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Mass divergences of wide-angle scattering amplitudes

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The mass divergences of wide-angle scattering amplitudes are examined in a class of field theories including those with elementary vector fields. Power-counting arguments are used to show that, apart from renormalization effects, the wide-angle gauge-singlet scattering amplitude has a well-defined zero-mass limit and hence satisfies the renormalization-group equations. The same techniques also reproduce the earlier results for Yukawa and ϕ^4 theories.

I. INTRODUCTION

In recent years there have been several attempts[1-4] to apply renormalization-group techniques to study wide-angle elastic scattering where all the external particles are on their mass shell. In this approach, the high-energy limit of the scattering amplitude is governed by the zero-mass singularities of the theory. This is explicitly seen in the work of Ref. 4. These authors show, using the improved renormalization-group equations, that asymptotically the on-shell T-matrix elements in the fixed-angle limit can be written as the product of (i) a function which has a well-defined zero-mass limit and (ii) the finite wave-function renormalization associated with the external on-shell particles to which the zero-mass singularities are confined. The relevance of this result to soft symmetry breaking in the fixed-angle limit is pointed out in Ref. 4. In Ref. 3, renormalization-group techniques are used to investigate whether the "quark counting" rules of Brodsky and Farrar[5] and Matveev *et al.*,[5] can be extracted from field theory. All these applications are in the context of field theories without fundamental vector fields and rely on the existence of a smooth zero-mass limit of the underlying field theory. It is therefore relevant to ask whether the limit $m \to 0$ exists in perturbation theory calculations, for scattering amplitudes in field theories involving elementary vector fields.

The purpose of this paper is to study mass divergences of the four-point scattering amplitude in the fixed-angle regime by directly examining the momentum-space behavior of Feynman integrals. It is found that, apart from renormalization effects, the scattering amplitude to all orders in perturbation theory has a well-defined zero-mass limit and hence satisfies the renormalization-group equations. This result is obtained for the case that the external particles can be described by canonical fields. It is valid for a large class of field theories including gauge theories *if* the external particles are of "neutral" charge. In obtaining this result, the behavior of mass divergences in totally massless field theories is estimated by using the power-counting procedure recently developed by Sterman.[6] The extension of this procedure to scattering processes has been recently studied by Libby and Sterman.[7] Previously, the zero-mass limit of wide-angle scattering amplitudes in φ^4 and in Yukawa theories has been studied by a number of authors.[1,2,8-10] High-energy color-singlet-color-singlet has been considered in ref. (11). It is shown by an explicit calculation up to sixth order in the quark-gluon coupling constant that the amplitude is infrared finite to this order.

In Sec. II, Sterman's power-counting procedure is reviewed and applied to the four-particle scattering amplitude. The result for scalar and Yukawa theories follows immediately from these power-counting arguments. For gauge theories, however, naive power counting suggests that the logarithmically divergent momentum configurations involve an arbitrary number of soft vector lines that attach to the external "self-energies" at three-point vertices in all possible ways. In Sec. III we consider the case of gauge theories in greater detail. We use arguments similar to Ref. 12 to show that if we consider gauge-singlet scattering, there is a suppression with respect to the naive power-counting estimates of Sec. II whenever there are soft vector particles associated with the amplitude. In Ref. 12 these arguments were used to show the finiteness of gauge-singlet production amplitudes in quantum chromodynamics (QCD). Using similar arguments we show that the "photon-photon" scattering amplitudes in massless QED and QCD also satisfy the renormalization-group equations.

II. POWER COUNTING FOR FOUR-PARTICLE SCATTERING AMPLITUDES

Consider a one-particle-irreducible (1PI) diagram which is given as an integral over internal loop momenta of products of Feynman propagators and vertex functions. We assume henceforth that the theory has been rendered ultraviolet finite by a convenient off-shell renormalization procedure. This integral therefore diverges only at those points where the Feynman denominators vanish. Such points are called singular points. Of these singular points only those where the undeformed contours are pinched (called pinch singular points) can give rise to mass singularities.[6] For each 1PI diagram, at the pinch singular point, a reduced diagram can be constructed by contracting all off-shell lines to a point. The internal lines of such a reduced diagram can be divided into sets of zero-momentum lines $\{k_i\}$ and finite-momentum lines $\{q_i\}$. Singularities from the Feynman denominators of zero-momentum lines always trap the contour at $k_i=0$. For finite-momentum lines, the contour is trapped whenever the Landau equations are satisfied for some choice of Feynman parameters α_j, i.e.,

$$q_j^2=m_j^2, \quad \sum \alpha_j q_j=0.$$

The sum is over all independent loops of finite-energy lines. Coleman and Norton[13,6] have given a physical interpretation to reduced diagrams at pinch singular points. According to them, all such reduced diagrams described physically realizable scattering processes in which particles propagate freely between vertices and energy always flows forward in time. Each vertex is associated with a space-time point, and the separation between two vertices is proportional to $\alpha_i q_i$. The vertices of a reduced diagram are now divided into two classes: soft vertices and hard vertices. A vertex is soft only if all the lines it connects are at threshold in incoming and outgoing channels and is otherwise hard. A soft vertex preserves the direction of momentum flow whereas a hard vertex does not. At a soft vertex, the number of constituents of a "jet" of parallel-moving lines may change. The energy and momenta of each such jet, however, is the same before as after the action of the soft vertex, where the number of jets is also individually conserved. At a hard vertex, the jets may change directions as well as their number.

