = -(\hat{I} + \nabla\cdot\nabla/\epsilon_0\omega^2)\frac{\exp(i\omega\sqrt{\epsilon_0}\,|\vec{r}-\vec{r}\,'|)}{4\pi|\vec{r}-\vec{r}\,'|}\,.$$

Here $\vec{E}_0(\vec{r})$ is the solution of Eq. (1) with homogeneous ϵ_0; $\hat{G}_0(\vec{r}-\vec{r}\,')$ and $\hat{G}(\vec{r},\vec{r}\,')$ are the tensor Green's functions of Eq. (1) with homogeneous ϵ_0 and inhomogeneous $\epsilon(\vec{r})$, respectively; $\hat{I}$ and $\nabla\cdot\nabla$ are the tensor unity and the second derivative. In Cartesian coordinates $\hat{I}_{\alpha\beta} = \delta_{\alpha\beta}$, $(\nabla\cdot\nabla)_{\alpha\beta} = \partial^2/\partial r_\alpha \partial r_\beta$, $\alpha,\beta = 1,2,3$. The condition for the Born approximation to be applicable to Eq. (5) is

$$|\vec{E}_0(\vec{r})| \gg \omega^2 \left|\int \hat{G}_0(\vec{r}-\vec{r}\,')\epsilon_1(\vec{r}\,')\vec{E}_0(\vec{r}\,')d\vec{r}\,'\right|. \tag{6}$$

Let us integrate this expression by parts twice by transferring the action of the differential operator in $\hat{G}_0$ to $\epsilon_1\vec{E}_0$ and then to $\vec{E}_0$, using (4). Let us omit the surface terms (in the assumption that at infinity $\epsilon_1 \to \infty$) and require the fulfillment of the given condition for all the waves $\exp(i\vec{q}\cdot\vec{r})$ into which the field $\vec{E}_0$ is decomposed. After omitting the vector factors in both the parts of the condition (unimportant in the relativistic case when the component $\vec{q}$ perpendicular to $\vec{v}$ is small in comparison with the longitudinal one, $|\vec{q}_\perp| \sim \omega/\gamma \ll q_\parallel = \omega/v$), we obtain finally

$$\frac{\omega^2}{4\pi}\left|\int \exp(i\omega\sqrt{\epsilon_0}y + i\vec{q}\cdot\vec{y})\epsilon_1(\vec{r}+\vec{y})y^{-1}d\vec{y}\right| \ll 1\,, \tag{7}$$

where $\vec{y} = \vec{r}\,' - \vec{r}$ and the point $\vec{r}$ is taken in the inhomogeneity region. An estimation of the integral in (7) with due regard for (3) is performed similarly to the determination of the Born approximation applicability condition for fast particles in quantum mechanics with the only distinction due to the "nonfree" wave $\exp(i\vec{q}\cdot\vec{r})$ in (7) with $\vec{q}^{\,2}$ not equal to $\epsilon_0\omega^2$ (but it is very close to $\epsilon_0\omega^2$ in the relativistic case). The integration of the rapidly oscillating function in (7) near the angle $\theta = \pi$ (under $\omega > 0$) gives

$$|\omega z_f \epsilon_1[\exp(ia/z_f) - 1]| \ll 1\,, \tag{8}$$

where $z_f = |\omega\sqrt{\epsilon_0} - |q||^{-1}$ is the formation zone of x-ray TR.

The condition (8) is more rigid than the condition (2) and it practically eliminates the possibility of using the Born approximation in the x-ray TR theory. In fact, it is easy to obtain from (8) that $\omega \gg \omega_B$. Therefore, the Born approximation [with due regard for (3)] can give only a high-energy part of the x-ray TR as has been discovered in Ref. 11 by the direct calculation. Let us note that the Born approximation is applicable for small inhomogeneities ($a\omega \ll 1$) because a similar estimation gives $|\epsilon_1| \ll 1$.

It is also easy to verify that the inequalities (2)–(4) are completely similar to the conditions of the eikonal approximation applicability in quantum mechanics. In conformity with this fact we take for the tensor Green's function $\hat{G}(\vec{r},\vec{r}\,')$ the following expression, typical of the eikonal approximation[12]:

$$\hat{G}(\vec{r},\vec{r}')=-(\hat{I}+\nabla\cdot\nabla/\epsilon(\vec{r})\omega^2)\times\exp[i\Psi(\vec{r},\vec{r}')](4\pi|\vec{r}-\vec{r}'|)^{-1}, \tag{9}$$

$$\Psi=\omega\int_0^{s_0}[\epsilon(s)]^{1/2}ds,\quad s_0\equiv|\vec{r}-\vec{r}'|,$$

where the integration is performed over a straight line that connects the points $\vec{r}$ and $\vec{r}'$. A scalar Green's function of this type has been used in Ref. 13 for the calculation of the intensity of the x-ray TR produced in interstellar dust by cosmic rays. The quasiclassical Green's function, close to the eikonal one (having an additional preexponential factor), has been used for solving one-dimensional problems in Ref. 5. After the substitution of (9) into (5) and integration by parts twice we obtain

$$\vec{E}_1(\vec{r})=\int\eta(\vec{r},\vec{q})(\hat{I}-\vec{q}\cdot\vec{q}/\epsilon(\vec{r})\omega^2)\vec{E}_0(\vec{q})\exp(i\vec{q}\cdot\vec{r})d\vec{q},$$

