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Classical quark statics

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I review and update the ideas on classical quark statics developed in two previous papers, with an emphasis on the Euclidean spacetime formulation of the equilibrium field equations, and complete the calculation of the order-g^2 static $q\bar{q}$ potential in a Prasad-Sommerfield background. This background field increases the order-g^2 potential, as compared with the Coulomb potential, indicating a focusing of color flux lines, but for widely separated quarks the focusing is not strong enough to give confinement. I briefly discuss the outlook for future developments within the classical framework.

I. INTRODUCTION

In two previous papers[1,2] I proposed a classical approach to the problem of quark statics, based on the idea of treating quark color correlations in an exact quantum-mechanical fashion within an otherwise classical framework. The purpose of the present paper is threefold. First of all, in Sec. II I give a review and update of the classical algebraic approach, which I hope will clarify some of the issues raised in Refs. 1 and 2. Second, in Sec. III and Appendix A I complete the calculation, begun in Ref. 2, of the order-g^2 static $q\bar{q}$ potential in a Prasad-Sommerfield background. While not giving a confining potential, the result of the calculation is nonetheless interesting in that it indicates a focusing of color flux lines. Finally, in Sec. IV I briefly indicate directions for further investigation in classical chromodynamics. In Appendix B I show the equivalence, to order g^2, of the methods of this paper and the Wilson loop criterion for quark confinement. In Appendix C I give a method for evaluating the spin-orbit potential.

II. REVIEW AND UPDATE

A. Classical algebraic chromodynamics

The basic idea of classical algebraic chromodynamics is to set up a non-Abelian analog of classical electrodynamics, in which the quark (and antiquark) color charges are finite matrices acting on a finite-dimensional color Hilbert space.[3] Taking the underlying color group as SU(n), and considering a system with N source charges, the color charges for distinct sources are assumed to commute,

$$[Q^a_{(j)}, Q^b_{(k)}]=0,\quad j\neq k,\quad a,b,c=0,\ldots,n^2-1, \tag{1a}$$

while the components of the color charge for a given source are assumed to obey

$$\begin{aligned}&[Q^0_{(j)}, Q^A_{(j)}]=0\,,\\ &Q^A_{(j)}Q^B_{(j)}=q^{ABC}Q^C_{(j)},\quad j\text{th particle a quark}\,,\\ &Q^A_{(j)}Q^B_{(j)}=-q^{BAC}Q^C_{(j)},\quad j\text{th particle an antiquark}\,,\\ &q^{ABC}=\tfrac{1}{2}(d^{ABC}+if^{ABC}),\quad A,B,C=1,\ldots,n^2-1\,.\end{aligned} \tag{1b}$$

In Eq. (1) I have allowed for the possibility that the color charges may contain color-singlet components $Q^0_{(j)}$, as well as color (n^2-1)-plet components $Q^A_{(j)}$, $A=1,\ldots,n^2-1$, which, in the respective cases of quark and antiquark, have the algebraic properties of the Gell-Mann matrices $\frac{1}{2}\lambda^A$ and $-\frac{1}{2}\lambda^{*A}$. The structure constants which appear are the usual f- and d-type SU(n) constants. The color charges act on a finite-dimensional color Hilbert space which is the direct product of the color Hilbert spaces for the N source particles, and which contains a color-singlet state specified by

$$\left(\sum_{j=1}^{N} Q^A_{(j)}\right)|\text{color singlet}\rangle=0,\quad A=1,\ldots,n^2-1\,. \tag{2}$$

In general, the fact that the charge components $Q^A_{(j)}$ and $Q^B_{(j)}$ are noncommuting for $A\neq B$ leads to severe operator-ordering problems when one attempts to use them as source charges in a gauge field theory. The key result of Sec. I of Ref. 1 is that a classical chromodynamics, with noncommuting source charges satisfying Eq. (1), can be set up when the conditions specified by the following definitions and theorem are satisfied:

Definition. An *outer product* $w=P(u,v)$ on the algebra of color charge operators is an antisymmetric, anti-Hermitian,[4] n^2-plet-valued bilinear form on the n^2-plet arguments u,v. That is, given any two Hermitian n^2-plets u^a, v^a, the bilinear map $w^a=P^a(u,v)=-P^a(v,u)$ defines a new anti-Hermitian n^2-plet w.

Definition. An *inner product* $w=S(u,v)$ on the algebra of color charge operators is a symmetric, Hermitian, singlet-valued bilinear form on the n^2-plet arguments u,v. That is, given any two Hermitian n^2-plets u^a, v^a, the bilinear map $w=S(u,v)=S(v,u)$ defines a Hermitian color-singlet operator w.

Definition. The *color-charge algebra* $\mathfrak{A}_{N_q,N_{\bar q}}$ is the minimal set of n^2-plet operators containing N_q quark charges, $N_{\bar q}$ antiquark charges, and closed under composition with the outer product P. Note that since the color charges are finite-dimensional matrices, the dimension of the algebra $\mathfrak{A}_{N_q,N_{\bar q}}$ is necessarily finite.

Definition. The outer product P has the *Jacobi property* if, for any triplet of elements u,v,w belonging to any given color-charge algebra $\mathfrak{A}_{N_q,N_{\bar q}}$, one has

$$P(u,P(v,w))+P(w,P(u,v))+P(v,P(w,u))=0 . \tag{3a}$$

Definition. The inner product S has the *trace property* if, for any triplet of elements u,v,w belonging to any given color-charge algebra $\mathfrak{A}_{N_q,N_{\bar q}}$, one has

$$S(u,P(v,w))=S(P(u,v),w) . \tag{3b}$$

The inner product S has the *restricted trace property* if, for any triplet of elements u,v,w belonging to any given color-charge algebra $\mathfrak{A}_{N_q,N_{\bar q}}$ for which color singlets can be realized, one has

$$\langle \text{color singlet}|S(u,P(v,w))|\text{color singlet}\rangle=\langle\text{color singlet}|S(P(u,v),w)|\text{color singlet}\rangle . \tag{3c}$$

Theorem. Given color charges $Q_q^a, Q_{\bar q}^a$, an outer product P with the Jacobi property, and an inner product S with the trace property (or with the restricted trace property), one can generalize classical SU(n) chromodynamics into a classical algebraic chromodynamics by the replacements

$$gf^{ABC}\to -igP,\quad \delta^{AB}\to S[\text{or } \delta^{AB}\to\langle\text{color singlet}|S|\text{color singlet}\rangle] . \tag{4}$$

All the standard derivations of classical chromodynamics remain valid under these replacements, including the proof of the existence of a conserved, gauge-invariant stress-energy tensor (or, in the restricted trace property case, a conserved, gauge-invariant color-singlet expectation of the stress-energy tensor).

Obviously, a structural theorem such as the one just stated has content only if the conditions of the theorem can be satisfied. Whether color charges $Q_q, Q_{\bar q}$, an outer product P, and an inner product S can be found with the requisite Jacobi and trace properties is at present an open question. In Sec. II of Ref. 1 I proposed a set of definitions which satisfy the conditions of the theorem in the 2-particle ($qq, q\bar q, \bar q\bar q$) cases. However, subsequent calculations by several groups[5] have shown that, in the qqq case, the definitions of Ref. 1 do not give the trace property, even when S is restricted to its color-singlet expectation. Another defect of my original definitions is that they give $S(Q_q,Q_q)\neq S(Q_{\bar q},Q_{\bar q})$, and hence do not have a manifest charge-conjugation symmetry between quarks and antiquarks. In an added note to Ref. 2, I suggested more general definitions of the color charges and the outer and inner products, which may remedy the defects of my original ansatz. The investigations of the algebraic properties of these generalized constructions is in progress.

In the rest of this paper, I will assume that a construction for the color charges and outer and inner products can be found which satisfies the conditions of the theorem, and that the construction, moreover, is manifestly symmetric between quarks and antiquarks. According to the analysis of Sec. II of Ref. 1, the color-charge algebra $\mathfrak{A}_{N_q,N_{\bar q}}$ is expected to diagonalize into a set of SU(j) Lie algebras, the overlying algebras for the $N_q, N_{\bar q}$ sector. In each overlying algebra, the quark charges will be represented by c-number SU(j) effective charges $Q_{(k)}^{a\,\text{eff}}$, $a=1,\ldots,j^2-1$, and the dynamical problem reduces to that of a classical SU(j) Yang-Mills field with external source charges. The local algebraic gauge invariance of the underlying classical algebraic theory is reflected in the local classical gauge invariance of each of the overlying algebras, and so the magnitudes, *but not the orientations*, of the effective charges have physical significance. (Statements which I made in Ref. 1 to the effect that the orientations of the effective charges have physical significance are incorrect. In fact, the gauge freedom to rotate the effective charges plays a role in orthogonalizing the source current to the zero modes, as described in more detail in Sec. II C below.) One could of course proceed directly to a study of classical Yang-Mills field theories with source charges, without the complicated algebraic preliminaries, but getting the classical Yang-Mills equations as overlying structures in an algebraic approach accomplishes three things. First, it gives a quantum-mechanical Hilbert-space construction for the color states and the color-singlet condition, as is required to make contact with quark-model wave functions. Second, it gives formulas for the magnitudes of the effective charges and for the effective Lagrangians[6] in the overlying classical field theories. And third, it shows that the dynamical groups [the overlying SU(j) Lie groups] are not in general the same as the underlying color group SU(n). In particular, the analysis of Refs. 1–3 in the $q\bar{q}$ case gives SU(2) as the dynamical group independent of the color group SU(n), and I believe that this result will remain unchanged in a fully satisfactory algebraic construction.