The power-counting procedure[6] is based on identifying a minimal set of variables necessary to put all lines on shell at a pinch singular point. These variables, called normal variables, are defined below. Consider a Feynman integral and make a change of variables. The new variables are divided into two categories: intrinsic variables and normal variables. The normal variables are such that they vanish at the pinch singular point and the Feynman denominators can be expressed as a homogeneous function of these variables by suppressing all terms quadratic in them relative to the linear terms. Thus, for power counting only the normal variables are relevant. For zero-momentum lines, the normal variables are the four components of the internal loop momenta. At the pinch singular point, all the lines of a jet are on-shell *and* parallel to each other. Using this, together with the fact that there is an overall freedom of direction, gives the number of normal variables associated with jet momenta as 2× (number of independent finite momenta) −1. This includes the contribution from the independent loop momenta for the internal jet loops and the independent normal variables associated with the external momenta of each jet. It is important to observe that when we consider a general scattering process all the normal variables associated with external jet momenta are not necessarily independent. This question will be discussed in detail later.

Consider a reduced diagram of a 1PI diagram at the pinch singular point. Suppose now that the normal variables have been identified and the homogeneous integral constructed. For power-counting purposes the relevant differentials are those corresponding to the normal variables as opposed to those corresponding to the intrinsic variables. Let P denote the difference in the scaling behavior of all the relevant momentum factors in the numerator and in the denominator. Mass divergences can exist if $P \leq 0$. Let J = number of finite energy lines in the reduced diagram, S = number of zero-momentum lines, L^s, L^j = number of soft loops and internal jet loops, respectively, and N = contribution from momentum factors in the numerator from propagators and vertices. As mentioned earlier, not all the normal variables associated with the external jet momenta are independent, whereas only an independent set is relevant for power counting. For this purpose, a quantity D is defined thus: λ^D is defined to be the volume in loop-momentum space where the absolute value of each normal variable for the external jet lines is of the order of a small parameter λ. When a change of variables from loop momenta to normal variables is made, λ^D, in the space of normal variables, defines the scaling behavior of the volume element multiplied by the Jacobian of the transformation. Then we can write

$$p = 4L^s + 2L^j + D - (2S + J) + N. \quad (1)$$

We next separate the effects of zero-momentum and finite-energy lines. For the zero-momentum lines we use dimensional arguments and for finite-energy lines we separate out the contributions from each jet to write

$$p = \sum_{i=1}^{K} (2l_i^j - j_i + n_i^j) + D + b + \tfrac{3}{2}f, \quad (2)$$

where K is the number of jets in the reduced diagram. The notation is obvious for the jets, and b and f are the number of soft boson and fermion lines, respectively. Now,

$$j_i = \tfrac{1}{2}\Big(\sum_{\alpha\geq 3} \alpha X_{i\alpha} + \sum_{\beta\geq 2} \beta Y_{i\beta} + \gamma_i\Big) - \delta_i, \quad (3)$$

$$V_i = \sum_{\alpha\geq 3} X_{i\alpha} + \sum_{\beta\geq 2} Y_{i\beta} + (W_i + K_i) - \delta_i, \quad (4)$$

$$l_i = j_i - V_i + \Delta_i, \quad (5)$$

where $\delta_i = 1$ if the ith jet contains an external line of the reduced diagram and is zero otherwise.

γ_i = total number of finite energy external lines of the ith jet.

W_i = the number of times that only one finite-energy external line of the ith jet is attached to a given hard vertex.

V_i = number of vertices of the ith jet (including the hard vertices to which jet lines may attach).

K_i = the number of times that more than one finite-energy external line of the ith jet line is attached to a single hard vertex.

Note that $\gamma_i \geq W_i$, $K_i \geq 0$. Further, $\gamma_i \geq (W_i + K_i)$, and $\gamma_i = W_i$ only when $K_i = 0$.

$X_{i\alpha}$ = total number of soft vertices in the ith jet with α jet lines attached.

$Y_{i\beta}$ = number of soft vertices in the ith jet with β jet lines and one or more soft lines attached.

Equation (5) is the Euler identity.

$\Delta_i = 1$ if the jet is completely connected, if not, Δ_i = number of disconnected pieces of the ith jet.

Equation (2) can now be written as

$$p = \tfrac{1}{2} \sum_{i=1}^{K} \Big(\sum_{\alpha\geq 3} (\alpha - 4) X_{i\alpha} + \sum_{\beta\geq 2} (\beta - 4) Y_{i\beta} + 4\Delta_i - 3\gamma_i + 4(\gamma_i - W_i - K_i) + 2n_i^j\Big) + (D+4) + b + \tfrac{3}{2}f. \quad (6)$$

For φ^4 theory, $\alpha \geq 4$, $\beta \geq 3$, and $n_i^j = 0$ automatically and hence

$$p = \tfrac{1}{2} \sum_{i=1}^{K} \Big(\sum_{\alpha\geq 4} (\alpha - 4) X_{i\alpha} + \sum_{\beta\geq 3} (\beta - 4) Y_{i\beta} + 4\Delta_i - 3\gamma_i + 4(\gamma_i - W_i - K_i)\Big) + (D+4) + b + \tfrac{3}{2}f. \quad (7)$$

For Yukawa theory it was shown in Ref. 6 that there exists a lower bound on n_i^j, i.e.,

$$n_i^j \geq \tfrac{1}{2} X_{i3} + \tfrac{1}{2} Z_i^{(0)}, \quad (8)$$

where $Z_i^{(0)}$ is the number of soft scalars emitted at three-point vertices. For gauge theories also, it was shown that the above equation is valid for each graph individually, provided one does the power counting in a noncovariant gauge in which the gauge-fixing term is

$$\frac{i}{2\alpha} (e^{i\theta} \partial_0 A_0 - \vec{\partial} \cdot \vec{A})^2$$

and $0 < \theta < \frac{1}{2}\pi$. With this choice of gauge, as will be seen below, each reduced diagram is at worst logarithmically divergent on a graph by graph basis even in gauge theories. Notice that in gauge theories there is no suppression associated with the vertex in which a soft vector is emitted from a finite-energy line.