$$\eta(\vec{r},\vec{q})=\omega^2\int\exp[i\psi(\vec{r},\vec{r}+\vec{y})+i\vec{q}\cdot\vec{y}]\times\epsilon_1(\vec{r}+\vec{y})(4\pi y)^{-1}d\vec{y}, \tag{10}$$

$$d\vec{q}=d\varphi\kappa d\kappa d\omega/v.$$

Then after the integration in $\eta(\vec{r},\vec{q})$ over the angular variables [as in formula (7) but taking into account both the signs of frequency] we have

$$\eta(\vec{r},\vec{q})=i\omega^2\int_0^\infty\exp[i\psi(\vec{r},\vec{r}+\vec{y})-iqy]\times\epsilon_1(\vec{r}+\vec{y})dy/2q, \tag{11}$$

where the odd function of frequency $q=|\vec{q}|\,\mathrm{sgn}\omega$, is introduced for convenience. The integration in (11) is carried out from the point $\vec{r}$ in the direction $-\vec{q}\,\mathrm{sgn}\omega$.

Formulas (10) and (11) are the main initial formulas of the eikonal approximation; they will be used in Secs. III and IV for the calculation of the charged particle work on the complete electromagnetic field. This magnitude is the total energy lost by a particle in an inhomogeneous medium and includes both the ionization energy loss and Cherenkov and transition radiation.

To calculate the intensity of TR alone one may use (10) at asymptotically large values of $|\vec{r}|$.[14] But it is easier to solve once more the equations for the Fourier transform of the vector potential $\vec{A}(\vec{r})$ in the Lorentz gauge using the scalar eikonal Green's function $G(\vec{r},\vec{r}')=-\exp[i\psi(\vec{r},\vec{r}')](4\pi|\vec{r}-\vec{r}'|)^{-1}$. Acting in a manner similar to the aforementioned one but not separating the zero term $\vec{A}_0(\vec{r})$ when solving the equations (i.e., considering the external current of the field source) we obtain the differential frequency-angular spectrum of the x-ray transition radiation $dS/d\omega d\Omega$:

$$dS/d\omega d\Omega=e^2v^2\sin^2\theta|M|^2\omega^2/4\pi^2,\quad M=\int_{-\infty}^{\infty}\exp[i\psi_\infty(\vec{n},\vec{v}t)+i\omega t]dt, \tag{12}$$

$$\Psi_\infty(\vec{n},\vec{v}t)=\lim_{|\vec{r}|\to\infty}\{\psi(\vec{r},\vec{v}t)-\psi(\vec{r},t=0)\}, \tag{13}$$

where $\vec{n}$ is the unit vector in the radiation direction $\Psi_\infty(\vec{n},\vec{v}t)$ is the phase difference which depends on $\vec{n}$ and $\vec{v}t$ but not on $|\vec{r}|$. θ is the angle between $\vec{n}$ and $\vec{v}$.

Formulas (12) and (13) allow us to obtain simple expressions for the intensity of the x-ray TR from radiators in the form of a ball, an inclined plate, etc. In the case of a particle traversing a system of parallel interfaces the eikonal approximation (12) and (13) coincides practically with the approximate method of finding the x-ray TR suggested by Garibian.[15] In fact, if a particle intersects the boundary surface at $t=0$, then

$$\psi_\infty(t<0)=-v\omega t(\sqrt{\epsilon}-\sqrt{\epsilon_0}\sin^2\theta)/\cos\theta,$$

$$\psi_\infty(t>0)=-v\omega t\sqrt{\epsilon_0}\cos\theta.$$

It differs from Garibian's result,

$$\psi_\infty(t<0)=-v\omega t(\epsilon-\epsilon_0\sin^2\theta)^{1/2},$$

by small terms θ^2, $\epsilon-\epsilon_0$ of relative order γ_0^{-2} at $t<0$. Moreover, the usual way of eikonal approximation improvement—the integration in ψ is performed not over a straight line but over a "classical" ray trajectory (i.e., with due regard for refraction at a boundary)—leads in the given case exactly to Garibian's method of calculations. But there is no necessity to make such a complication because it can be shown that both the simple and improved eikonal approximations (just as a quasiclassical approximation and Garibian's method) have the same accuracy: All of these methods give correctly on the first term of the $dS/d\omega d\Omega$ expansion in powers of quantities which are small in the relativistic case, $\theta^2\sim|\epsilon-\epsilon_0|\sim\gamma_0^{-2}$. With due regard for this fact we can use the simple eikonal approximation in the essential angle and frequency range $\theta^2\sim|\epsilon-\epsilon_0|\sim\gamma_0^{-2}\ll1$ and omit other corrections of order γ_0^{-2} [by assuming, for example, $\sin\theta\approx\theta$ in (12)].

One can show that the transverse dimensions of the inhomogeneities which play an effective role in making TR take on values of order $\sigma^{-1/2}$ (see also Ref. 13). So under the condition (3a), TR from any inhomogeneity with piecewise-constant $\epsilon(\vec{r})$ coincides within the accuracy considered with TR from the plate as thick as the part of the particle trajectory within the inhomogeneity.