B. The classical equilibrium equations

Given a classical SU(j) Yang-Mills theory with fixed source charges, what are the equations for equilibrium field configurations analogous to the equations of classical electrostatics? In Sec. I and Appendix B of Ref. 2 I argued that the equations of classical "chromostatics" are not the specialization of the Minkowski-space Euler-Lagrange equations to vanishing time derivatives, but rather are a related set of equations differing by a change of sign in one term in the "curl B" equation. Specifically, in the case of an SU(2) classical overlying algebra, the equilibrium equations which I wrote down (with coupling constant g scaled out) are[7]

$$\vec{E}^j = -D_j\vec{b}^0\,,$$
$$\vec{B}^j = \epsilon^{jkl}\left(\frac{\partial}{\partial x^k}\vec{b}^l + \tfrac{1}{2}\vec{b}^k\times\vec{b}^l\right), \quad D_j\vec{B}^j = 0\,, \tag{5a}$$

$$D_j\vec{w} = \frac{\partial}{\partial x^j}\vec{w} + \vec{b}^j\times\vec{w}\,,$$
$$D_j\vec{E}^j = g^2\vec{J}^0\,,$$
$$\epsilon^{mlj}D_l\vec{B}^j = g^2\vec{J}^m_{\text{spin}} - \vec{b}^0\times D_m\vec{b}^0\,, \tag{5b}$$

with source currents given by[8]

$$\vec{J}^0 = \sum_{n=1}^{2}\vec{Q}^{\text{eff}}_{(n)}\,\delta^3(x-x_n)\,,$$
$$\vec{J}^k_{\text{spin}} = \epsilon^{klm}D_l\vec{\mathcal{J}}^m_{\text{spin}}\,, \tag{6}$$
$$\vec{\mathcal{J}}^m_{\text{spin}} = i\sum_{n=1}^{2}\vec{Q}^{\text{eff}}_{(n)}\,\frac{\sigma^m_{(n)}}{2m_q}\,\delta^3(x-x_n)\,.$$

As noted in Ref. 2, by using the relation $\epsilon^{mlj}D_mD_l\vec{B}^j=0$ and Eqs. (5) and (6), one finds that the quark effective charges and locations are constrained[8] by the "compatibility conditions"

$$\vec{b}^0(x_n)\times\vec{Q}^{\text{eff}}_{(n)} + \vec{B}^m(x_n)\times\frac{i\vec{Q}^{\text{eff}}_{(n)}\sigma^m_{(n)}}{2m_q} = 0\,, \quad n=1,2\,. \tag{7a}$$

Adopting the natural Euclidean four-dimensional notation

$$\vec{b}^\mu = (\vec{b}^0, \vec{b}^j)\,, \quad \vec{J}^\mu = (\vec{J}^0, \vec{J}^j_{\text{spin}})\,,$$
$$D^\mu = (D^0, D^j)\,, \quad D^0\vec{w} = \vec{b}^0\times\vec{w}\,, \tag{7b}$$
$$D^j\vec{w} = D_j\vec{w} = \frac{\partial}{\partial x^j}\vec{w} + \vec{b}^j\times\vec{w}\,,$$

the compatibility conditions of Eq. (7a) just correspond to the statement that the source current $\vec{J}^\mu$ is covariantly conserved,

$$D^\mu\vec{J}^\mu = 0\,. \tag{7c}$$

I gave two arguments for the choice of Eqs. (5) as the equilibrium equations. First, when the surface term at infinity can be neglected, these equations satisfy the principle of virtual work,[6]

$$\delta E_{\text{gluon}} = \delta D\int d^3x\,\frac{1}{2g^2}\,(\vec{E}^j\cdot\vec{E}^j + \vec{B}^j\cdot\vec{B}^j)$$
$$= \delta V_{\text{static}} = D\int d^3x\,\delta\vec{b}^\mu\cdot\vec{J}^\mu\,. \tag{8a}$$

That is, any infinitesimal variation in the field energy can be reinterpreted as a variation in the potential energy of the sources, a condition which *must* be satisfied by an equilibrium field configuration. Second, these equations describe the field Cauchy data $b^0(x,t=0)$, $\vec{b}^j(x,t=0)$ which maximize the quantum transition amplitude into a state at time $t=0$ with a specified source charge distribution. Hence they are the relevant equations for describing physically occurring states.

I wish to give here a third argument for the validity of Eqs. (5), based on the observation that the concept of "static potential" is ambiguous with respect to the choice between "static with respect to real time" and "static with respect to imaginary time." The ambiguity can be resolved by noting that while physics takes place in Minkowski space-time, the correct procedure to do calculations in quantum field theory is to first calculate in Euclidean spacetime, where functional integrals and perturbation-theory integrals have no mass-shell singularities, and then to return to Minkowski spacetime by an appropriate continuation procedure. *Quantities not depending on mass-shell conditions, such as static potentials, are given directly by the Euclidean calculation.*[9] In Abelian quantum electrodynamics, this statement has no practical effect since the static (vanishing time derivative) specializations of the Minkowski and of the Euclidean field equations are the same. However, there are field theories, such as both non-Abelian gauge theories and the Abelian Higgs model, where the static specializations of the Minkowski and of the Euclidean field equations differ. *In such theories, the fields which exist in equilibrium with infinitely massive source charges, and the corresponding static potentials, are determined by the Euclidean static equations.* Equations (5) are of course just the Euclidean static equations for an SU(2) Yang-Mills theory.[10]

To continue these general observations a bit further, I note the following:

(i) The Euclidean static equations satisfy the principle of virtual work because they are the time-independent specializations of the variational equations for the Euclidean action

$$S_E = D\int d^4x\left[\frac{1}{2g^2}(\vec{E}^j\cdot\vec{E}^j+\vec{B}^j\cdot\vec{B}^j) - \vec{b}^0\cdot\vec{J}^0-\vec{B}^m\cdot\vec{\mathcal{J}}^m_{\rm spin}\right], \tag{8b}$$

and because the field-strength-squared term of S_E has the same functional form as the Minkowski field energy.[11] That is, for static field configurations satisfying

$$\delta S_E = -D\int d^4x(\vec{b}^0\cdot\delta\vec{J}^0+\vec{B}^m\cdot\delta\vec{\mathcal{J}}^m_{\rm spin}), \tag{8c}$$

variations of the Minkowski field energy necessarily reduce to surface terms around the source charges (and also a possible surface term at infinity, which will be further considered in Sec. II C below).

(ii) Euclidean static field configurations are analogs, in non-Abelian gauge theories, of the equilibrium points x_c, where $V'(x_c)=0$, in the quantum mechanics of a particle moving in a one-dimensional potential $V(x)$. Recall[12] that such equilibrium points are static solutions of the Euclidean equation of motion

$$\frac{d^2x}{dt^2}=V'(x), \tag{9}$$

and hence give the leading classical approximation to the functional integral for one-dimensional particle mechanics. In a similar fashion, the Euclidean static field configurations will give the leading classical approximation to the functional integral for a non-Abelian gauge theory with external sources.

(iii) As discussed in detail in Ref. 2, Euclidean static configurations are in general realized in Minkowski spacetime at only one time slice along a system world line; at subsequent time slices along the world line, they undergo a complicated time evolution in cases, such as non-Abelian gauge theories, where the Euclidean and Minkowski static equations differ. However, the gluon energy and the quark static potential remain constant as a function of Minkowski time, as a consequence of energy conservation. Hence, from a Minkowski viewpoint, talking of "static configurations" and "static equations" is misleading; the terminology "equilibrium configurations" and "equilibrium equations" is more descriptive of the physics involved.

C. The background-field approximation

Since the semiclassical approximation to quantum field theory is a small-g^2 approximation,[13] it is consistent with semiclassical ideas to look for solutions to the static equilibrium equations in the form of a series expansion in the coupling g^2. Writing

$$\begin{aligned}\vec{b}^0&=\vec{b}^0_0+g^2\vec{b}^0_1+\cdots, & \vec{E}^j&=\vec{E}^j_0+g^2\vec{E}^j_1+\cdots,\\ \vec{b}^j&=\vec{b}^j_0+g^2\vec{b}^j_1+\cdots, & \vec{B}^j&=\vec{B}^j_0+g^2\vec{B}^j_1+\cdots\end{aligned} \tag{10a}$$

and substituting into Eq. (5), the zeroth-order potentials $\vec{b}^0_0, \vec{b}^j_0$ evidently satisfy the source-free equations

$$\begin{aligned}&\vec{E}^j_0=-D_{0j}\vec{b}^0_0,\\ &\vec{B}^j_0=\epsilon^{jkl}\left(\frac{\partial}{\partial x^k}\vec{b}^l_0+\tfrac{1}{2}\vec{b}^k_0\times\vec{b}^l_0\right),\quad D_{0j}\vec{B}^j_0=0,\end{aligned} \tag{10b}$$

$$\begin{aligned}&D_{0j}\vec{w}=\frac{\partial}{\partial x^j}\vec{w}+\vec{b}^j_0\times\vec{w},\\ &D_{0j}\vec{E}^j_0=0,\\ &\epsilon^{mlj}D_{0l}\vec{B}^j_0=-\vec{b}^0_0\times D_{0m}\vec{b}^0_0.\end{aligned} \tag{10c}$$

The source currents first appear in the equations for the first-order potentials, which are most conveniently written in the Euclidean four-dimensional notation

$$\vec{b}_1^\mu = (\vec{b}_1^0, \vec{b}_1^j)\,, \tag{11a}$$

$$D_0^\mu = (D_0^0, D_0^j)\,,$$

$$D_0^0 \vec{w} = \vec{b}_0^0 \times \vec{w}\,, \tag{11b}$$

$$D_0^j \vec{w} = D_{0j}\vec{w} = \frac{\partial}{\partial x^j}\vec{w} + \vec{b}_0^j \times \vec{w}\,.$$

In the "natural" or "background-field" gauge[14] where

$$D_0^\mu \vec{b}_1^\mu = 0\,, \tag{12}$$

the first-order equations take the form

$$(D^2_{(2)} b_1)^\mu \equiv D_0^\sigma D_0^\sigma \vec{b}_1^\mu + 2\vec{f}_0^{\mu\tau} \times \vec{b}_1^\tau = -\vec{J}^\mu\,, \qquad \vec{f}_0^{0j} = -\vec{f}_0^{j0} = \vec{E}_0^j, \quad \vec{f}_0^{kj} = \epsilon^{kjm}\vec{B}_0^m\,. \tag{13}$$

Obviously, one solution to the above set of equations is obtained by expanding around the zero-background-field solution to Eq. (10),

$$\vec{b}_0^0 = \vec{b}_0^j = 0\,, \tag{14}$$

for which Eqs. (11) and (12) reduce to

$$\frac{\partial}{\partial x^j}\vec{b}_1^j = 0\,, \qquad \nabla^2 \vec{b}_1^\mu = -\vec{J}^\mu\,, \tag{15}$$

with the Coulombic or quasi-Abelian solution

$$\vec{b}_1^0 = \sum_{n=1}^{2} \frac{\vec{Q}^{\text{eff}}_{(n)}}{4\pi|\vec{x}-\vec{x}_n|}\,, \qquad \vec{b}_1^j = \sum_{n=1}^{2} \frac{i}{2m_q}\epsilon^{jlm}\frac{\partial}{\partial x^l}\left[\frac{\vec{Q}^{\text{eff}}_{(n)}\sigma^m_{(n)}}{4\pi|\vec{x}-\vec{x}_n|}\right]. \tag{16}$$

This case is the non-Abelian analog of the usual Abelian solution, and evidently does not give quark confinement.