Using these and the inequalities

$$\tfrac{1}{2}(b + f) \geq \tfrac{1}{2} \sum_{i=1}^{K} (Y_{i3} + Y_{i2}), \quad (9)$$

$$\tfrac{1}{2} f \geq \tfrac{1}{2} \sum_{i=1}^{K} (Y_{i2} - Z_i^{(0)} - Z_i^{(1)}), \quad (10)$$

where $Z_i^{(1)}$ are the number of soft vectors emitted in ith jet at three-point vertices, we obtain

$$p \geq \tfrac{1}{2} \sum_{i=1}^{K} \Big(\sum_{\alpha\geq 4} (\alpha - 4) X_{i\alpha} + \sum_{\beta\geq 4} (\beta - 4) Y_{i\beta} + 4\Delta_i - 3\gamma_i + 4(\gamma_i - W_i - K_i)\Big) + (D+4) + \tfrac{1}{2} \sum_{i=1}^{K} (b_i^{(0)} + b_i^{(1)} - Z_i^{(1)}) + \tfrac{1}{2} f, \quad (11)$$

where $b_i^{(0)}$ and $b_i^{(1)}$, respectively, denote the total number of soft scalars and vectors attached to jet lines of the ith jet. From (11) we see that the most singular case would correspond to the situation when $\Delta_i = 1$, $\gamma_i = W_i$, and $K_i = 0$, i.e., there are no disconnected jets, and all the lines that meet at a given hard vertex belong to different jets. For this case, which is the most singular configuration, e.g. (11) becomes

$$p \geq \tfrac{1}{2} \sum_{i=1}^{K} \Big(\sum_{\alpha\geq 4} (\alpha - 4) X_{i\alpha} + \sum_{\beta\geq 4} (\beta - 4) Y_{i\beta} - 3W_i + 4\Big) + (D+4) + \tfrac{1}{2} \sum_{i=1}^{K} (b_i^{(0)} + b_i^{(1)} - Z_i^{(1)}) + \tfrac{1}{2} f, \quad (12a)$$

and for φ^4 field theory we have

$$p \geq \frac{1}{2}\sum_{i=1}^{K}\Big(\sum_{\alpha\geq 4}(\alpha-4)X_{i\alpha} + \sum_{\beta\geq 4}(\beta-4)Y_{i\beta} - 3W_i + 4\Big) + (D+4) + \frac{1}{2}\sum_{i=1}^{K} b_i^{(0)} + f\,. \tag{12b}$$

We now come to the most significant part of this paper. We show with reference to Eqs. (12) that, at a pinch singular point, for all reduced diagrams except those where scattering takes place at one hard vertex.

$$D > \sum_{i=1}^{K}\left[\tfrac{3}{2}(W_i - 2) + 1\right] - 4, \tag{13}$$

and that the logarithmically divergent configurations are only those where scattering takes place at a single hard vertex. We will ignore soft lines for purposes of this argument. This is justified for the following reason: It is clear from equations (11) and (12) that the soft lines are "decoupled" from the jet lines for purposes of power counting. It will be seen later [see Eq. (15)] that the condition for logarithmic divergence is that there are *only* soft vector lines which attach to jet lines at three-point vertices. If we ignore soft lines (and the three-point soft vertices where they attach to jet lines) then the momenta flowing through the jet lines are unchanged even though the number of internal jet lines will be reduced. Thus, if we are interested in finding the minimum number of independent normal variables associated with external jet momenta we are not overcounting if we exclude soft lines from the argument.

Consider the set of ordered reduced diagrams, R_p, of a four-particle scattering process. An ordered diagram is defined to be one in which energy always flows forward in time and (equivalently) there are no subdiagrams in which energy flows around in a loop. By the Coleman-Norton theorem, a reduced diagram of a scattering process at the pinch singular point is an ordered diagram. Hence these are included in R_p.

We now proceed to show the validity of inequality (13). The first step is the construction of a "jet reduced" diagram. The usefulness of such a construction will be apparent later.

An *ordered* "jet reduced" diagram $\hat{R}_p$ can be constructed by contracting all the jet lines in R_p to a point except $\sum_{i=1}^{K}(W_i - 1)$ external jet lines chosen arbitrarily. This is always possible when R_p is ordered. (For a detailed proof see Ref. 7.) $\hat{R}_p$ now has only hard vertices, which are of order four or more. In $\hat{R}_p$ one of the external jet momenta of each jet is chosen as the "reference" momentum which specifies the jet direction, and one normal variable is associated with each. The $(W_i - 2)$ other external jet momenta of, say, the ith jet are treated on the same footing as the internal jet loop momenta and two normal variables are associated with each of them. These normal variables are $(K_\alpha^i)^2$ and

$$\frac{K_\alpha^i \cdot K_1^i}{(K_\alpha^i)_0 (K_1^i)_0} = 1 - \cos\theta_\alpha^i \approx \tfrac{1}{2}(\theta_\alpha^i)^2,$$

where K_1^i denotes the reference momentum of the ith jet and θ_α^i is the angle between this reference momentum and a momentum K_α^i ($\alpha > 1$) in the ith jet. In order to establish the validity of (13) we first need an estimate of the volume in loop-momentum space where the following conditions are satisfied:

$$|K_\alpha^i|^2 < \lambda, \tag{14a}$$

$$|\theta_\alpha^i|^2 < \lambda. \tag{14b}$$

The rules for obtaining this estimate are discussed in Ref. 7. The volume associated with the variables $(K_\alpha^i)^2$ is

$$\lambda^{\sum_{i=1}^{K}(W_i - 1) - 4},$$

since they are obviously independent. There is a (−4) in the exponent because if the jet contains an external line of the diagram, its momentum is fixed on shell. The angular normal variables, however, may not all be independent and a lower bound on the volume associated with them is obtained using the jet reduced diagram $\hat{R}_p$. For this purpose, the l_{12} lines ($l_{12} \geq 2$) connecting the first two (hard) vertices of $\hat{R}_p$ are put on shell and we are reduced to the problem of finding the volume in phase space where (14b) is satisfied. Once this is done, the l_{12} lines of $\hat{R}_p$ are contracted and the lines l_{12}' connecting the first two vertices of the resulting diagram are treated in a similar manner. The process is repeated till the diagram $\hat{R}_p$ is reduced to a single vertex. Proceeding in this manner, the bound on the phase-space volume obtained in Ref. 7 is λ^A where $A \geq \frac{1}{2}(W_i - 2)$. Combining this with the volume where (14a) is satisfied, it is seen that the volume in loop-momentum space where equations (14) are satisfied is bounded by λ^D, where

$$D \geq \sum_{i=1}^{K}(W_i - 1) + \tfrac{1}{2}\sum_{i=1}^{K}(W_i - 2) - 4 = \sum_{i=1}^{K}\left[\tfrac{3}{2}(W_i - 2) + 1\right] - 4.$$

The equality holds when all the internal lines of $\hat{R}_p$ are nonreference (hence there are only four jets) and there are only two lines connecting any

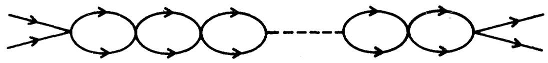

FIG. 1. Typical ordered jet reduced diagram for which $D=D_{\min}$. Here and in the following, the arrow points in the direction of energy flow.

two (hard) vertices.

We next invesitgate whether those diagrams for which

$$D=D_{\min}=\sum_{i=1}^{K}\left[\tfrac{3}{2}(W_i-2)+1\right]-4$$

gives rise to mass divergences. Two essential features of those $\hat{R}_p$ (henceforth called $\hat{R}_p'$) for which $D=D_{\min}$ are

(i) $\hat{R}_p'$ is an ordered diagram and

(ii) there are only four-point vertices in $\hat{R}_p'$.

(ii) follows from the fact that there are two incoming and two outgoing lines, and any two hard vertices are connected by just two lines.

Putting these together we see that with each hard vertex of $\hat{R}_p'$, a four-particle scattering process can be associated in which there are two incoming and two outgoing particles. This places severe restrictions on the possible types of the diagram $\hat{R}_p'$. Since there are only four-point hard vertices in $\hat{R}_p'$ and at each vertex there are two incoming and two outgoing lines, then for the diagram to be connected, the only possible construction is one in which the outgoing lines of a preceding hard vertex must be the incoming lines of the succeeding hard vertex. This reduces us to the "chain diagrams" shown in Fig. 1. All other diagrams contain at least one situation where energy flows around in a loop and hence cannot be ordered. For example, consider the diagrams shown in Figs. 2(a) and 2(b). Since, as mentioned earlier, we want to have only four jets in order to get the minimum value of D, and since we want a four-particle scattering process at each vertex, one possible way for this to happen is shown in Figs. 3(a) and 3(b), with the corresponding jet assignments. In each case, the diagram is not ordered.

Thus, the only possible types of ordered jet reduced diagrams for which $D=D_{\min}$ are those

(a) (b)

FIG. 2. Examples of jet reduced diagrams not included in Fig. 1.

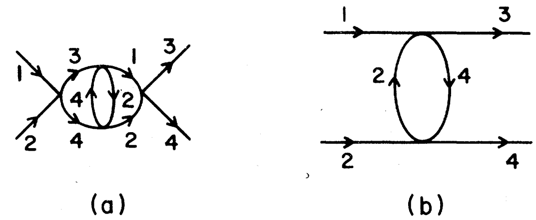

FIG. 3. Jet reduced diagrams of Fig. 2(a) and 2(b) in which the lines are distributed among the four jets labeled 1, 2, 3, 4. These are drawn such that at each vertex there are two incoming and two outgoing lines.

shown in Fig. 1 in which the internal lines are all nonreference. It is easily seen that such diagrams are suppressed with respect to power counting. However, we will now show that none of these jet reduced diagrams correspond to a reduced diagram *at a pinch singular point* and hence are automatically excluded from considerations of mass singularities.

We would like to emphasize that a jet reduced diagram is not the same as a reduced diagram at a pinch singular point and no physical interpretation (in the sense of the Coleman-Norton theorem) can be directly applied to it. However, the number of hard vertices in the jet reduced diagram is the same as in the corresponding reduced diagram.

We will use this to show that the jet reduced diagrams of Fig. 1 in which all the lines are distributed among four jets, cannot be obtained from a reduced diagram at a pinch singular point.

Consider first a jet reduced chain diagram with only one loop (i.e., with two hard vertices labeled H_1 and H_2). This is shown in Fig. 4. The corresponding reduced diagram, R_p', also has only two hard vertices. Both the diagrams are ordered and hence in the reduced diagram also, H_2 must be the last hard vertex. The preceding *hard* vertex of R_p' is H_1 and there may be an arbitrary number of soft vertices in between. The jet directions can change only at hard vertices, thus since the internal lines of Fig. 5 belong only to jets 1 and 2, there must be one situation in R_p' where, at H_2, incoming lines from jets 1 and 2 change direction and become outgoing lines belonging to jets 3 and 4. A possible form of this vertex is shown in Fig. 5 where 1′ and 2′ belong to jets 1 and 2, respectively. Since there are no more

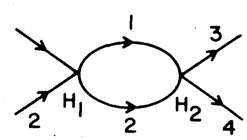

FIG. 4. Same as Fig. 1 with only two hard vertices H_1 and H_2 and one possible jet assignment for the different lines of the diagram.