III. ENERGY LOST BY A PARTICLE TRAVERSING A PLATE

Let us calculate in the eikonal approximation the work $\mathcal{P}$ performed by a charged particle on the complete electromagnetic field in normal traversal of a plate with a thickness a. According to (5) the complete electric field is $\vec{E}=\vec{E}_0+\vec{E}_1$. The work $\mathcal{P}_0$ on the field $\vec{E}_0$ is equal to usual ionization loss and Cherenkov radiation in the medium ϵ_0. The work $\mathcal{P}_1$ on the field $\vec{E}_1$ is equal to

$$\mathcal{P}_1=-e\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}d\omega\,\vec{E}_1(x=y=0,z,\omega)\vec{v}\exp(-i\omega z/v)v^{-1}. \tag{14}$$

Using (11), we calculate

$$\eta(\vec{r},\vec{q}),$$

$$\eta(z<0)=0,$$

$$\eta(0\leqslant z<a)=\epsilon_1 v\omega[\exp(i\alpha z)-1]/2\alpha,$$

$$\eta(z\geqslant a)=\epsilon_1 v\omega\exp[i\alpha_0(z-a)][\exp(i\alpha a)-1],$$

$$\alpha=(\omega\sqrt{\epsilon}-q)qv/\omega,\quad \alpha_0=(\omega\sqrt{\epsilon_0}-q)qv/\omega.$$

After the substitution of these expressions into (10) and (14) and the integration over z we obtain

$$\mathcal{P}_1=\frac{ie^2}{\pi v^2}\int_{-\infty}^{\infty}\omega\,d\omega\int_0^{\kappa_{\max}}\kappa\,d\kappa\left(\frac{1-\epsilon v^2}{\epsilon\Lambda}-\frac{1-\epsilon_0 v^2}{\epsilon_0\Lambda_0}\right)\left[a+[1-\exp(i\alpha a)]\left(\frac{i\xi}{\alpha_0}-\frac{i}{\alpha}\right)\right]\mu,$$

$$\Lambda=\vec{q}^{\,2}-\epsilon\omega^2,\quad \mu=\frac{q+\omega\sqrt{\epsilon}}{2q}, \tag{15}$$

$$\xi=\frac{\epsilon}{\epsilon_0}\,\frac{1+[\omega(\epsilon_0 v^2-1)/v\kappa]^2}{[1+\omega^2(\epsilon_0 v^2-1)(\epsilon v^2-1)/v^2\kappa^2]}.$$

The first term $\mathcal{P}_{\rm ion}(a)$ in (15) proportional to a is equal to the difference between ionization loss of energy and Cherenkov radiation in the medium ϵ at the distance a and the same magnitude in the medium ϵ_0 with an accuracy to the factor μ. The extra factor μ, connected with the eikonal approximation, differs greatly from unity when $|\vec{q}|\gg|\omega\sqrt{\epsilon}|$. The influence of the factor μ on the integral over κ is not important because high values of κ give only a logarithmic contribution to $\mathcal{P}_{\rm ion}(a)$. Explicit calculation shows that the error introduced by μ is below 2%. Therefore we assume $\mu=1$ hereafter.

The second item $\mathcal{P}_{\rm tr}(a)$ in (15) describes the total transition effect, i.e., the TR and corrections to the local ionization loss of energy near the boundary surfaces of two media.[16] In accordance with what has been said about the accuracy of the eikonal approximation, we can take $\mathcal{P}_{\rm tr}(a)$ in a simpler form by keeping only the terms essential to the relativistic case; namely, let us assume $v\approx 1$, $\xi=1$, $\mu=1$, $\alpha=-\Lambda/2\omega$, $\alpha_0=-\Lambda_0/2\omega$. Then

$$\mathcal{P}_{\rm tr}(a)=2e^2\int_{-\infty}^{\infty}\omega^4 d\omega\int_0^{\infty}\kappa^3 d\kappa\,\epsilon_1^{\,2}[1-\exp(i\alpha a)]/\pi\Lambda_0^{\,2}\Lambda^2. \tag{16}$$

The integration in (16) is also extended to the frequencies $\omega\sim\omega_i$ when the conditions (2)–(4) can be disturbed. However, one can see that the corresponding error in (16), as well as in all following integral magnitudes, has the order of γ_0^{-1}.[17]

In the frequency range where ϵ and ϵ_0 are real the change to integration over positive frequencies in (16) leads to the relativistic limit of the usual formula for the total energy of the x-ray TR. The corresponding part of the work $\mathcal{P}_{\rm tr}(a)$ is emitted in the radiation.

It should be interesting to calculate (16) for arbitrary complex ϵ as Imϵ may be of importance for heavy substances in the region of frequencies considered. The most correct way of calculating the integrals of the type (16) is to take account of analytic properties of the dielectric constant ϵ as a function of complex ω. But it is impossible to employ the usual method of such calculations (closing of the integration path over ω through the upper semiplane) in (16) owing to an unlimited growth of the factor $\exp(i\alpha a)$ at $\omega\to i\infty$. The closing of the path through the lower semiplane is inconvenient owing to multiple singularities of the integrand. Let us note that it is altogether impossible to use the analytic properties of ϵ while calculating the total radiation: The corresponding integrand has the singularities over the whole complex plane ω because it includes both $\epsilon(\omega)$ and $\epsilon^*(\omega)=\epsilon(-\omega)$. Nevertheless, an attempt to calculate (16) analytically has besides esthetic grounds some truly pragmatic ones: The numerical calculations of (16) for complex ϵ are unwieldy and unreliable owing to the complicated form of ϵ in a range of frequencies where the absorption is substantial.