The possibility of interesting nonperturbative effects, within the classical equilibrium framework, arises from the fact that Eqs. (10) admit nontrivial background-field solutions with finite order-(−1) energy $E_{-1\ \text{gluon}}$,

$$E_{\text{gluon}} = (g^2)^{-1}E_{-1\ \text{gluon}} + E_{0\ \text{gluon}} + g^2 E_{1\ \text{gluon}} + \cdots\,,$$

$$E_{-1\ \text{gluon}} = \tfrac{1}{2}D\int d^3x\,(\vec{E}_0^j\cdot\vec{E}_0^j + \vec{B}_0^j\cdot\vec{B}_0^j)\,. \tag{17}$$

When classified by their large-x asymptotic behavior, these solutions group naturally into families characterized by a parameter κ with dimension $(\text{length})^{-1}$, and an integer-valued topological index n. To see this in a general way, let us consider static Euclidean potentials and field strengths as in Eqs. (10b), with finite energy integral Eq. (17), but which need not satisfy the field equations of Eq. (10c). I will assume that for large $x = |\vec{x}|$, the potentials and field strengths can be expanded in asymptotic series in inverse powers of x. In order for the integral of Eq. (17) to converge, $\vec{E}_0^j$ and $\vec{B}_0^j$ must vanish as $1/x^2$ at infinity. The asymptotic behavior of $\vec{B}_0^j$, together with Eq. (10b), implies only that $\vec{b}_0^j$ must vanish as $1/x$ at infinity. Since it is not necessary for $\vec{b}_0^j$ to have the asymptotic form of a gauge transformation $G(\partial/\partial x^j)G^{-1}$ [with G an SU(2)-matrix function of angular variables], the asymptotic behavior of $\vec{b}_0^j$ does not define a mapping $S^2 \to S^3$ analogous to the mapping $S^3 \to S^3$ defined by the asymptotic behavior of Euclidean finite-action solutions.[15] Things get more interesting, however, when we analyze the requirement that the electric field contribution to Eq. (17) be finite.[16] From $\vec{E}_0^j = -D_0^j\vec{b}_0^0 \sim 1/x^2$ we get $D_0^j(\vec{b}_0^0\cdot\vec{b}_0^0) = (\partial/\partial x^j)(\vec{b}_0^0\cdot\vec{b}_0^0) \sim 1/x^2$, which when integrated along the arc connecting radius vectors $x\hat{x}_1, x\hat{x}_2$ gives

$$\vec{b}_0^0\cdot\vec{b}_0^0\big|_{x\hat{x}_1} - \vec{b}_0^0\cdot\vec{b}_0^0\big|_{x\hat{x}_2} \sim \frac{1}{x}\xrightarrow[x\to\infty]{} 0\,. \tag{18}$$

Hence $|\vec{b}_0^0|$ must approach the same value in all directions,[17] defining a dimensional constant κ,

$$\lim_{x\to\infty} |\vec{b}_0^0| = \kappa\,. \tag{19}$$

The final step[18] in the classification is to define a unit isotopic vector $\hat{b} = \vec{b}_0^0/|\vec{b}_0^0|$, which in the asymptotic region defines a mapping $S^2 \to S^2$ with the winding number

$$n = \lim_{x\to\infty}\frac{1}{8\pi}\int d^2S_x^i \epsilon^{ijk}\epsilon^{abc}\hat{b}^a\frac{\partial}{\partial x^j}\hat{b}^b\frac{\partial}{\partial x^k}\hat{b}^c$$

$$= \frac{1}{8\pi}\int d^3x\,\epsilon^{ijk}\epsilon^{abc}\frac{\partial}{\partial x^i}\hat{b}^a\frac{\partial}{\partial x^j}\hat{b}^b\frac{\partial}{\partial x^k}\hat{b}^c\,. \tag{20}$$

Consider now a finite-energy, static Euclidean field configuration with asymptotic parameters κ and n. By an identity due to Bogomol'nyi[19] and to Coleman *et al.*,[19] the energy integral of Eq. (17) can be rewritten in terms of the topological quantum number n,

$$E_{-1\ \text{gluon}} = D\left[\int d^3x\,\tfrac{1}{2}(\vec{B}_0^j \pm \vec{E}_0^j)^2 \mp 4\pi\kappa n\right]. \tag{21}$$

Hence for the special class of self-dual solutions of Eq. (10c) satisfying

$$\vec{B}_0^j = \pm\vec{E}_0^j\,, \tag{22}$$

the energy is determined solely by the asymptotic parameters,

$$E^{\text{self-dual}}_{-1\ \text{gluon}} = D4\pi\kappa|n|\,, \tag{23}$$

and has no dependence on other parameters appearing in the solution. I suspect that self-dual solutions are in fact the only finite-$E_{-1\ \text{gluon}}$ solutions of the static Euclidean equations, just as self-dual solutions are believed to be the only finite-Euclidean-action solutions of the full time-dependent Euclidean Yang-Mills equations.[20]

Let me now examine the implications for self-dual background solutions of the principle of virtual work. Varying Eq. (17) and using Eq. (10c) gives

$$\delta E_{-1\ \text{gluon}} = \lim_{x\to\infty} -D\int dS_x^j \vec{E}_0^j \cdot \delta\vec{b}_0^0 , \tag{24a}$$

indicating that for background solutions, changes in energy can only arise from a transfer of energy through the sphere at infinity. Since self-dual solutions have $\delta E_{-1\ \text{gluon}}=0$ for fixed κ, all deformations (i.e., zero modes[21] of the operator $D^2{}_{(2)}$) which have the asymptotic scale parameter κ fixed must satisfy the boundary condition

$$\lim_{x\to\infty}\int dS_x^j \vec{E}_0^j \cdot \delta\vec{b}_0^0 = 0 . \tag{24b}$$

[It is easy to check that the dilatation $\delta\vec{b}_0^0 = (\partial/\partial\kappa)\vec{b}_0^0$ violates the condition of Eq. (24b).] Hence in order $(g^2)^{-1}$, self-dual background solutions with given asymptotic parameters κ, n describe energy in equilibrium in the finite region of space, with no energy transfer across the sphere at infinity. This will continue to be the case when source charges are added, provided that the surface terms in orders $(g^2)^{0,1}$, given by

$$\begin{aligned} \delta E_{0\ \text{gluon}}\big|_\infty &= \lim_{x\to\infty} -D\int dS_x^j(\vec{E}_0^j\cdot\delta\vec{b}_1^0 + \vec{E}_1^j\cdot\delta\vec{b}_0^0) , \\ \delta E_{1\ \text{gluon}}\big|_\infty &= \lim_{x\to\infty} -D\int dS_x^j \vec{E}_1^j\cdot\delta\vec{b}_1^0 , \end{aligned} \tag{25}$$

also vanish. Since we in general expect $\vec{b}_1^0 \sim 1/x$ (or smaller) asymptotically we have $\delta\vec{b}_1^0\sim 1/x$, $\vec{E}_1^j\sim 1/x^2$, $\vec{E}_0^j\sim 1/x^2$ and the terms involving $\delta\vec{b}_1^0$ all vanish. The term involving $\delta\vec{b}_0^0$ vanishes because $\delta\vec{b}_0^0$ vanishes asymptotically in general for deformations at fixed κ. (This is easily seen to be true in the undistorted background-field gauge $D_0^\mu\delta\vec{b}_0^\mu = 0$, where the asymptotic part of $\delta\vec{b}_0^0$ must satisfy $\vec{b}_0^0\times\delta\vec{b}_0^0 = \vec{b}_0^0\cdot\delta\vec{b}_0^0 = 0$, and will be true in any other gauge obtained by transformation[11] with a gauge function $\vec{\phi}$ which is bounded asymptotically.) Thus, when source charges are inserted in a self-dual background field, the virtual work equation of Eq. (8a) is satisfied. Note that it is important in this connection that the boundary condition of Eq. (19) arises without the presence of a Higgs potential, since when a Higgs potential is included in the equilibrium equation, the background-field energy can no longer be expressed entirely in terms of the asymptotic parameters κ, n.[22]

To recapitulate the main conclusions of the preceding discussion, we have seen the following: (1) Static Euclidean background-field configurations of finite energy[16] necessarily involve a dimensional parameter κ. Thus, they provide a natural means for introducing a spontaneous breakdown of scale invariance, which is essential to get the possibility of quark confinement. (2) Static Euclidean configurations of charges built perturbatively on a self-dual background field satisfy the principle of virtual work, thus permitting the field energy integral to be reinterpreted as a quark potential energy.

As a final general observation, I note that the background fields always occur in parity-conjugate pairs, corresponding in the self-dual case to the two choices of sign in Eq. (22). Since the background field associated with a hadronic $q\overline{q}$ system must have a definite parity, it will, in the classical limit, be an equal-probability-amplitude superposition of background-field configurations $\vec{B}_0^j = \vec{E}_0^j$ and $\vec{B}_0^j = -\vec{E}_0^j$. In such a state, the expectations of odd-parity quantities vanish, while the expectations of even-parity quantities are those calculated with either member of the background field pair alone.