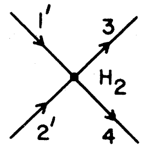

FIG. 5. Possible form of the vertex H_2 in the reduced diagram corresponding to Fig. 4.

hard vertices between H_1 and H_2, then in R_p' also there must be lines belonging to jets 1 and 2 that emerge from the preceding hard vertex H_1. Now, the line belonging to jet 1, for example, that emerges from H_1 must be connected to the line 1′ via other lines of jet 1 that are attached to these at soft vertices. The same is true for the lines in jet 2. Thus we are led to the conclusion that in the reduced diagram R_p' there must exist at least one subdiagram of the type shown in Fig. 6. The Coleman-Norton theorem, however, forbids the appearance of such a loop in a reduced diagram at a pinch singular point. This is for the following reason: By the Coleman-Norton theorem, particles between the vertices of a reduced diagram at a pinch singular point propagate freely. If this is so then a loop like the one shown in Fig. 6 cannot occur because the particles in jets 1 and 2 would be propagating freely and since they belong to different jets, they cannot meet together at H_2. (It is easily seen that the spatial component of the Landau equation $\sum \alpha_j q_j = 0$ can never be satisfied for such a loop.) This same conclusion holds if the lines of Fig. 4 are distributed among the four jets in any other way.

Consider next the jet reduced chain diagrams with more than one loop. In the corresponding reduced diagram, again, there will be at least one situation where a loop like the one shown in Fig. 6 occurs. This will happen whenever there are two hard vertices like H_1 and H_2. From H_1 only lines from, say, jets 1 and 2 emerge and at the succeeding hard vertex H_2 lines from jets 1 and 2 change direction and come out as lines in, say, jets 3 and 4. Thus we see that even though the jet reduced diagrams shown in Fig. 1 are ordered and all lines are on shell, they do not correspond to a reduced diagram at a pinch singular point. (This is possible if all lines are on shell but only the time component of the Landau equations can be satisfied and not the spatial components.)

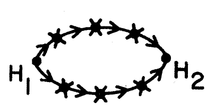

FIG. 6. A possible subdiagram in the reduced diagram corresponding to Fig. 4. Here the crosses denote soft vertices.

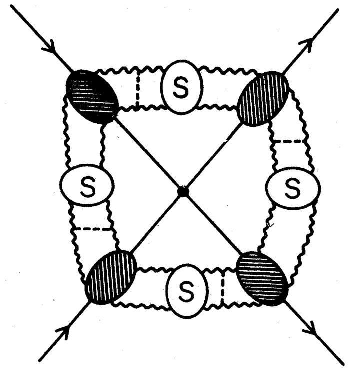

FIG. 7. Reduced diagram at the divergence point for gauge theories according to Eq. (15). Wavy lines denote soft vector lines. All on-shell lines in S are soft.

In summary, we have shown that for those diagrams $\hat{R}_p$ which contain more than one loop between two hard vertices, inequality (13) is valid. Then equation (12) tells us that there are no mass divergences. For those $\hat{R}_p$ shown in Fig. 1 and in which all the lines are distributed among only four jets there is a possibility for D to equal

$$\sum_{i=1}^{K}[\tfrac{3}{2}(W_i-2)+1]-4,$$

however, these do not correspond to reduced diagrams at a pinch singular point and hence do not contribute to mass divergences. The only reduced diagrams that give rise to mass divergence are therefore those in which the scattering takes place at a single hard vertex. The conditions for this are easily read off from Eqs. (12). They are

$$Y_{i\alpha}=X_{i\alpha}=0 \ (\alpha>4), \tag{15a}$$

$$b_i^{(0)}=f=0, \tag{15b}$$

$$b_i^{(1)}=Z_i^{(1)}, \tag{15c}$$

which are the conditions for logarithmic mass di-

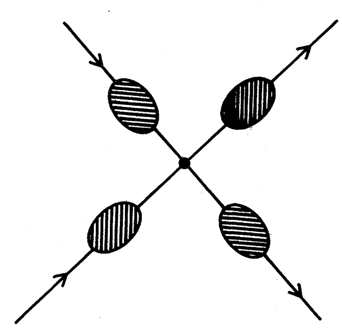

FIG. 8. Reduced diagram at the divergence point in theories without fundamental vector fields.

vergence. For gauge theories, these correspond to the reduced diagram shown in Fig. 7 and for scalar φ^4 and Yukawa theories to the reduced diagram in Fig. 8. In Fig. 8 the shaded blobs are the self-energies associated with the external lines and in Fig. 7 all on-shell lines included in the shaded blobs are jet lines to which soft vectors attach to three-point vertices. In the absence of the vectors the shaded blobs in Fig. 7 would also represent self-energies. Note that since we are considering only wide-angle scattering there are no finite-energy lines connecting the external lines of the diagram since they belong to four *different* jets. For gauge theories, however, they can be connected by zero-momentum vector lines. For scalar φ^4 and Yukawa theories we directly obtain the result given in the Introduction. For gauge theories, we show in the next section that if the incoming and outgoing particles are gauge singlets then there is a further suppression in the power-counting arguments and the reduced diagram at the divergence point again assumes the form shown in Fig. 8.