We propose the following way out of this situation. The exponential factor complicating the calculation is connected with the rigid definiteness of the plate thickness. If a radiator of the x-ray TR is described by a plate with the random thickness, then $\exp(i\alpha a)$ can be replaced by a factor more convenient for the calculations. For example, in order to describe random media of the foam plastic type one may use the gamma distribution,[18]

$$W_{K,\mu}(a) = (a/\mu)^K \exp(-a/\mu)/\mu K!, \quad K = 0, 1, 2, \ldots .$$

In this case when calculating the average work

$$\langle \mathcal{P}_{\rm tr}(K, \mu)\rangle_{\rm av} = \int_0^\infty W_{K,\mu}(a)\mathcal{P}_{\rm tr}(a)da$$

the exponential factor $\exp(i\alpha a)$ will be replaced by $(1 - i\alpha\mu)^{-K-1}$. Let us take the simplest distribution $W_{0,\mu}$ as an illustration, then $\langle a\rangle_{\rm av} = \mu$ and

$$\langle \mathcal{P}_{\rm tr}\rangle_{\rm av} = 2e^2 \int_0^\infty \omega^4 \epsilon_1^2 d\omega \int_0^\infty \kappa^3 d\kappa \Big/ \pi\Lambda_0^2\Lambda(\Lambda - 2i\omega\langle a\rangle_{\rm av}^{-1}). \tag{17}$$

The integrand in (17) has in the upper semiplane two single poles and one dual pole located on the imaginary axis. The calculation of (17) is performed similarly to Landau's method for calculation of the total energy loss of charged particles in a homogeneous medium.[19] Let us close the integration path over ω by an infinite semicircle in the upper semiplane, find residues of function in the poles (the dual pole is temporarily replaced by two close single poles), and change the integration over κ to an integration over the coordinates of the corresponding poles on the imaginary axis: $\omega = i\nu$. After regrouping the terms we obtain finally [the sign of the average value in $\langle a\rangle_{\rm av}$ will be omitted in (18)–(22) and in (24)–(26)]

$$\langle \mathcal{P}_{\rm tr}\rangle_{\rm av} = ae^2\left[-\int_{\nu_0}^{\nu_1} \nu d\nu(v^{-2} - \epsilon) + \int_{\nu_0}^{\nu_a} \nu d\nu(v^{-2} - \epsilon_a)(\epsilon - \epsilon_0)^2(\epsilon_a - \epsilon_0)^{-2}\right], \tag{18}$$

where $\epsilon_a(i\nu) = \epsilon(i\nu) + 2/a\nu$ and the integration limits are equal either to positive roots of the equations, $\epsilon_0(i\nu_0) = v^{-2}$, $\epsilon(i\nu_1) = v^{-2}$, $\epsilon_a(i\nu_a) = v^{-2}$, or to zero if the corresponding equation has no positive solution. The calculation $\langle \mathcal{P}_{\rm tr}\rangle_{\rm av}$ for $K \geq 1$ is performed in a similar manner.

Let us analyze (18) considering some particular cases. If ϵ and ϵ_0 can be taken as $\epsilon = 1 + \sigma/\nu^2$ and $\epsilon_0 = 1 + \sigma_0/\nu^2$ over the whole integration range in (18), then the calculation of the integrals gives

$$\langle \mathcal{P}_{\rm tr}\rangle_{\rm av} = ae^2\sigma\left[(1-\eta)\left(\frac{\xi\sqrt{\eta} - 1 + \eta}{\xi\sqrt{\eta} + X_a} - \frac{1}{2}\right) + \frac{(1-\eta)^2 + \xi^2}{\xi^2}\ln\left|\frac{1 - \eta + X_a}{1 - \eta + \xi\sqrt{\eta}}\right| - \ln\frac{X_a}{\xi}\right],$$
$$\eta = \sigma_0/\sigma, \quad \xi = 2\gamma/a\sqrt{\sigma}, \quad X_a = 2\nu_a/a\sigma = \xi[\xi/2 + (1 + \xi^2/4)^{1/2}]. \tag{19}$$

In any case formula (19) is applicable for

$$\gamma \gg \gamma_f = \max\left[\frac{\omega_i}{\sqrt{\sigma}}, \frac{\omega_i^0}{\sqrt{\sigma_0}}\right],$$

where the total ionization losses in the media ϵ and ϵ_0 reach the corresponding ionization plateaus. For $\xi \ll 1$ we obtain from (19) $\langle \mathcal{P}_{\rm tr}\rangle_{\rm av} = 2e^2\gamma(\sqrt{\sigma} - \sqrt{\sigma_0})^2/3(\sqrt{\sigma} + \sqrt{\sigma_0})$; this expression coincides with Garibian's result for two interfaces located at a distance much larger than the formation zone $2\gamma/\sqrt{\sigma}$.[4] For $\eta \ll 1$ and $\xi \sim \eta^{-1/2}$, Eq. (19) gives

$$\langle \mathcal{P}_{\rm tr}\rangle_{\rm av} = (ae^2\sigma/2)\{\ln[\xi^2(1 + \xi\sqrt{\eta})^{-2}] - 1\},$$
$$a\sigma/2\sqrt{\sigma_0} \sim \gamma \gg a\sqrt{\sigma}/2. \tag{20}$$