In the remainder of this section I review (and correct in significant ways) the procedure developed in Ref. 2 for calculating the zeroth- and first-order static potentials. I take the sources to be a quark and an antiquark, with equal effective charge magnitudes $|\vec{Q}_{(1)}^{\text{eff}}| = |\vec{Q}_{(2)}^{\text{eff}}| = Q$ (cf. the discussion of Sec. II A above), and with the quark-antiquark separation fixed at $\kappa|\vec{x}_1 - \vec{x}_2| = 2\rho$. It is first of all necessary to fix the $q,\overline{q}$ spatial locations and their effective charge orientations relative to the fixed choice of background field. [Note that it is possible to introduce an arbitrary SU(2) rotation between the effective charges and the background field by first subjecting all physical quantities to an arbitrary local SU(2) rotation, and then subjecting the background field alone to the inverse rotation, bringing it back to its originally assumed form. I neglected this fact in the discussion of Eqs. (44)–(55) of Ref. 2, which is incorrect.] The first condition to be imposed in orienting the quarks is the zeroth-order approximation to the "compatibility conditions" of Eq. (7),[8]

$$\vec{b}_0^0(x_n)\times\vec{Q}_{(n)}^{\text{eff}} + \vec{B}_0^m(x_n)\times\frac{i\vec{Q}_{(n)}^{\text{eff}}\sigma_{(n)}^m}{2m_q} = 0 , \quad n=1,2 . \tag{26}$$

The remaining conditions are that, in the natural inner product

$$(b_1, b_2) \equiv \int d^3x\, \vec{b}_1^\mu \cdot \vec{b}_2^\mu \tag{27}$$

associated with the differential operator $D^2_{(2)}$ of Eq. (13), the source current must be orthogonal to all zero modes $b_{(s)}$ of $D^2_{(2)}$ [even if not normalizable in the inner product of Eq. (27)] which satisfy the boundary condition of Eq. (24b),

$$(J, b_{(s)}) = 0 \Longleftrightarrow \begin{cases} D^2_{(2)} b_{(s)} = 0\,, \\ \lim\limits_{x\to\infty} \int dS_x^j \vec{E}_0^j \cdot \vec{b}_{(s)}^0 = 0\,. \end{cases} \tag{28}$$

The relevant zero modes are obtained by differentiating the zeroth-order solution with respect to all parameters (other than κ) on which it depends, and always include the normalizable translational modes

$$\vec{b}_{(s)}^{\mu\ \mathrm{trans}} = D_0^\mu \psi_{(s)}^{\mathrm{trans}} + \frac{\partial}{\partial a^s} \vec{b}_0^\mu(x+a)\bigg|_{a=0}\,. \tag{29}$$

The gauge term $D_0^\mu \psi_{(s)}^{\mathrm{trans}}$ is needed for the modes $\vec{b}_{(s)}^{\mu\ \mathrm{trans}}$ to satisfy Eq. (12), but by virtue of Eq. (26) it makes no contribution to the orthogonality conditions of Eq. (28). The conditions of Eq. (26) and Eq. (28) will in general fix the source locations and effective charge orientations up to an overall sign in the effective charges; a criterion for resolving this remaining ambiguity (which, as we shall see, does not affect the static potential through order g^2) is discussed below in Sec. II D.

With the quark spatial locations and effective charge orientations fixed as a function of the separation 2ρ, I can now apply the principle of virtual work to get explicit expressions for the order-0 and order-1 contributions to the static potential,

$$V_{\mathrm{static}} = V_{0\ \mathrm{static}} + g^2 V_{1\ \mathrm{static}} + \cdots\,. \tag{30a}$$

The order-0 contribution to Eq. (8) gives

$$\delta V_{0\ \mathrm{static}} = D \int d^3x\, \delta\vec{b}_0^\mu \cdot \vec{J}^\mu\,, \tag{30b}$$

which vanishes since $\delta\vec{b}_0^\mu$ is a zero mode and thus is orthogonal to the source current. Hence the static potential has vanishing order-0 contribution, and it is also easy to see that the constant of integration in Eq. (8a) is such that the order-0 contribution to E_{gluon} vanishes as well,[23]

$$E_{0\ \mathrm{gluon}} = V_{0\ \mathrm{static}} = 0\,. \tag{30c}$$

To determine $V_{1\ \mathrm{static}}$ (and higher-order terms), let me make a particularly simple choice of variation δ in Eq. (8a): I rescale the source current by a scale factor λ, $\vec{J}^\mu \to \lambda \vec{J}^\mu$, and take δ to be $d/d\lambda$. Then Eq. (8a) becomes

$$\frac{dV_{\mathrm{static}}}{d\lambda} = \lambda D \int d^3x \frac{d\vec{b}_{(\lambda)}^\mu}{d\lambda} \cdot \vec{J}^\mu\,, \tag{30d}$$

with $\vec{b}_{(\lambda)}^\mu$ the solution of Eqs. (5a) and (5b) with the source current rescaled by λ. Integrating Eq. (30d) now gives an exact expression for V_{static} to all orders in g^2,

$$D^{-1} V_{\mathrm{static}} = \int_0^1 \lambda\, d\lambda \int d^3x \frac{d\vec{b}_{(\lambda)}^\mu}{d\lambda} \cdot \vec{J}^\mu\,. \tag{30e}$$

Since from the definition of our perturbation expansion we have $d\vec{b}_{(\lambda)}^\mu/d\lambda = g^2 \vec{b}_1^\mu + O(g^4)$, Eq. (30e) gives an independent derivation of $V_{0\ \mathrm{static}} = 0$, obtained by using variations which do not deform the background field. Consistency with Eq. (30b), which holds for general variations, is possible only when the source current is orthogonal to *all* zero modes (not just normalizable ones) satisfying the boundary condition of Eq. (24b). Another way of saying this is that unless the zero-mode orthogonality condition is imposed, the background-field approximation procedure does not generate a true solution of the original differential equation system given in Eqs. (5).

From Eq. (30e) we get the expression for $V_{1\ \mathrm{static}}$,

$$D^{-1} V_{1\ \mathrm{static}} = \tfrac{1}{2} \int d^3x\, \vec{b}_1^\mu \cdot \vec{J}^\mu\,. \tag{30f}$$

The first-order perturbation b_1 can be calculated from the source current by

$$b_1^{a\mu}(x) = \int d^3y\, G^{a\mu,\,b\nu}(x,y) J^{b\nu}(y)\,, \tag{31}$$

where $G^{a\mu,\,b\nu}(x,y)$ is the propagator defined by

$$\begin{aligned} &[D_{0x}^\sigma D_{0x}^\sigma \vec{G}^{\mu,\,b\nu}(x,y) + 2\vec{f}_0^{\,\mu\tau}(x) \times \vec{G}^{\tau,\,b\nu}(x,y)]^a = -Q^{a\mu,\,b\nu}(x,y)\,, \\ &Q^{a\mu,\,b\nu}(x,y) = \delta^{ab}\delta^{\mu\nu}\delta^3(x-y) - \sum_{\substack{\text{normalizable}\\ \text{zero modes } s}} b_{(s)}^{a\mu}(x) b_{(s)}^{b\nu}(y)\,. \end{aligned} \tag{32}$$

In evaluating the first-order static potential, it is convenient to define a subtracted Green's function by separating off the leading short-distance singularity[24] and removing divergent or ambiguous terms involving the zero modes,

$$G^{a\mu,b\nu}(x,y)_{\rm sub}=G^{a\mu,b\nu}(x,y)-\frac{\delta^{ab}\delta^{\mu\nu}}{4\pi|\vec{x}-\vec{y}|}-\sum_{\substack{\text{zero modes}\\ s}} C_{(s)}b^{a\mu}_{(s)}(x)b^{b\nu}_{(s)}(y)\,, \tag{33}$$

giving for the first-order static potential

$$\begin{aligned}D^{-1}V_{1\ \rm static}=&\frac{\vec{Q}^{\rm eff}_{(1)}\cdot\vec{Q}^{\rm eff}_{(2)}}{4\pi|\vec{x}_1-\vec{x}_2|}\\&-\frac{1}{4\pi}\frac{\vec{Q}^{\rm eff}_{(1)}\cdot\vec{Q}^{\rm eff}_{(2)}}{4m_q^2}\left[\frac{8\pi}{3}\sigma^m_{(1)}\sigma^m_{(2)}\delta^3(x_1-x_2)+\frac{3\sigma^m_{(1)}(x_1-x_2)^m\sigma^n_{(2)}(x_1-x_2)^n}{|\vec{x}_1-\vec{x}_2|^5}-\frac{\sigma^m_{(1)}\sigma^m_{(2)}}{|\vec{x}_1-\vec{x}_2|^3}\right]\\&+\frac{1}{4\pi}\frac{1}{4m_q^2}(\vec{Q}^{\rm eff}_{(1)}\times\vec{Q}^{\rm eff}_{(2)})\cdot\{\vec{b}^l_0(x_2)[(x_1-x_2)^l\sigma^m_{(1)}\sigma^m_{(2)}-\sigma^l_{(1)}\sigma^n_{(2)}(x_1-x_2)^n]\\&\qquad+\vec{b}^l_0(x_1)[(x_1-x_2)^l\sigma^m_{(1)}\sigma^m_{(2)}-\sigma^l_{(2)}\sigma^n_{(1)}(x_1-x_2)^n]\}\frac{1}{|\vec{x}_1-\vec{x}_2|^3}\\&-\frac{1}{4\pi}\frac{1}{4m_q^2}[\vec{Q}^{\rm eff}_{(1)}\cdot\vec{Q}^{\rm eff}_{(2)}\vec{b}^l_0(x_1)\cdot\vec{b}^n_0(x_2)-\vec{Q}^{\rm eff}_{(1)}\cdot\vec{b}^n_0(x_2)\vec{Q}^{\rm eff}_{(2)}\cdot\vec{b}^l_0(x_1)](\delta^{ln}\sigma^m_{(1)}\sigma^m_{(2)}-\sigma^n_{(1)}\sigma^l_{(2)})\frac{1}{|\vec{x}_1-\vec{x}_2|}\\&+\tfrac{1}{2}\int d^3x\,d^3y\,J^{a\mu}(x)G^{a\mu,b\nu}(x,y)_{\rm sub}J^{b\nu}(y)\,.\end{aligned} \tag{34}$$

In writing Eq. (34), I have followed the usual procedures of dropping the divergent quark self-energies arising from the divergent part $\delta^{ab}\delta^{\mu\nu}/(4\pi|\vec{x}-\vec{y}|)$ of the propagator, and of supplementing the naive spin potential by the standard contact term.[25] I have also made explicit use of the fact that the source current has been orthogonalized to the zero modes. The individual terms in Eq. (34) are not gauge invariant because the separation of $G^{a\mu,b\nu}(x,y)$ into subtracted and divergent parts defined in Eq. (34) is not a manifestly gauge-invariant procedure. Note, however, that the divergent quark self-energies $\vec{Q}^{\rm eff}_{(1)}\cdot\vec{Q}^{\rm eff}_{(1)}/(4\pi|\vec{x}_1-\vec{x}_1|)$, $\vec{Q}^{\rm eff}_{(2)}\cdot\vec{Q}^{\rm eff}_{(2)}/(4\pi|\vec{x}_2-\vec{x}_2|)$ which have been dropped are gauge invariant (the spin self-energies are irrelevant, since we recall[8] that only spin terms bilinear in the q and $\bar{q}$ spins are determined by our procedure), and so the sum of all terms in Eq. (34) gives a gauge-invariant static potential. Quark confinement, in the background-field approximation, would be signaled by an indefinite increase of the final term of Eq. (34) as the quark separation is increased to infinity, indicating a focusing of the quark flux lines into a stringlike configuration as a result of the influence of the background field. In Appendix B, I show that Eqs. (30f)–(34) are the same equations for $V_{1\ \rm static}$ as are obtained when the Wilson loop method for computing the static potential is applied to a static Euclidean background field. A method for evaluating the spin-orbit potential is given in Appendix C.