III. ELIMINATION OF SOFT LINES IN GAUGE THEORIES FOR GAUGE-SINGLET SCATTERING

Consider the reduced diagram of the four-particle "photon-photon" scattering amplitude. In Ref. 12 it was shown that whenever there are zero-momentum vector lines associated with an amplitude involving a gauge-singlet particle, Ward identities could be used to show a suppression of the integrand with respect to naive power-counting estimates. We use similar arguments to show the suppression first, for the case of photon-photon scattering in QED. Next, we briefly outline the arguments which can be used to show the suppression for the case of color-singlet scattering in QCD. For this purpose, the axial-gauge Ward identities[14] in QCD are used.

The power counting in the preceding section was done in a noncovariant gauge and it was found that mass divergences were at worst logarithmic whether the external lines of the diagram are gauge singlets or not. What happens to these arguments if any other gauge is used? For individual graphs. the divergence may be worse than logarithmic, however, as argued in Ref. 12, if one considers a gauge-invariant sum of graphs then the divergence *is* logarithmic and in particular, in the axial gauge, contributions from gauge denominators do not effect power counting.

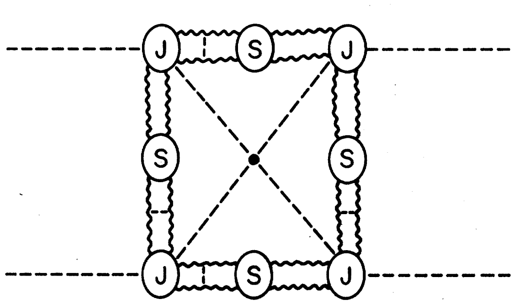

FIG. 9. Reduced diagram at the divergence point according to Eq. (15) for gauge-singlet scattering. Here, wavy lines collectively denote all soft vector lines and broken lines denote gauge singlets only.

Let us first look at the "photon-photon" scattering amplitude more closely. The only reduced diagrams that have a logarithmic mass divergence are shown in Fig. 9. It is worthwhile to emphasize here, that, at the divergence point, by Eq. (15b), only soft vector lines and no soft scalars or fermions attach to the "self-energies" associated with the hard "photon" line. The soft vector lines are explicitly shown. J and S may contain vector lines and closed loops of fermions and scalars. J contains only jet lines and S only soft lines. From the above each jet separately is of the form shown in Fig. 10. The general philosophy behind the method used to eliminate the soft vector lines completely from the reduced diagram is the following: Referring to Fig. 10 we show a suppression with respect to our earlier power-counting estimates for the amplitude for the emission of n soft vectors as their momenta are scaled together to zero. Thus when we integrate over the momenta of the soft vector particles to get the contribution to the scattering amplitude we get a finite answer since the overall amplitude is naively only logarithmically divergent. We consider, first, the case of photon-photon scattering in QED with massless electrons.

In QED, the differentiation of the propagator of an electron having charge e and momentum k is equivalent to the insertion of a zero-energy photon according to the equation[15]

$$e\frac{\partial}{\partial p_\mu} S_F^{-1}(p) = e\Gamma_\mu(p,p,0). \qquad (16)$$

$\Gamma_\mu(p,p,0)$ is the proper vertex for the emission of one zero-energy photon and to lowest order in just γ_μ. This is the Ward identity for QED and will be used below.

Consider the amplitude for the emission of two

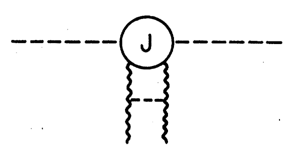

FIG. 10. Typical form of each jet in the reduced diagram of Fig. 9. Broken line denotes a hard gauge singlet and wavy lines denote soft vectors.

soft photons from a hard photon line via virtual interactions. This is shown in Fig. 11. The amplitude is not amputated in the (hard) photon line carrying momentum p'. The important feature here is that the electron lines in K always occur only in closed loops. From the Ward identity, the amplitude

$$G_{\mu_1\mu_2\mu_3\mu_4}(4;p,k_1,k_2=0)$$

can be obtained from $G_{\mu_1\mu_2\mu_3}(3;p,k_1)$. If we represent $G_{\mu_1\mu_2\mu_3}(3;p,k_1)$ by

$$G_{\mu_1\mu_2\mu_3}(3;p,k_1) = \int \prod_{i=1}^{N} \frac{d^4 l_i}{(2\pi)^4} F_{\mu_1\mu_2\mu_3}(p,\{l\},k_1), \quad (17)$$

where l_i denote (say) N independent loop momenta of the charged fermion lines. [It is to be noted that $G_{\mu_1\mu_2\mu_3}(3;p,k_1)=0$ identically by Furry's theorem when a gauge-invariant sum of all such graphs is considered.] Then,

$$G_{\mu_1\mu_2\mu_3\mu_4}(4;p,k_1,k_2=0) \sim \sum_i \left(\int d^4l_1 \cdots d^4l_{i-1} d^4l_{i+1} \cdots d^4l_N \times \int d^4l_i \frac{\partial}{\partial l^i_{\mu_4}} F_{\mu_1\mu_2\mu_3}(p,l_1,\ldots,l_i,l_N,k_1) \right),$$

where the sum is over all independent loop momenta of the charged fermion lines. In an obvious notation we write the above as

$$G_{\mu_1\mu_2\mu_3\mu_4}(4;p,k_1,k_2=0) \sim \sum_i \int \prod_j{}' \, d^4l_j \int d^4l_i \frac{\partial}{\partial l^i_{\mu_4}} F_{\mu_1\mu_2\mu_3}(p,\{l\},k_1). \quad (18)$$