Finally, for $\xi \gg |1 - \eta|\eta^{-1/2}$, $\langle \mathcal{P}_{\rm tr}\rangle_{\rm av}$ reaches the plateau irrespective of a value of η,

$$\langle \mathcal{P}_{\rm tr}\rangle_{\rm av} = (ae^2\sigma/2)(-\ln\eta - 1 + \eta),$$
$$\gamma \gg a|\sigma - \sigma_0|/2\sqrt{\sigma_0}. \tag{21}$$

Formulas (20) and (21) give the ultrarelativistic limit of (18) in the general case. If $\sigma_0 = 0$, $\langle \mathcal{P}_{\rm tr}\rangle_{\rm av}$ has an unlimited logarithmic growth,

$$\langle \mathcal{P}_{\rm tr}\rangle_{\rm av} = (ae^2\sigma/2)(\ln\xi^2 - 1),$$
$$\gamma \gg a\sqrt{\sigma}/2. \tag{22}$$

In the above calculation of $\langle \mathcal{P}_{\rm tr}\rangle_{\rm av}$ the simplest gamma distribution $W_{0,\mu}(a)$ has been used. The average values $\langle \mathcal{P}(K, \mu)\rangle_{\rm av}$ at $K \geq 1$ can be obtained without new integration over ω and κ. Indeed, the distributions $W_{K,\mu}(a)$ satisfy the recurrence relations

$$W_{K+1,\mu}(a) = W_{K,\mu}(a) + \mu dW_{K,\mu}(a)/d\mu(K+1);$$

similar relations are fulfilled for $\langle \mathcal{P}(K, \mu)\rangle_{\rm av}$. Hence, $\langle \mathcal{P}(K, \mu)\rangle_{\rm av}$ are obtained from the abovementioned formulas for $\langle \mathcal{P}(0, \mu)\rangle_{\rm av}$ by means of a finite number of differentiations with respect to $\mu = \langle a\rangle_{\rm av}$. For example, at $K \geq 1$, instead of (22) we have

$$\langle \mathcal{P}_{\rm tr}(K,\mu)\rangle_{\rm av} = (\langle a\rangle_{\rm av} e^2\sigma/2)(\ln \xi_{\rm av}{}^2 - 1 + A_K)\,,$$

$$\langle a\rangle_{\rm av} = \mu(K+1);\ \xi_{\rm av} = 2\gamma/\langle a\rangle_{\rm av}\sqrt{\sigma}\ ;$$

$$A_K = 2\left[\ln(K+1) + 1 - \sum_{n=1}^{K+1} n^{-1}\right]\,, \qquad (22a)$$

$$\lim_{K\to\infty} A_K \approx 0.85\,.$$

As

$$\langle (a - a_{\rm av})^2\rangle_{\rm av}^{1/2}/a \xrightarrow[K\to\infty]{} 0$$

formula (22a) at $K\to\infty$ gives actually the work done in traversing a plate of fixed thickness. We can calculate the same quantity directly provided we interpret the integration with the function $\mu^{-1}\exp(-a/\mu)$ considered above as a Heaviside transformation. After an inverse transformation

$$\mathcal{P}(a) = \int_{\delta - i\infty}^{\delta + i\infty} (2\pi i P)^{-1}\langle \mathcal{P}(0,\mu = P^{-1})\rangle_{\rm av}\exp(Pa)da \quad (\delta > 0)$$

formula (22) turns into (22a) for $K\to\infty$, formula (21) does not change, and formula (20) turns into

$$\mathcal{P}_{\rm tr}(a) = (ae^2\sigma/2)\{\ln\xi^2 - 1 + 0.85 + 2{\rm Ei}(-\beta) - 2\beta^{-1}[1 - \exp(-\beta)]\}\,, \qquad (20a)$$

$$\beta = 1/\xi\sqrt{\eta}\ ,$$

where $\mathrm{Ei}(X)$ is an integral exponential function. The use of Heaviside's inverse transformation in (20)–(22) applicable to the part of the real axis $P = \mu^{-1}$ provides the coincidence of the range of applicability of formulas (20)–(22) and the corresponding formulas (20a)–(22a) (with the change $\langle a\rangle_{\rm av}\to a$).

The aforementioned formulas show clearly the relation between the total transition effects and total ionization loss (including Cherenkov radiation). $\langle \mathcal{P}_{\rm tr}\rangle_{\rm av}$ has the same logarithmic growth as $\langle \mathcal{P}_{\rm ion}\rangle_{\rm av}$ in (15),

$$\langle \mathcal{P}_{\rm ion}\rangle_{\rm av} = (\langle a\rangle_{\rm av} e^2\sigma/2)[\ln(2m\gamma^2 E_{\rm max}/\Omega^2) - v^2 + \nu_1{}^2/\sigma\gamma^2]\,, \qquad (23)$$