As discussed in detail in Ref. 2, for self-dual background fields the techniques of Brown, Carlitz, Creamer, and Lee[26] can be used to give an explicit construction of the vector propagator of Eq. (32) in terms of the scalar propagator $\Delta^{ab}(x,y)$ defined by

$$D^\sigma_{0x}D^\sigma_{0x}\Delta^{ab}(x,y)=-\delta^{ab}\delta^3(x-y)\,, \tag{35}$$

with the result [the signs ($\mp$) correspond to the signs $\pm$ in Eq. (22)]

$$\begin{aligned}G^{a\mu,b\nu}(x,y)&=q^{(\mp)\mu\nu\lambda\kappa}\int d^3z[D^\lambda_x\Delta(x,z)]^{ac}\times[D^\kappa_y\Delta(y,z)]^{bc}\,,\\q^{(\mp)\mu\nu\lambda\kappa}&=\delta^{\mu\lambda}\delta^{\nu\kappa}+\delta^{\mu\nu}\delta^{\lambda\kappa}-\delta^{\mu\kappa}\delta^{\nu\lambda}\mp\epsilon^{\mu\nu\lambda\kappa}\,,\\\epsilon^{0123}&=-1\,.\end{aligned} \tag{36}$$

In the self-dual case, there are also powerful techniques[27] which enable one to construct the scalar propagator itself. In Sec. III, I give an explicit calculation illustrating the methods described above when the background field is the $n=1$ spherically symmetric Prasad-Sommerfield solution.

D. Discussion

I turn finally to a discussion of two issues raised by the procedure for calculating a static quark potential suggested above.

1. Scale fixing and quantum stability of the confined state

The first issue concerns aspects of a theory of confinement which are not purely classical, but where inputs from the underlying quantum field theory are needed. The perturbative construction described in Secs. II B and II C clearly does not give a unique classical equilibrium configuration, but rather a discrete set of such configurations, corresponding to allowed[28] self-dual background-field solutions with different values of topological index n, and to an overall effective-charge sign ambiguity in the allowed source charge insertions in each background. Furthermore, as we have seen, each configuration with $n \neq 0$ involves an arbitrary dimensional scale parameter κ. A necessary condition for there to be a classical description of confinement is that there exist a self-dual background configuration which gives a confining static potential in order g^2. A full theory of confinement, however, will require two essential inputs from the underlying quantum field theory: (i) First, the quantum theory must guarantee that, in the limit of large quark separations (and infinitely massive quarks, so that quark pair creation can be ignored), the equilibrium configuration built about the confining background field is the true ground state and is stable against quantum decay. The quasi-Abelian equilibrium solution of Eq. (16), corresponding to vanishing background fields, must be a metastable state, which is unstable against making a quantum transition into the confining configuration when furnished with the requisite transition energy. Another way of putting this is that if a $q\bar{q}$ pair is created in the quasi-Abelian state with very small separation, the system must become unstable against making a first-order phase transition to the confining state when the quark-antiquark separation is increased sufficiently. I give below a simple mechanism which accomplishes this, based on Euclidean path-integral quantization ideas. (ii) Second, the quantum field theory must relate the scale parameters κ appearing in the various sectors with different particle content, together with scale parameters associated with renormalization points, in such a way that only one overall scale remains undetermined. A plausible way to get this scale fixing is to require that the asymptotic condition $\lim_{x\to\infty}\vec{\mathfrak{b}}^0\cdot\vec{\mathfrak{b}}^0=\kappa^2$ be maintained in the presence of quantum corrections in the form $\lim_{x\to\infty}\langle\vec{\mathfrak{b}}^0\cdot\vec{\mathfrak{b}}^0\rangle=\kappa^2$; that is, radiative corrections should not renormalize the scale parameter κ.

Returning to the question of quantum stability, I make the assumption that when quark loops are neglected, the procedure for quantizing algebraic chromodynamics is to quantize each overlying classical SU(j) Yang-Mills theory by the usual path-integral procedure. Hence I will study the Euclidean path-integral quantization of an SU(2) gauge theory with external sources. Just as in the discussion associated with Eq. (9) in Sec. II B above, it is useful to first consider the analogous Euclidean path integral for the quantum mechanics of a particle moving in a one-dimensional classical potential $V(x)$, which is formulated[12] in terms of the functional integral

$$Z=N\int d[x]e^{-S_E},\qquad S_E=\int dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2+V(x)\right]. \tag{37}$$

In the neighborhood of an equilibrium point x_c where $V'(x_c)=0$, the Euclidean action can be approximated by

$$S_E=\int dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2+\tfrac{1}{2}(x-x_c)^2V''(x_c)+V(x_c)\right], \tag{38}$$

which gives a convergent Gaussian functional integral provided that $V''(x_c)>0$. Suppose, as discussed in Ref. 12, that there are two equilibrium points x_{c_1}, x_{c_2}, both with $V''>0$ but with $V(x_{c_1})\neq V(x_{c_2})$. Which of the two is the true quantum-mechanical vacuum which is stable against quantum decay? The answer[12] is given by the following rule: Regard S_E as a free energy functional of the type familiar in statistical mechanics. Then the classical equilibrium configuration of minimum free energy is the true quantum vacuum. In the above example, this rule gives the intuitively obvious answer that the vacuum is the equilibrium point with the smaller (smallest if there are more than two candidates) value of $V(x_c)$.

Now let me return to the case of an SU(2) gauge theory, with external sources, where the relevant functional integral is

$$Z=N\int d[\vec{\mathfrak{b}}^\mu]e^{-S_E} \tag{39}$$

with S_E the action functional given in Eq. (8b) above. As emphasized in Sec. II B, the solutions to the classical equilibrium equations Eqs. (5) are analogs of the equilibrium points x_c in potential theory, so let me label them by a subscript c. The total potential $\vec{b}^\mu$ is then the sum of a classical and a quantum part, $\vec{b}^\mu = \vec{b}_c^\mu + \vec{b}_q^\mu$, and since the classical solutions are extrema of S_E (for fixed quark variables), we have

$$S_E = \text{quadratic in } \vec{b}_q^\mu + S_{Ec}\,, \tag{40}$$

$$S_{Ec} = DT \int d^3x \left[\frac{1}{2g^2} (\vec{E}_c^j \cdot \vec{E}_c^j + \vec{B}_c^j \cdot \vec{B}_c^j) - \vec{b}_c^0 \cdot \vec{J}^0 - \vec{B}_c^m \cdot \vec{\mathcal{J}}_{\text{spin}}^m \right],$$

where I have made explicit use of the fact that the equilibrium fields are Euclidean-time-independent. In order to define the functional integral of Eq. (39), it is of course necessary to add to the action a gauge-fixing term $\xi(D_c^\mu \vec{b}_q^\mu)^2$ and a compensating ghost term, but these are simply additional parts of the quadratic fluctuation term in the action, and play no role in the stability analysis. I now apply the rule stated above, according to which S_E plays the role of a free-energy functional in our analysis. Again, this rule predicts that *the classical equilibrium configuration of minimum free energy* S_{Ec} *is the true quantum vacuum*. Developing $T^{-1}S_{Ec}$ in a perturbation expansion in coupling g^2 as was done in Sec. II C above, and using [cf. Eqs. (23), (30c), (30f), and Ref. 8]

$$\begin{aligned} E_{\text{gluon}} &= D \int d^3x \frac{1}{2g^2} (\vec{E}_c^j \cdot \vec{E}_c^j + \vec{B}_c^j \cdot \vec{B}_c^j) \\ &= \frac{D4\pi\kappa|n|}{g^2} + g^2 V_{1\ \text{static}}\,, \\ -D \int d^3x\, g^2 (\vec{b}_{c1}^0 \cdot \vec{J}^0 + \vec{B}_{c1}^m \cdot \vec{\mathcal{J}}_{\text{spin}}^m) &= -Dg^2 \int d^3x\, \vec{b}_{c1}^\mu \cdot \vec{J}^\mu \\ &= -2g^2 V_{1\ \text{static}}\,, \end{aligned} \tag{41}$$

the classical action becomes

$$T^{-1}S_{Ec} = D\frac{4\pi\kappa|n|}{g^2} - D\int d^3x (\vec{b}_{c0}^0 \cdot \vec{J}^0 + \vec{B}_{c0}^m \cdot \vec{\mathcal{J}}_{\text{spin}}^m) - g^2 V_{1\ \text{static}} + O(g^4)\,. \tag{42}$$

There is one possible additional contribution to Eq. (42), arising from the fact that it is always permissible to add a total derivative to the action without changing the equations of motion. Thus, since $\vec{E}^j \cdot \vec{B}^j$ is a total derivative, a term[29]

$$\Delta S_E = \phi D \int d^3x \frac{1}{g^2} \vec{E}^j \cdot \vec{B}^j \tag{43}$$

(with ϕ a complex constant) can be added to the Euclidean action of Eq. (8b), leading to an additional term in $T^{-1}S_{Ec}$ given by [cf. Eqs. (21) and (22)]

$$T^{-1}\Delta S_{Ec} = \phi D \int d^3x \frac{1}{g^2} \vec{E}_c^j \cdot \vec{B}_c^j = \phi D \frac{4\pi\kappa n}{g^2}\,. \tag{44}$$

Only the real part of Eq. (44) is relevant to the stability analysis, and so the effective free-energy functional is given by

$$\begin{aligned} \mathfrak{F}_c &\equiv T^{-1}(S_{Ec} + \text{Re}\Delta S_{Ec}) \\ &= D\frac{4\pi\kappa}{g^2}(|n| + n\,\text{Re}\phi) \\ &\quad - D\int d^3x (\vec{b}_{c0}^0 \cdot \vec{J}^0 + \vec{B}_{c0}^m \cdot \vec{\mathcal{J}}_{\text{spin}}^m) \\ &\quad - g^2 V_{1\ \text{static}} + O(g^4)\,. \end{aligned} \tag{45}$$

Equation (45) exhibits a number of very interesting features. First of all, the term $g^2 V_{1\ \text{static}}$ appears with a negative coefficient.[30] Hence if there is a confining equilibrium solution, for which $g^2 V_{1\ \text{static}}$ becomes infinite as the q-$\bar{q}$ separation approaches infinity, this solution (and not the quasi-Abelian solution) will be the true quantum vacuum for large q-$\bar{q}$ separations. Second, the presence in $\mathfrak{F}_c$ of a zeroth-order term linear in the effective charges resolves the effective charge sign ambiguity discussed above, requiring effective charge alignment to give $\int d^3x\, \vec{b}_0^0 \cdot \vec{J}^0 > 0$, and justifying the procedure of dropping parity-odd quantities in averaging over parity-conjugate background solutions. Finally, the stability analysis at small q-$\bar{q}$ separation will be dominated by the g^{-2} term in Eq. (45); the details here will depend crucially on the value[29] of the constant Reϕ in the total derivative term ΔS_E of Eq. (43). For fixing the value of Reϕ, just as for fixing the value of the scale parameter κ, a deeper analysis of the quantum structure of the theory is needed. (See added note.)

