Singularities and sources of static gauge fields

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It is argued that for static spherically symmetrical solutions, if the source current J^a_μ tends to $O(1/r^{3-\epsilon})$ as $r \to 0$, then singularities of the gauge field at the origin do not require external sources to sustain them. Singularities and sources of some known static solutions are discussed.

I. INTRODUCTION

Recently there has been much interest in constructing solutions for the sourceless Yang-Mills field equation. In particular, self-dual gauge fields are necessarily sourceless and may play a fundamental role in the confinement of quarks.[1] However, if the solution obtained has singularities, then it is necessary to study these singularities carefully since these singularities may require external sources to sustain them and as a result the solution is not a true sourceless solution at these singularities. For example, the Wu-Yang solution[2] has a singularity at the origin and it necessitates the presence of a source.[3] Does a singular solution *always* imply the existence of a source? The answer is no.

In this paper we argue that for a point singularity of a solution, if the source current J^a_μ vanishes everywhere but behaves as $O(r^{-3+\epsilon})$ near the singularity, then the solution is valid everywhere including the singular point. We then apply this criterion to the singular Prasad-Sommerfield-type solution obtained in Ref. 4, and other known static solutions. We find that the singularity at $r=0$ for the solution in Ref. 4 does not require the presence of an external source, thus verifying the validity of that solution at $r=0$. In fact, the solution in Ref. 4 is related to the original Prasad-Sommerfield solution[5] by a singular gauge transformation. This explains why the properties of these two solutions are the same.

II. SOURCES

The source function associated with the SU(2) Yang-Mills field is defined as[2]

$$J^a_\mu = \partial^\nu F^a_{\mu\nu} + g\epsilon_{abc} A^{b\nu} F^c_{\mu\nu},$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc} A^b_\mu A^c_\nu. \tag{1}$$

J^a_μ vanishes everywhere except at the singularities of the solution A^a_μ.

We now make use of distribution theory and regard J^a_μ as a functional of tensor-valued test functions $\phi^a_\mu(x)$. ϕ^a_μ are indefinitely differentiable functions with bounded support. If $J^a_\mu[\phi^a_\mu]$ is nonzero for some ϕ^a_μ, then we say the singularity of A^a_μ implies the existence of an external source, otherwise A^a_μ is sourceless in spite of the fact that it may carry electric or magnetic charges.[6,7]

Suppose R is the whole space excluding the singularities of A^a_μ, and S consists of various small regions containing these singularities. Then

$$J^a_\mu[\phi_\mu] = \int_R J^a_\mu \phi^{a\mu} d^3r + \int_S J^a_\mu \phi^{a\mu} d^3r = \int_S J^a_\mu \phi^{a\mu} d^3r , \tag{2}$$

since, in R, J^a_μ vanishes. Depending on regions of singularities, in general it is not straightforward to determine whether the functional $J^a_\mu[\phi^a_\mu]$ vanishes or not in S. However, if the regions of singularities just consist of points $\vec{b}_i$, one can choose S to be small spheres surrounding the singularities $\vec{b}_i$. Then from Eq. (2) we see that as test functions ϕ^a_μ are regular everywhere, $J^a_\mu[\phi^a_\mu]$ will vanish, i.e., the solution is sourceless if as $\vec{r} \to \vec{b}_i$, $J^a_\mu \to O(|\vec{r} - \vec{b}_i|^{-3+\epsilon})$. If the point singularity $\vec{b}_i$ is at the origin, then the condition $J^a_\mu \to O(r^{-3+\epsilon})$ as $r \to 0$ will guarantee no source at the origin. This is a sufficient condition. We note that from Eq. (2), if the singularity of J^a_μ is of the δ-function type, the functional $J^a_\mu[\phi^a_\mu]$ is nonzero, which means that the singularity requires an external source to sustain itself.

III. SINGULARITIES

In Ref. 4, a singular Prasad-Sommerfield-type solution for the Yang-Mills field coupled to the Higgs field Φ^a in the limit that the self-interaction potential of the Higgs triplet vanishes is given by

$$A^a_i = \epsilon_{iab} n_b \frac{K(r)-1}{gr} + (n_i n_a - \delta_{ia}) \frac{W(r)}{gr} ,$$
$$A^a_0 = n^a J(r)/gr, \quad \Phi^a = n^a H(r)/gr , \tag{3}$$

where n_a is the unit vector and

$$K/\sin\alpha = W/\sin\alpha = pr/\sinh pr ,$$
$$J/\sinh\gamma = H/\cosh\gamma = 1 ,$$

$$I = pr\coth pr - 1\,.$$

Here α and γ are constants. Only A_i^a is singular at $r=0$. To check for the validity of solution (3) at $r=0$, we note that it is derived from the complex spherically symmetrical solution for the sourceless Yang-Mills equation, with A_i^a given exactly by Eq. (3) and

$$A_0^a = in_a I(r)/(gr)\,. \qquad (4)$$

We thus can evaluate the source current J_μ^a for solution (4). After some computation we find that as $r \to 0$, $J_\mu^a \to O(r^{-1})$. Thus, the last integral in Eq. (2) vanishes, and solution (4) and hence solution (3) are valid at the origin.

For the Wu-Yang (WY) solution,[2] J_j^a behaves as $O(r^{-3})$ near the origin. This implies that the singularity of the WY solution requires an external source. Of course the singularity of J_j^a for the WY solution as $r \to 0$ can also be regarded as the δ-function type, which again, by Eq. (2), indicates the presence of a source.[3] Applying the same calculation to the solution of Ref. 8, we find the solution is not sourceless. Thus, the solution of Ref. 8 is not valid at the origin.

IV. REMARKS

The singular spherically symmetrical solution (3) possesses the same finite energy, same electric and magnetic charges as those of Ref. 5, and is also stable against deformation.[4] That this is so is not an accident, for one can regard solution (3) as the gauge transform of the Prasad-Sommerfield solution.[9] The required gauge transformation is $\omega = \exp(\frac{1}{2} i\beta\sigma_j n^j)$, where β is a constant and σ_j are the Pauli matrices. As

$$\omega\sigma_j\omega^{-1} = \sigma_j\cos\beta + \epsilon_{jlk} n^l \sigma_k \sin\beta + 2n_j n_l \sigma_l \sin^2(\tfrac{1}{2}\beta)\,, \qquad (5)$$

the result is easily verified. Note that the gauge transformation is singular in the sense that $\partial_j\omega\omega^{-1}$ has a singularity at $r=0$. Since we have just shown that the singularity in expression (3) does not lead to the existence of a source, this singular gauge transformation ω is physically acceptable. Singular gauge transformations such as those discussed in Ref. 10 introduce string-type singularities which require sources to sustain them. Consequently, they may alter the dynamics of the theory[11] and cannot be accepted physically.

The nonmonopole-type self-dual static solutions found in Ref. 12 also possess singularities. The regions of singularities are surfaces and we are uncertain whether these singularities do lead to the presence of external sources.

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