where

$$\ln\Omega^2 = \sum_i f_i \ln(\nu_1{}^2 + \omega_i{}^2)\,,$$

f_i are the oscillator strengths of the atoms in the medium ϵ, m is the electron mass, $E_{\rm max}$ is the maximum registered energy transferred during one collision of the particle with an atom (in track detectors $E_{\rm max}\sim 10$–100 keV). For simplicity the total ionization loss in the medium ϵ_0 considered to be constant is omitted. The ionization plateau is reached in $\langle \mathcal{P}_{\rm ion}\rangle_{\rm av}$ at $\gamma \gtrsim I/\sqrt{\sigma}$ (I is the average ionization potential of the medium ϵ) and the logarithmic dependence in $\langle \mathcal{P}_{\rm tr}\rangle_{\rm av}$ begins at $\gamma \sim a\sqrt{\sigma}/2$ (if $\sigma_0 \ll \sigma$). If $I/\sqrt{\sigma} < a\sqrt{\sigma}/2$, the total work $\langle \mathcal{P}_1\rangle_{\rm av} = \langle \mathcal{P}_{\rm ion} + \mathcal{P}_{\rm tr}\rangle_{\rm av}$ at $I/\sqrt{\sigma} < \gamma < a\sqrt{\sigma}/2$ has a limited plateau after which the logarithmic growth resumes, and for the second time reaches the plateau at $\gamma > a\sigma/2\sqrt{\sigma_0}$ (see Ref. 4). If $I/\sqrt{\sigma} > a\sqrt{\sigma}/2$, then $\langle \mathcal{P}_1\rangle_{\rm av}$ has the uninterrupted logarithmic growth. These remarks are illustrated in Fig. 1 where the value $2\langle \mathcal{P}_1\rangle_{\rm av}/e^2 a\sigma$ calculated according to formulas (18) and (23) for $\epsilon_0 = 1$,

$$\epsilon(i\nu) = 1 + \sigma/(\nu^2 + I^2)$$

as a function of $\sqrt{\sigma}\gamma/I$ at different values of the parameter $U = 2I/a\sigma$ is given. The additive parameter is

$$\ln(2mE_{\rm max}/\sigma) = 20\,.$$

If $\epsilon_0(i\nu) = 1 + \sigma_0/\nu^2$, the work $\langle \mathcal{P}_1\rangle_{\rm av}$ reaches the plateau determined by formula (21) and shown in Fig. 1 by the dashed line for $\sigma/\sigma_0 = 10^4$. In conclusion of this section let us note that the estimation of plate thickness, where the density effect is suppressed, given in Ref. 4, means actually $U \gg 1$; a violation of this condition does not abolish the logarithmic growth of $\langle \mathcal{P}_1\rangle_{\rm av}$ but shifts it only to a range of greater γ (see Fig. 1). At the same time we can show that under $U \gg 1$ the work $\langle \mathcal{P}_1\rangle_{\rm av}$ is for the most part the correction to the ionization capability of the particle near the plate front boundary $\langle \mathcal{P}_1\rangle_{\rm av} \simeq a\Delta(dE/dz)|_{z\to 0}$.[16] In this case TR takes on a value of order $ae^2\sigma(a\sigma/I)$ and is very small. On the contrary, for $U \ll 1$ the whole correction to the particle ionization capability takes on a value of the order of $e^2 I$, $\langle \mathcal{P}_1\rangle_{\rm av}$ with the same accuracy coincides with TR.

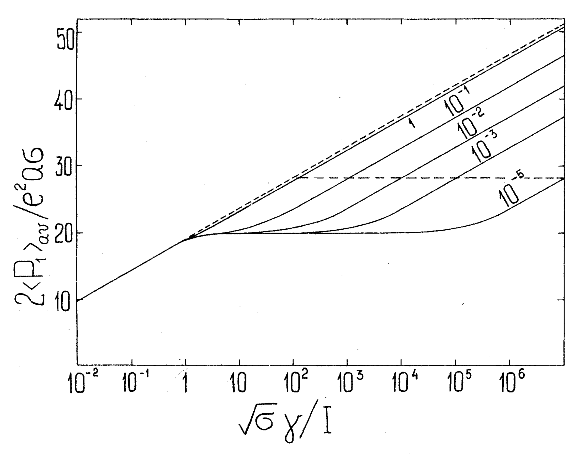

FIG. 1. The average total work $\langle \mathcal{P}_1\rangle_{\rm av}$ performed by a charged particle in normal traversal of a plate in vacuum as a function of $\sqrt{\sigma}\gamma/I$ at different values of $U = 2I/a\sigma$. The dashed line is the plateau for $\sigma/\sigma_0 = 10^4$.

IV. ENERGY LOST BY A CHARGED PARTICLE IN A RANDOM STACK OF PLATES AND IN A RANDOM MEDIUM

Let a relativistic charged particle traverse normally a stack of N parallel plates, and let thicknesses $\{a_i\}$ and spacing $\{b_i\}$ be random. Similarly to Sec. II we receive for the average $\langle \mathcal{P}_1\rangle_{\text{av}}$ and $\langle \mathcal{P}_{\text{tr}}\rangle_{\text{av}}$ formulas (15) and (16) with the substitution

$$a \to N\langle a\rangle_{\text{av}},$$

$$[1-\exp(i\alpha a)] \to S = N(1-h_a h_b)^{-1}(1-h_a)(1-h_b) + (1-h_a h_b)^{-2}(1-h_a)^2(1-h_a^N h_b^N)h_b,$$

where

$$h_a = \langle \exp(i\alpha a)\rangle_{\text{av}},$$

$$h_b = \langle \exp(i\alpha_0 b)\rangle_{\text{av}}.$$

In a range of frequencies where ϵ and ϵ_0 are real the change to integration over positive frequencies in these formulas leads to the formulas for TR obtained from a random stack of plates.[18]