2. *Classical stability considerations*

The second issue concerns the implications for the validity of the background-field approximation of the very interesting classical stability theorems[31] proved by Coleman, Deser, and Weder.[32] These theorems state that there are no finite-energy, nonsingular solutions of the Minkowski spacetime Yang-Mills equations for which the energy within a sphere of radius R remains greater than any positive ϵ for all time. In colloquial langauge, classical "lumps" or gluon bound states are not possible. As applied to the background-field configurations discussed above, the theorems state that when such configurations are introduced as Cauchy data in Minkowski spacetime at $t=0$, the energy which is initially concentrated in a finite radius rapidly disperses. At first sight this would seem to be a disaster for the program outlined above, but I believe that in fact it is not really a problem, for the following two reasons. First of all, even for a pure background configuration without source charges, it is clear that despite the spreading of energy the topological quantum number n of Eq. (20), which depends only on the asymptotic scalar potential, does not change when the time evolution is calculated with a sufficiently regular gauge fixing.[33] Hence a topological gluon bound state,[34] even when spread very thin, retains its distinguished topological character. Second, the insertion of source charges was an essential ingredient of the construction of Secs. II B and II C, and when source charges are present the Coleman-Deser-Weder theorems no longer apply. The physical reason for the failure of the theorems in the presence of the source charges is evident: Since the presence of the background field changes the static potential of the source charge configuration by a finite increment, and since energy conservation in Minkowski spacetime requires this increment to be constant in time, the configuration of sources plus background cannot relax to a Coulombic source solution with all of the background energy at infinity. What the Coleman-Deser-Weder theorems do imply is that the procedure of perturbing about a background solution, while valid on the initial time slice in Minkowski spacetime, cannot be uniformly valid for all Minkowski times. No matter how small the coupling g^2, the background configuration will eventually spread to the point where the effect of the source charges can no longer be treated as a small perturbation.

III. STATIC POTENTIAL IN A PRASAD-SOMMERFIELD BACKGROUND FIELD

I turn now to the task of completing the calculation, begun in Ref. 2, of the static $q\overline{q}$ potential in a Prasad-Sommerfield background field. The background potentials, which are readily verified to give a solution of Eq. (22), are

$$b_0^{a0} = \mp \frac{x^a}{x^2}(1-\kappa x \coth \kappa x),$$
$$b_0^{ai} = \frac{\epsilon^{aij}x^j}{x^2}\left(1-\frac{\kappa x}{\sinh \kappa x}\right). \tag{46}$$

In order to keep things relatively simple, I will neglect spin forces throughout, since they are only a small perturbation on the dominant charge interaction terms. As in Sec. II C, I take the quark and antiquark effective charges to have equal magnitudes Q, and fix the quark-antiquark separation at $\kappa|\vec{x}_1-\vec{x}_2|=2\rho$. The first step in the calculation is to fix the $q,\overline{q}$ spatial locations and effective charge orientations. Substituting Eq. (46) into the zeroth-order compatibility condition of Eq. (26), and dropping spin terms, gives the conditions

$$\hat{x}_n \times \vec{Q}^{\text{eff}}_{(n)} = 0, \quad n=1,2. \tag{47}$$

That is, the quark effective charges must be oriented parallel or antiparallel to their radius vectors. The next step is to orthogonalize the source current to the zero modes. The translational zero modes may be calculated from Eq. (29) [or equivalently, by substituting $\vec{\phi}=\vec{b}_0^0$ into Eq. (85) of Ref. 2, which gives them directly in the background-field gauge], with the resulting condition on the source charges

$$\sum_{n=1}^{2} \hat{x}_n^j\left(\frac{1}{x_n^2}-\frac{1}{\sinh^2 x_n}\right)\hat{x}_n \cdot \vec{Q}^{\text{eff}}_{(n)} = 0. \tag{48}$$

Since the function $f(z)=1/z^2-1/\sinh^2(z)$ is monotone decreasing for $0<z<\infty$, the only simultaneous solutions of Eqs. (47) and (48) are

$$\hat{x}_1=-\hat{x}_2, \quad \kappa x_1 = \kappa x_2 = \rho,$$
$$\vec{Q}^{\text{eff}}_{(1)} = Q\hat{x}_1, \quad \vec{Q}^{\text{eff}}_{(2)} = Q\hat{x}_2,$$
or
$$\vec{Q}^{\text{eff}}_{(1)} = -Q\hat{x}_1, \quad \vec{Q}^{\text{eff}}_{(2)} = -Q\hat{x}_2, \tag{49}$$

and so the source locations and effective charge orientations are fixed up to the sign ambiguity in the effective charges discussed in Sec. II. I will assume, in the remaining discussion, that there are no other zero modes (such as distortions of the Prasad-Sommerfield solution) to which the source current must be orthogonalized, although this is an important open question.[35]

To evaluate Eq. (34) for the order-g^2 static potential, it is necessary to calculate the vector propagator $G^{a\mu,b\nu}(x,y)$ in the Prasad-Sommerfield background solution. By Eq. (36), this propagator may be expressed in terms of the scalar propagator in the same background field. An explicit contour-integral formula for the scalar propagator, in the singular gauge where the background potential takes the form $b_0^{a\mu}(x) = -\eta^{(-)\mu\nu a}\partial^\nu \ln \pi(x)$, was constructed in Appendix A of Ref. 2.[36] The evaluation of the contour integrals is basically straightforward, but tedious. The only minor subtlety that arises is that it is important to take the triangle inequality into account in determining the locations of singularities relative to the integration contours. For example, in the integral

$$\int dv \frac{e^{2iv}}{v+\frac{1}{2}i(\Delta+y-x)}, \quad \Delta=|\vec{x}-\vec{y}| \tag{50}$$

the pole $v=-\frac{1}{2}i(\Delta+y-x)$ always lies in the lower half complex plane, and so makes no contribution when the integration contour is closed up. The result of doing the integrations is a very lengthy expression for the scalar propagator in the singular gauge. The final step in the calculation is to use the complex matrix of Eq. (A4) of Ref. 2 to rotate this propagator to the physical gauge, where the potentials are given by Eq. (46). The rotation not only cancels away all imaginary terms in the singular gauge propagator but also greatly simplifies the remaining real terms as well, yielding the relatively compact expressions for the physical-gauge propagator Δ^{ab} given in Appendix A below. (The simplification which occurs in doing the gauge rotation suggests that there should be a better way of doing the calculation, in which one transforms directly to the physical gauge before doing explicit evaluation of integrals.) As a check on the final result for Δ^{ab}, I used numerical methods to verify the scalar propagator differential equation for a representative set of values $\vec{x},\vec{y}$, approximating derivatives by finite differences on a very fine mesh. Relevant formulas for the differential operator $D^{\sigma}_{0x}D^{\sigma}_{0x}$ are also given in Appendix A.

When spin terms are neglected, and the geometry of quark orientations and the scaling property of Eq. (A1) are taken into account, Eq. (34) can be rewritten in the form

$$g^2V_{1\ \text{static}}=\kappa\frac{g_{\text{eff}}^{\ 2}}{4\pi}\left[v_1(\rho)-\frac{1}{2\rho}\right], \quad g_{\text{eff}}^{\ 2}\equiv g^2DQ^2, \tag{51}$$

with

$$v_1(\rho)=4\pi[G^{LL}(\rho\hat{x}_1,\rho\hat{x}_1)_{\text{sub}} + G^{LL}(\rho\hat{x}_1,-\rho\hat{x}_1)_{\text{sub}}],$$

$$G^{LL}(x,y)_{\text{sub}}\equiv\hat{x}^a\hat{y}^bG^{a0,b0}(x,y;\kappa=1)_{\text{sub}}. \tag{52}$$

In order to obtain a convenient expression for $G^{LL}(x,y)_{\text{sub}}$, I use Eq. (36) to get

$$\hat{x}^a\hat{y}^bG^{a0,b0}(x,y;1)=\int d^3z\,\Delta^j(x,z;1)^{Lb} \times\Delta^j(y,z;1)^{Lb}, \tag{53}$$

with

$$\Delta^j(x,y;1)^{Lb} \equiv\hat{x}^a\left[\frac{\partial}{\partial x^j}\Delta^{ab}(x,y)+\epsilon^{ars}b_0^{rj}(x)\Delta^{sb}(x,y)\right]\bigg|_{\kappa=1}. \tag{54}$$

An explicit expression for $\Delta^j(x,y;1)^{Lb}$ is given in Appendix A. From the formulas given there, it is easy to see that the leading singularity of the integrand of Eq. (53), when $\vec{x}-\vec{z}$ and $\vec{y}-\vec{z}$ are both small, is

$$\Delta^j(x,z;1)^{Lb}\Delta^j(y,z;1)^{Lb}\underset{\substack{\vec{x}-\vec{z}\sim 0\\ \vec{y}-\vec{z}\sim 0}}{\simeq}\frac{1}{(4\pi)^2}\hat{x}\cdot\hat{y} \times\frac{(\vec{z}-\vec{x})\cdot(\vec{z}-\vec{y})}{|\vec{z}-\vec{x}|^3|\vec{z}-\vec{y}|^3}, \tag{55}$$

which gives the expected Coulombic singularity when integrated over z,

$$\int d^3z\frac{1}{(4\pi)^2}\hat{x}\cdot\hat{y}\frac{(\vec{z}-\vec{x})\cdot(\vec{z}-\vec{y})}{|\vec{z}-\vec{x}|^3|\vec{z}-\vec{y}|^3} =\hat{x}\cdot\hat{y}\frac{1}{4\pi|\vec{x}-\vec{y}|}. \tag{56}$$