Consider now the amplitude

$$G_{\mu_1\mu_2\mu_3\mu_4}(4;p,k_1,k_2).$$

Power-counting arguments of Sec. II can be used to show that this is quadratically divergent as k_1, $k_2 \to 0$ at $p^2=0$. From Eq. (18) it is seen that by differentiating the integrand of Eq. (17) with respect to the independent fermion loop momenta in turn, a zero-energy photon is in turn attached to each corresponding jet loop. When $k_1 \to 0$, and $k_2 = 0$, then at the corresponding pinch singular point, the jet lines of the loops go on shell and their momenta become proportional to p. By consideration of the integral in Eq. (18) in the neighborhood of this pinch singular point of interest we will now argue that $G_{\mu_1\mu_2\mu_3\mu_4}(4;p,k_1,k_2=0)$ is less than quadratically divergent as $k_1 \to 0$, i.e., there is a suppression with respect to the power-counting estimates of the preceding section.

In Eq. (18) a term in the sum, corresponding to, say, the ith loop is selected. Consider a par-

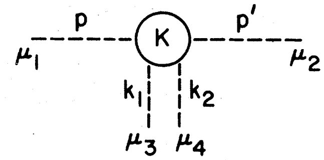

FIG. 11. Momentum configuration for the amplitude $G_{\mu_1\mu_2\mu_3\mu_4}(4;p,k_1,k_2)$ in QED. The blob denotes the sum of all gauge-invariant graphs. A broken line denotes a photon.

ticular component $\mu_4=\alpha$. Let d^3l_i denote collectively the differentials corresponding to the other three components. Suppose that in the l^i_α integration, the contour is trapped at $l^i_\alpha=(l^i_\alpha)^c$ as $k_1 \to 0$. In the neighborhood of this pinch singular point, we can write for the corresponding term on the right-hand side of Eq. (18)

$$\int \prod_j{}' d^4l_j \int d^3l_i \int_{l^i_\alpha=(l^i_\alpha)^0-\Delta}^{l^i_\alpha=(l^i_\alpha)^0+\Delta} dl^i_\alpha \frac{\partial}{\partial l^i_\alpha} \times F_{\mu_1\mu_2\mu_3}(\{l\},p,k_1) \equiv G'^{\Delta}_{\mu_1\mu_2\mu_3\mu_4}(4;p,k_1,k_2=0), \quad (19)$$

where $\Delta \neq 0$ defines the region near the pinch singular point where the integral is being investigated. It is chosen such that in the region considered above, no other pinch singular point corresponding to the vanishing of other momenta are included. Equation (19) can be written as

$$G'^{\Delta}_{\mu_1\mu_2\mu_3\mu_4}(4;p,k_1,k_2=0) = \int \prod_j{}' d^4l_j \times \int d^3l_i \big(F_{\mu_1\mu_2\mu_3}(\{l\},p,k_1)\big|_{l^i_\alpha=(l^i_\alpha)^0+\Delta} - F_{\mu_1\mu_2\mu_3}(\{l\},p,k_1)\big|_{l^i_\alpha=(l^i_\alpha)^0+\Delta}\big). \quad (20)$$

Thus, the fact that the integrand in Eq. (19) is a perfect differential enables us to do the dl^i_α integration, and in the integrand of Eq. (20), l^i_α assumes values only at the boundary.

For $\Delta \neq 0$, this corresponds to an off-shell value for l^i_α and hence when the remaining integrals in (20) are evaluated the quantity

$$G'^{\Delta}_{\mu_1\mu_2\mu_3\alpha}(4;p,k_1,k_2=0)$$

so obtained will be more convergent as $k_1 \to 0$ than suggested by the naive power-counting arguments of the preceding section. This same argument is true in the neighborhood of a pinch singular point for each component of μ_4 and for each term on the right-hand side of Eq. (18). Thus we see that term by term,

$$\lim_{\lambda\to 0} \lambda^2 G_{\mu_1\mu_2\mu_3\mu_4}(4;p,\lambda k_1,k_2=0)=0. \quad (21)$$

This is derived with k_2 fixed at zero and k_1 scaled to zero, but it is obviously also true if k_1 is fixed at zero and k_2 is scaled to zero. We therefore consider Eq. (21) indicative that $G_{\mu_1\mu_2\mu_3\mu_4}(4;p,k_1,k_2)$ is less than quadratically divergent as both k_1 and k_2 are together scaled to zero. These arguments, in conjunction with the Ward identity can now be used similarly to show that the amplitude for the emission of n soft photons is less than nth-order divergent as the soft photon momenta are together scaled to zero. Hence, when we integrate over the soft photon momenta to get the contribution to the entire four-particle scattering amplitude, which is naively logarithmically divergent, we get a finite answer. Thus, we see that whenever there are soft photons associated with the photon-photon scattering amplitude we get a suppression in the naive power-counting estimates and the corresponding reduced diagram has no logarithmic mass singularity. The only reduced diagram that gives rise to a logarithmic mass divergence is shown in Fig. 8 which shows that apart from renormalization effects the amplitude has a well defined zero-mass limit.