The second term in S describes the transition effect owing to the finiteness of stack, and for $N \gg 1$ it can be omitted. The first term is equal to the transition work in an infinite stack and after averaging over the distributions $W_{0,\mu}(a)$ and $W_{0,\mu'}(b)$ we obtain for the work within one period a formula similar to (17) with the other pole denominator

$$\Lambda\Lambda_0[\Lambda\Lambda_0 - 2i\omega(\Lambda\langle b\rangle_{\text{av}}^{-1} + \Lambda_0\langle a\rangle_{\text{av}}^{-1})].$$

The further calculations are similar to the above mentioned ones and give the following formula instead of (18):

$$\langle \mathcal{P}_{\text{tr}}\rangle_{\text{av}} = e^2\left(b\int_{\nu_0}^{\nu_2} \nu\, d\nu(v^{-2}-\epsilon_0) - a\int_{\nu_2}^{\nu_1} \nu\, d\nu(v^{-2}-\epsilon) + 2\int_{\nu_2}^{\nu_3} d\nu(v^{-2}-\epsilon_3)(\epsilon-\epsilon_0)^2/(\epsilon_3-\epsilon_0)(\epsilon_3-\epsilon_2)(\epsilon_3-\epsilon)\right), \quad (24)$$

where ϵ_2 and ϵ_3 are found from the quadratic equation

$$(\epsilon_{2,3}-\epsilon)(\epsilon_{2,3}-\epsilon_0) = 2(\epsilon_{2,3}-\epsilon_0)/a\nu + 2(\epsilon_{2,3}-\epsilon)/b\nu,$$

and the integration limits ν_2 and ν_3 are equal either to positive roots of the equations $\epsilon_{2,3}(i\nu_{2,3}) = v^{-2}$, or to zero if the corresponding equation has no positive solution. At $b \to \infty$ formula (24) turns into (18) because $\epsilon_3 \to \epsilon_a$, $\epsilon_2 \to \epsilon_0$, $\nu_3 \to \nu_a$, $\nu_2 \to \nu_0$ and the first integral in (24) vanishes.

In the ultrarelativistic limiting case $\gamma \gg \gamma_f$ one may assume $\epsilon = 1+\sigma/\nu^2$, $\epsilon_0 = 1+\sigma_0/\nu^2$ in the integrand of (24). In this case the calculation of (24) gives a rather large formula, where ν_2 and ν_3 are found from an equation of degree four. At $\xi \ll 1$ the known expression for TR from two interfaces results from this formula (see Sec. III). At

$$1 \ll \xi \sim \sqrt{\sigma}/\sqrt{\langle\sigma\rangle} \sim \sqrt{b}/\sqrt{a}$$

we obtain the following formula similar to (20):

$$[\langle\sigma\rangle = (a\sigma + b\sigma_0)/(a+b)]$$

$$\langle \mathcal{P}_{\text{tr}}\rangle_{\text{av}} = \frac{e^2}{2}\left[a\sigma\left(\ln\frac{\xi^2}{(X_2+1)^2} - 1\right) + b\sigma_0\left(\ln\frac{\eta\xi^2}{X_2^2} - 1\right) + \frac{b\sigma X_2{}^2}{\xi^2}\right],$$

$$\frac{a\sigma}{2\sqrt{\langle\sigma\rangle}} \sim \gamma \gg \frac{a\sqrt{\sigma}}{2}, \quad (25)$$

$$X_2 = \frac{2\nu_2}{a\sigma} = \frac{\xi\sqrt{\langle\sigma\rangle}}{\sqrt{\sigma}} - \frac{a\sigma}{2b\langle\sigma\rangle}.$$

Under

$$\gamma \gg \frac{|\sigma-\sigma_0|ab}{2(a+b)\sqrt{\langle\sigma\rangle}},$$

$\langle \mathcal{P}_{\text{tr}}\rangle_{\text{av}}$ reaches the plateau irrespective of the values of a/b and $\langle\sigma\rangle/\sigma$

$$\langle \mathcal{P}_{\text{tr}}\rangle_{\text{av}} = (e^2/2)[a\sigma\ln(\sigma/\langle\sigma\rangle) + b\sigma_0\ln(\sigma_0/\langle\sigma\rangle)]. \quad (26)$$

In contrast to $\langle \mathcal{P}_{\text{tr}}\rangle_{\text{av}}$ in (22) $\langle \mathcal{P}_{\text{tr}}\rangle_{\text{av}}$ in (25) reaches the plateau

$$(e^2 a\sigma/2)\ln[(a+b)/a]$$

at $\sigma_0 = 0$,

$$\gamma \gg b\sqrt{a\sigma}/2(a+b)^{1/2}.$$

For illustration the value $2\langle \mathcal{P}_{\text{tr}}\rangle_{\text{av}}/e^2 a\sigma$ calculated according to formula (24) for $\epsilon_0 = 1$, $\epsilon(i\nu) = 1+\sigma/\nu^2$ as a function of $\xi = 2\gamma/a\sqrt{\sigma}$ at different values of the parameter $t = a/(a+b)$ if given in Fig. 2.