Similarly, the leading large-z behavior of the integrand of Eq. (53) is

$$\Delta^j(x,z;1)^{Lb}\Delta^j(y,z;1)^{Lb}\underset{z\to\infty}{\simeq}\frac{1}{(4\pi)^2}\hat{x}\cdot\hat{y}\frac{1}{z^2}\left(\frac{1}{x^2}-\frac{1}{\sinh^2x}\right)\left(\frac{1}{y^2}-\frac{1}{\sinh^2y}\right), \tag{57}$$

which when integrated over z is proportional to a linearly divergent constant times a sum of projected translational zero modes,

$$\int d^3z \frac{1}{(4\pi)^2} \hat{x}\cdot\hat{y} \frac{1}{z^2}\left(\frac{1}{x^2}-\frac{1}{\sinh^2 x}\right)\left(\frac{1}{y^2}-\frac{1}{\sinh^2 y}\right) \propto \sum_s \hat{x}^a b_{(s)}^{a0\ \mathrm{trans}}(x)\hat{y}^b b_{(s)}^{b0\ \mathrm{trans}}(y) \,. \tag{58}$$

Hence comparing Eq. (33) with Eqs. (52)–(58), we see that

$$G^{LL}(x,y)_{\mathrm{sub}} = \int d^3z [\Delta^j(x,z;1)^{Lb}\Delta^j(y,z;1)^{Lb}]_{\mathrm{sub}},$$

$$[\Delta^j(x,z;1)^{Lb}\Delta^j(y,z;1)^{Lb}]_{\mathrm{sub}} = \Delta^j(x,z;1)^{Lb}\Delta^j(y,z;1)^{Lb} - \frac{1}{(4\pi)^2}\hat{x}\cdot\hat{y}\frac{(\vec{z}-\vec{x})\cdot(\vec{z}-\vec{y})}{|\vec{z}-\vec{x}|^3|\vec{z}-\vec{y}|^3} - \frac{1}{(4\pi)^2}\hat{x}\cdot\hat{y}\frac{1}{z^2}\left(\frac{1}{x^2}-\frac{1}{\sinh^2 x}\right)\left(\frac{1}{y^2}-\frac{1}{\sinh^2 y}\right). \tag{59}$$

The integral in Eq. (59) is now convergent since the potential logarithmic divergences arising from the regions $|\vec{z}-\vec{x}|\sim 0$, $|\vec{z}-\vec{y}|\sim 0$, and $z\sim\infty$ all vanish after the respective angular integrations $\int d\Omega_{\vec{z}-\vec{x}}$, $\int d\Omega_{\vec{z}-\vec{y}}$, $\int d\Omega_{\vec{z}}$ are done.[24] Equation (59) cannot be expressed in terms of elementary functions, so I evaluated it by numerical integration, with due attention to the need for symmetric angular integrations in the neighborhood of the points $\vec{z}=\vec{x}$, $\vec{z}=\vec{y}$, $\vec{z}=\infty$. In order to minimize truncation errors, I rewrote Eq. (59) in terms of subtracted versions of the individual factors appearing in the integrand, defined by explicitly removing leading short-distance and large-distance behaviors,

$$\Delta^j(x,z;1)^{Lb}_{\mathrm{sub}} \equiv \Delta^j(x,z;1)^{Lb} - \frac{1}{4\pi}\frac{\hat{x}^j\hat{z}^b}{z}\left(\frac{1}{x^2}-\frac{1}{\sinh^2 x}\right) - \frac{1}{4\pi}\frac{\hat{x}^b(z^j-x^j)}{|\vec{z}-\vec{x}|^3},$$

$$[\Delta^j(x,z;1)^{Lb}\Delta^j(y,z;1)^{Lb}]_{\mathrm{sub}} = \Delta^j(x,z;1)^{Lb}_{\mathrm{sub}}\,\Delta^j(y,z;1)^{Lb}_{\mathrm{sub}} + \cdots. \tag{60}$$

The results of the numerical integration for $v_1(\rho)$ are shown in Fig. 1, together with the Coulombic potential $-1/(2\rho)$ and the total order-g^2 potential $v_1(\rho)-1/(2\rho)$. The fact that $v_1(\rho)$ is everywhere positive indicates that the effect of the background field is always to focus, rather than defocus, the quark color flux lines. Note that an interpretation of $V_{1\,\mathrm{static}}$ in terms of the density of quark color flux lines is warranted, because from Eqs. (8a) and (10a) we have

$$V_{1\ \mathrm{static}} = \int d^3x \tfrac{1}{2} D(\vec{E}_1^j\cdot\vec{E}_1^j + \vec{B}_1^j\cdot\vec{B}_1^j + 2\vec{E}_0^j\cdot\vec{E}_2^j + 2\vec{B}_0^j\cdot\vec{B}_2^j) - (\text{divergent quark self-energies}), \tag{61}$$

with $\vec{E}_{1,2}^j$ and $\vec{B}_{1,2}^j$ the changes in the color electric and magnetic fields produced by adding the quarks to the background-field configuration. However, the focusing effect is not strong enough when the quarks are far apart to give a confining potential. The numerical results (for $\rho\leq 10$) indicate that $v_1(\rho)\simeq 1/\rho$ for large ρ, giving $v_1(\rho)-1/(2\rho)\simeq 1/(2\rho)$ in the asymptotic region. What evidently happens is that there is a significant focusing effect when the quarks are located in the central strong-background-field region $\rho\lesssim 2$, but not when the quarks are in the asymptotic region of the background, where the background fields $\vec{E}_0^j, \vec{B}_0^j$ vanish as $1/\rho^2$. In order to get the possibility of confinement, it appears that one needs a family of background-field configurations characterized by four parameters $\kappa, \vec{a}$, in which the bulk of the background-field energy is concentrated around the points $\pm\vec{a}$, and for which orthogonalization of the quark source current to the deformational zero modes selects a value of $\vec{a}$ which, after translational orientation, puts the q and $\bar{q}$ in the strong

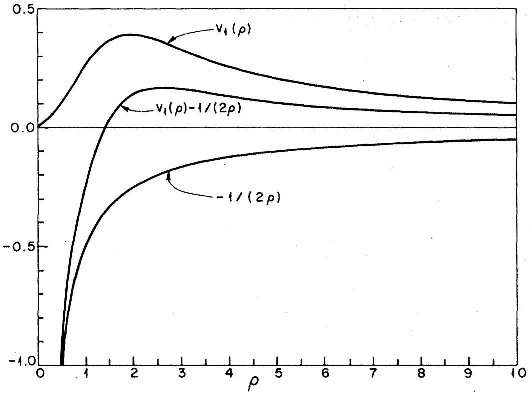

FIG. 1. Order-g^2 potentials in a Prasad-Sommerfield background field, as defined in Eqs. (51) and (52) of Sec. III.

background-field regions. Such a family of background fields would obviously not be spherically symmetric, but it could still have axial symmetry. It is not at present known whether there are axially symmetric solutions to the self-dual background-field equations.[37]

IV. OUTLOOK

In order for the classical approach to quark statics, outlined in this paper and in Refs. 1 and 2, to give a theory of confinement, two well-defined problems must be solved. First, as reviewed in Sec. II A, a construction for the quark color charges and outer and inner products must be found which has the Jacobi and (restricted) trace properties.[38] Second, as discussed in Secs. II C and III, a self-dual background-field solution (very likely, an axially symmetric one) must be found which gives a confining potential in order g^2. (A particularly interesting question in this regard is whether the $n=1$ spherically symmetric Prasad-Sommerfield solution can be embedded in a more general axially symmetric family of $n=1$ self-dual background-field solutions.[35]) A solution to the second problem alone would permit the calculation of the $q\bar{q}$ static potential in terms of the two parameters κ and $g_{\rm eff}{}^2$. A solution to the first problem is needed to get a color Hilbert-space interpretation, and to open the way to the construction of an underlying quantum field theory.

Note added in proof

There is a simple and natural modification of the definition of the Euclidean action functional which eliminates the order-$(g^2)^{-1}$ terms which complicate the stability analysis of Sec. II D 1. The modification is obtained by replacing Eq. (40) by

$$S_{Ec}=DT\int d^3x\left[\frac{1}{2g^2}(\vec{E}_c^j\cdot\vec{E}_c^j+\vec{B}_c^j\cdot\vec{B}_c^j)-\vec{b}_c^0\cdot\vec{J}^0-\vec{B}_c^m\cdot\vec{\mathcal{J}}^m_{\rm spin}\right]+\frac{DT}{g^2}\int_{\substack{\rm sphere\\ \rm at\ \infty}} d^2S^j\vec{b}_c^0\cdot\vec{E}_c^j. \qquad \text{(N1)}$$

The added surface term at infinity, when combined with the charge interaction term, can be reinterpreted as the volume integral of a total divergence,

$$DT\left(-\int d^3x\,\vec{b}_c^0\cdot\vec{J}^0+\frac{1}{g^2}\int_{\substack{\rm sphere\\ \rm at\ \infty}} d^2S^j\vec{b}_c^0\cdot\vec{E}_c^j\right)$$
$$=DT\left(\frac{1}{g^2}\int_{\substack{\rm small\ spheres\\ \rm around\ source\\ \rm charges}} d^2S^j\vec{b}_c^0\cdot\vec{E}_c^j+\frac{1}{g^2}\int_{\substack{\rm sphere\\ \rm at\ \infty}} d^2S^j\vec{b}_c^0\cdot\vec{E}_c^j\right)$$
$$=\frac{DT}{g^2}\int d^3x\,\frac{\partial}{\partial x^j}(\vec{b}_c^0\cdot\vec{E}_c^j), \qquad \text{(N2)}$$

and so it is not really consistent to include the charge interaction term without including the surface term at ∞ as well. With S_{Ec} given by Eq. (N1), the order-$(g^2)^{-1}$ contribution becomes

$$T^{-1}S_{-1Ec}=\frac{D}{g^2}\left[\int d^3x\,\tfrac{1}{2}(\vec{E}_{c0}\pm\vec{B}_{c0})^2+\int_{\substack{\rm sphere\\ \rm at\ \infty}} d^2S^j\vec{b}_{c0}^0\cdot(\vec{E}_{c0}^j\pm\vec{B}_{c0}^j)\right], \qquad \text{(N3)}$$

and vanishes for both self-dual and anti-self-dual background fields. The added surface term produces no change in the higher-order contributions to S_{Ec}. Equation (45) now becomes

$$\mathfrak{F}_c=D\frac{4\pi\kappa}{g^2}n\,{\rm Re}\phi-D\int d^3x(\vec{b}_{c0}^0\cdot\vec{J}^0+\vec{B}_{c0}^m\cdot\vec{\mathcal{J}}^m_{\rm spin})-g^2V_{1\,\rm static}+O(g^4), \qquad \text{(N4)}$$

and the order-$(g^2)^{-1}$ term vanishes when the conventional choice[29] ${\rm Re}\phi=0$ is made for the coefficient of the pseudoscalar total derivative term. Thus, $\mathfrak{F}_c$ contains no $(g^2)^{-1}$ term and there is now no danger of a phase transition out of the confining state at small q-$\bar{q}$ separations. That is, with the redefined Euclidean action and free energy, there is no free-energy penalty associated with having topologically nontrivial background-field configurations.