For the case of color-singlet scattering in QCD we again consider each jet individually and start with the amplitude for emission of two soft gluons from the hard photon line to lowest order in the electromagnetic interaction. This is shown in Fig. 12. In this the incident photon of momentum p emits two soft gluons via virtual interactions and is connected to the rest of the graph by an internal gluon line of momentum p'. The amplitude

$$G^{a_1a_2;b}_{\mu_1\mu_2;\nu_1\nu_2}(2;p,k_1,k_2)$$

is not amputated in the (hard) gluon line carrying momentum p' and is quadratically divergent as $k_1, k_2 \sim 0$ at $p^2=0$. As before we now show that there is a suppression with respect to this power-counting estimate. The arguments we outline below are similar to Ref. 12 where use is made of the axial-gauge differential Ward identities. We use the differential Ward identities in the form shown in Fig. 13. The notation is that of Ref. 14. In this form, the Ward identity is an algebraic

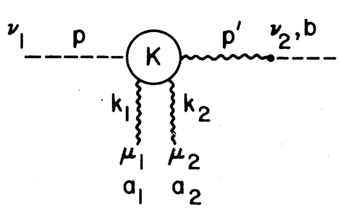

FIG. 12. Same as Fig. 11 for the amplitude $G^{a_1a_2;b}_{\mu_1\mu_2;\nu_1\nu_2}(2;p,k_1,k_2)$ in QCD. Wavy lines here are gluons and broken lines are "photons."

identity and is valid at each point in loop-momentum space. The first term on the left-hand side of Fig. 13 may be written as

$$-ig\sum_{\substack{\text{ext}\\\text{lines}}} f^{aca'}\frac{\partial}{\partial q_\mu}G^{\cdots a'\cdots}_{\cdots\alpha\cdots}(\cdots q\cdots),$$

where the sum is over derivatives with respect to all independent external lines that carry color and $f^{aca'}$ are the structure constants. In the second term the sum is over derivatives with respect to all independent loop momenta of colored fermions and gluons. For gluon loops, for example, a typical term included in the sum is of the form

$$-ig\sum\int{\prod_j}' d^4l_j\int d^4l_i f^{aca'}\frac{\partial}{\partial l^i_\mu}\times F^{\cdots a,a'\cdots}_{\cdots}(\cdots l^i\cdots),\qquad(22)$$

where the notation is as before and F is a function of independent external momenta and other loop momenta.

The term on the right-hand side of Fig. 13 represents the insertion of a zero-momentum line with group and spatial indices c and μ, respectively. Using the Ward identity, the amplitude

$$G^{a_1a_2;b}_{\mu_1\mu_2;\nu_1\nu_2}(2;p,k_1,k_2=0)$$

can be related to $G^{a_1;b}_{\mu_1;\nu_1\nu_2}(1;p,k_1)$. Thus,

$$\begin{aligned}G^{a_1a_2;b}_{\mu_1\mu_2;\nu_1\nu_2}&(2;p,k_1,k_2=0)\\&=-igf^{a_2a_1c}\frac{\partial}{\partial k_1^{\mu_2}}G^{c;b}_{\mu_1;\nu_1\nu_2}(1;p,k_1)\\&\quad-igf^{a_2bc}\frac{\partial}{\partial p^{\mu_2}}G^{a_1;c}_{\mu_1;\nu_1\nu_2}(1;p,k_1)\\&\quad+\text{terms with derivatives with respect to the independent loop momenta.}\end{aligned}\qquad(23)$$

But by charge-conjugation invariance $G^{a_1;b}_{\mu_1;\nu_1\nu_2}(\cdots)=0$, and hence

$$G^{a_1a_2;b}_{\mu_1\mu_2;\nu_1\nu_2}(2;p,k_1,k_2=0)$$

equals a sum of terms of the type (22). In their derivation of the Ward identity, Sugamoto *et al.*[14] show that such terms vanish when integrated over the entire range of loop momenta. In what follows we are, however, only interested in a particular region where the integration contour is pinched as the k_i's $\to 0$. This region is chosen to contain

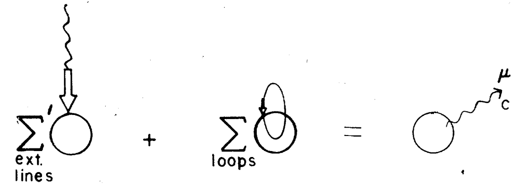

FIG. 13. Differential Ward identity for QCD in the axial gauge.

no other pinch singular points. By a power-counting examination of the integrals of the type (22) in the neighborhood of the pinch singular point as $k_1 \to 0$, we can argue that

$$G^{a_1 a_2; b}_{\mu_1 \mu_2; \nu_1 \nu_2}(2; p, k_1, k_2 = 0)$$

is less than quadratically divergent as $k_1 \to 0$. The procedure is exactly analogous to what was discussed earlier for massless QED. Thus,

$$\lim_{\lambda \to 0} \lambda^2 G^{a_1 a_2; b}_{\mu_1 \mu_2; \nu_1 \nu_2}(2; p, \lambda k_1, k_2 = 0) = 0.$$

This is indicative that the amplitude for two-gluon emission is less than quadratically divergent when $k_1, k_2 \sim 0$. An iterative argument based on repeated use of the Ward identity in a similar manner as above can now be used to show that the amplitude for emission of n gluons is less than nth-order divergent. For example, the amplitude for r-gluon emission can be related through the Ward identity to that for $(r+1)$-gluon emission. If the former amplitude is less than rth-order divergent then one argues that the amplitude for $(r+1)$-gluon emission is less than $(r+1)$th-order divergent. The suppression is thus established and just as for the case of QED discussed earlier we see that for color-singlet scattering in QCD, the only reduced diagrams that have a logrithmic mass divergence are those shown in Fig. 8 which implies that the wide-angle "photon-photon" scattering amplitudes in gauge theories also satisfy the renormalization-group equations.

In conclusion, we would like to emphasize that in this work we have everywhere treated the external particles as fundamental fields. It is not yet clear how the formalism of this paper can be applied to composite particles and hence this work may be regarded as instructive but not complete. It is hoped that a suitable modification of this formalism can also be used to study the wide-angle scattering of composite particles.

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