The values $\langle \mathcal{P}_{\text{tr}}\rangle_{\text{av}}$ calculated according to formulas (18) and (24) do not practically depend on the model representation of ϵ and ϵ_0. At the same time the division of these TR magnitudes into the total energy of the x rays and the corrections to the ionization loss near the boundary surfaces depends substantially on the form of ϵ and ϵ_0. A similar fact is well known for Cherenkov radiation and ionization loss in the homogeneous medium. The relativistic limits (19) and (25) of the corresponding formulas and the ionization plateau for the total loss in the homogeneous medium do not include any characteristics of substance at all, except the plas-

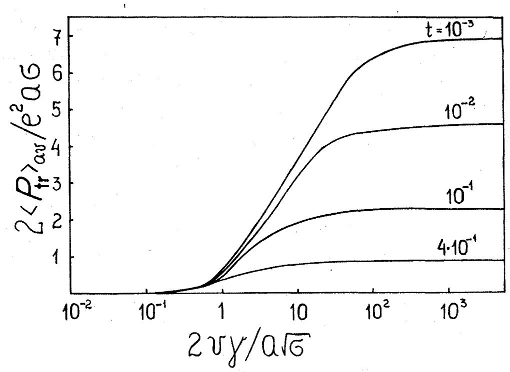

FIG. 2. The average transition work $\langle \mathcal{P}_{tr}\rangle_{av}$ performed by a charged particle in a random medium within one period as a function of $2\gamma/a\sqrt{\sigma}$ at different values of $t=a/(a+b)$.

ma frequency.

In this paper the principal attention is paid to the calculation of the work $\langle \mathcal{P}_1\rangle_{av}$ because of the possibility to use analytic methods here. The formulas obtained are the upper limits TR. Besides, in some experiments one observes the total energy released in an inhomogeneous medium rather than the energy emitted by a radiator. For example, in Ref. 8 the experimental setup is suggested and investigated in which a paraffin rod with superconductive tin granules embedded into it serves simultaneously as a radiator and as a detector. The number of granules N_g converted to the normal state by the passage of the particle is measured. The relation between N_g and energy loss or, possibly, the number of collisions of the charged particle in the range of frequencies 10–100 keV is highly complicated. At any rate, the spectrum of collisions $dN/d\omega$ of the particle with the medium taking into account both the interference phenomena (influence of the formation zone) and the influence of absorption, especially at K electrons, should be the input for the N_g calculation. The spectrum used above for deriving formulas (24) and (25) can be a good approximation for $dN/d\omega$:

$$dN/d\omega = 4e^2\omega^3\,\mathrm{Re}\left\{\epsilon_1^2\int_0^\infty \kappa^3 d\kappa/\pi\Lambda\Lambda_0[\Lambda\Lambda_0 - 2i\omega(\Lambda_0\langle a\rangle_{av}^{-1} + \Lambda\langle b\rangle_{av}^{-1})]\right\}.$$

The authors are grateful to B. M. Bolotovsky, G. G. Bagratuny, G. M. Garibian, and V. K. Ermilova for fruitful discussions.

[1] J. Cobb, C. W. Fabian, S. Iwata, C. Kourkoumelis, A. J. Lankford, G.-C. Moneti, A. Nappi, R. Palmer, P. Rehak, W. Struczinski, and W. W. Willis, Nucl. Instrum. Methods 140, 413 (1977).

[2] T. Prince, D. Muller, G. Hartmann, and M. Cherry, Nucl. Instrum. Methods 123, 231 (1975).

[3] V. L. Ginzburg and I. M. Frank, J. Phys. USSR 9, 353 (1945).

[4] G. M. Garibian, Zh. Eksp. Teor. Fiz. 37, 527 (1959) [Sov. Phys.—JETP 10, 372 (1960)].

[5] M. L. Ter-Mikaelian, *High-Energy Electromagnetic Processes in Condensed Media* (Wiley, New York, 1972).

[6] C. W. Fabian, Nucl. Instrum. Methods 146, 343 (1977).

[7] J. E. Bateman, Nucl. Instrum. Methods 103, 565 (1972).

[8] A. K. Drukier and L. C. L. Yuan, Nucl. Instrum. Methods 138, 213 (1976).

[9] One can take for ϵ_0 a volume average value $\langle \epsilon(\vec{r})\rangle_{av}$ or the dielectric constant of gaps between inhomogeneities.

[10] Such diffuseness of the boundary does not influence the TR; see J. V. Lepore and R. J. Riddell, Phys. Rev. D 13, 2300 (1976).

[11] G. M. Garibian and C. Yang, Zh. Eksp. Teor. Fiz. 61, 930 (1971) [Sov. Phys.—JETP 34, 495 (1972)].

[12] M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964).

[13] L. Durand, Phys. Rev. D 11, 89 (1975).

[14] It is not correct to suppose $|\vec{r}|\to\infty$ in (11), because a range of its applicability is $|\vec{r}| \ll a^2\omega$.

[15] G. M. Garibian, Zh. Eksp. Teor. Fiz. 39, 332 (1960) [Sov. Phys.—JETP 12, 237 (1961)].

[16] V. A. Chechin, Dokl. Akad. Nauk USSR 221, 813 (1975) [Sov. Phys.-Doklady 20, 269 (1975)].

[17] The first two terms of (16) expansion in powers of a are equal within the accuracy considered to the corresponding terms of the exact expression for the work $\mathcal{P}_{tr}$ (see Ref. 4).

[18] G. M. Garibian, L. A. Gevorkian, and C. Yang, Nucl. Instrum. Methods 125, 133 (1975).

[19] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, New York, 1960).