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APPENDIX A: PROPAGATOR FORMULAS

In giving formulas for the scalar propagator Δ^{ab} in the Prasad-Sommerfield background field, it is convenient to set $\kappa = 1$; to change to general κ one uses the scaling law

$$\Delta^{ab}(x,y)_{\text{general }\kappa} \equiv \Delta^{ab}(x,y;\kappa) = \kappa\Delta^{ab}(\kappa x, \kappa y; 1)\,. \tag{A1}$$

Writing

$$\Delta^{ab}(x,y;1) = \frac{1}{4\pi}\,\frac{x}{\sinh x}\,\frac{y}{\sinh y}\,\Sigma^{ab}\,, \tag{A2}$$

I find the following expression for Σ^{ab}:

$$\Sigma^{ab} = \sum_{i=1}^{5} \sigma_i^{ab}(x,y)\lambda_i(x,y)\,,$$

$$\begin{aligned}
\sigma_1^{ab} &= \delta^{ab} + \frac{\vec{x}\cdot\vec{y}}{x^2y^2}\,x^a y^b - \frac{x^a x^b}{x^2} - \frac{y^a y^b}{y^2}\,,\\
\sigma_2^{ab} &= x^a y^b\,,\\
\sigma_3^{ab} &= x^b y^a - \delta^{ab}\vec{x}\cdot\vec{y}\,,\\
\sigma_4^{ab} &= \frac{x^a}{x^2}\left(x^b - y^b\,\frac{\vec{x}\cdot\vec{y}}{y^2}\right),\\
\sigma_5^{ab} &= \frac{y^b}{y^2}\left(y^a - x^a\,\frac{\vec{x}\cdot\vec{y}}{x^2}\right),\\
\lambda_1 &= \frac{1}{2\Delta}\left[f_2(z_{++}) + f_2(z_{--}) + f_2(z_{+-}) + f_2(z_{-+})\right]\\
&= \frac{2}{\Delta}\int_0^1 d\alpha(1-\alpha)e^{-\alpha\Delta}\cosh\alpha x\cosh\alpha y\,,\\
\lambda_2 &= \frac{1}{x^2y^2}\left(\frac{\cosh x\cosh y - e^{-\Delta}}{\Delta} - \frac{\sinh x}{x}\,\frac{\sinh y}{y}\right),\\
\lambda_3 &= \frac{1}{2xy\Delta}\left[-f_2(z_{++}) - f_2(z_{--}) + f_2(z_{+-}) + f_2(z_{-+})\right]\\
&= -\frac{2}{\Delta}\int_0^1 d\alpha(1-\alpha)e^{-\alpha\Delta}\,\frac{\sinh\alpha x}{x}\,\frac{\sinh\alpha y}{y}\,,
\end{aligned} \tag{A3}$$

$$\begin{aligned}
\lambda_4 &= \frac{1}{2x\Delta}\{e^{x}[f_1(z_{-+}) + f_1(z_{--})] - e^{-x}[f_1(z_{+-}) + f_1(z_{++})]\}\\
&= \frac{2}{\Delta}\int_0^1 d\alpha\, e^{-\alpha\Delta}\cosh\alpha y\,\frac{\sinh(1-\alpha)x}{x}\,,\\
\lambda_5 &= \frac{1}{2y\Delta}\{e^{y}[f_1(z_{+-}) + f_1(z_{--})] - e^{-y}[f_1(z_{-+}) + f_1(z_{++})]\}\\
&= \frac{2}{\Delta}\int_0^1 d\alpha\, e^{-\alpha\Delta}\cosh\alpha x\,\frac{\sinh(1-\alpha)y}{y}\,,\\
\Delta &= |\vec{x}-\vec{y}|\,,\\
z_{++} &= x+y-\Delta,\quad z_{+-} = x-y-\Delta,\\
z_{-+} &= -x+y-\Delta,\quad z_{--} = -x-y-\Delta\,,\\
f_1(z) &= \frac{e^z-1}{z}\,,\quad f_2(z) = \frac{e^z-1-z}{z^2}\,.
\end{aligned} \tag{A4}$$

It is easy to verify that, despite the factors x^{-2} and y^{-2} in the σ's, Eq. (A3) is in fact analytic near $x=0$ and near $y=0$. Near $\vec{x}=\vec{y}$, Eq. (A2) has the expected short-distance singularity

$$\Delta^{ab}(x,y;1) \underset{\Delta\to 0}{\sim} \frac{1}{4\pi}\,\frac{\delta^{ab}}{\Delta} + O(1)\,, \tag{A5}$$

while the limiting behavior for large y, with x fixed, is

$$\Delta^{ab}(x,y;1) \underset{y\to\infty}{\sim} \frac{1}{4\pi}\,\frac{x^a}{x}\left(\coth x - \frac{1}{x}\right)\frac{y^b}{y^2} + O\!\left(\frac{1}{y^2}\right), \tag{A6}$$

which has $b_0^{a0}(x)$ as the x-dependent factor. The expression for the differential operator $D_x^\mu D_x^\mu$ used in verifying the propagator differential equation is

$$\begin{aligned}
&\left(\frac{\sinh x}{x}\,D_{0x}^\sigma D_{0x}^\sigma\,\frac{x}{\sinh x}\,\vec{\phi}\right)^a = \left(\frac{\partial}{\partial x^j}\right)^2\phi^a + C_1\phi^a + C_2\hat{x}^j\,\frac{\partial}{\partial x^j}\,\phi^a + C_3\hat{x}^a\hat{x}^j\phi^j + C_4\hat{x}^j\,\frac{\partial}{\partial x^a}\,\phi^j + C_5\hat{x}^a\frac{\partial}{\partial x^j}\,\phi^j\,,\\
&C_1 = \frac{2}{x}\left(\frac{1}{\sinh x} - \coth x\right),\quad C_2 = 2\left(\frac{1}{x} - \coth x\right),\quad C_3 = 1 + C_1\,,\\
&C_4 = -2\left(\frac{1}{x} - \frac{1}{\sinh x}\right),\quad C_5 = -C_4\,.
\end{aligned} \tag{A7}$$

The projected covariant derivative of the scalar Green's function, defined by Eq. (46) of the text, is given by the following formulas:

$$\Delta^j(x,y;1)^{Lb}=\frac{1}{4\pi}\,\frac{y}{\sinh y}\sum_{j=1}^{5}\sigma^{jb}(x,y)\tau_j(x,y)\,,$$

$$\begin{aligned}
\tau_1(x,y)&=\frac{1}{\sinh x}\left(\lambda_4-\frac{\vec{x}\cdot\vec{y}}{\Delta}\,\frac{\partial\lambda_4}{\partial\Delta}\right)-\frac{x}{\sinh^2 x}\,\lambda_1\,,\\
\tau_2(x,y)&=\left(\frac{2}{\sinh x}-\frac{x\cosh x}{\sinh^2 x}\right)\lambda_2+\frac{x}{\sinh x}\,\frac{\partial\lambda_2}{\partial x}+\frac{x^2-\vec{x}\cdot\vec{y}}{\Delta\sinh x}\,\frac{\partial\lambda_2}{\partial\Delta}\,,\\
\tau_3(x,y)&=-\frac{1}{\Delta\sinh x}\,\frac{\partial\lambda_4}{\partial\Delta}-\frac{x}{\sinh^2 x}\,\lambda_3\,,\\
\tau_4(x,y)&=\left(\frac{1}{\sinh x}-\frac{x\cosh x}{\sinh^2 x}\right)\lambda_4+\frac{x}{\sinh x}\,\frac{\partial\lambda_4}{\partial x}+\frac{x^2-\vec{x}\cdot\vec{y}}{\Delta\sinh x}\,\frac{\partial\lambda_4}{\partial\Delta}\,,\\
\tau_5(x,y)&=-\frac{x^2y^2}{\Delta\sinh x}\,\frac{\partial\lambda_2}{\partial\Delta}-\frac{x}{\sinh^2 x}\,\lambda_5\,,
\end{aligned}\tag{A8}$$

with

$$\begin{aligned}
\frac{\partial\lambda_4}{\partial\Delta}&=-\left(1+\frac{1}{\Delta}\right)\lambda_4+\frac{1}{2x\Delta}\{e^x[f_2(z_{-+})+f_2(z_{--})]-e^{-x}[f_2(z_{+-})+f_2(z_{++})]\}\,,\\
\frac{\partial\lambda_4}{\partial x}&=-\frac{1}{x}\,\lambda_4+\frac{1}{2x\Delta}\{e^x[f_2(z_{-+})+f_2(z_{--})]+e^{-x}[f_2(z_{+-})+f_2(z_{++})]\},\\
\frac{\partial\lambda_2}{\partial\Delta}&=\frac{1}{x^2y^2}\left(-\frac{\cosh x\cosh y-e^{-\Delta}}{\Delta^2}+\frac{e^{-\Delta}}{\Delta}\right),\\
\frac{\partial\lambda_2}{\partial x}&=-\frac{2}{x}\,\lambda_2+\frac{1}{x^2y^2}\left[\frac{\sinh x\cosh y}{\Delta}-\left(\frac{\cosh x}{x}-\frac{\sinh x}{x^2}\right)\frac{\sinh y}{y}\right].
\end{aligned}\tag{A9}$$

In the region $y\gg x$, $y\gg 1$ the following formula is useful:

$$\begin{aligned}
\Delta^j(x,y;1)^{Lb}&=\frac{1}{4\pi}\left[x^jy^b\tau_2^s+\frac{y^b}{y^2}\left(y^j-x^j\,\frac{\vec{x}\cdot\vec{y}}{x^2}\right)\tau_5^s\right]+O(e^{-y})\,,\\
\tau_2^s&=\frac{1}{y^2}\,\frac{1}{x^3}-\frac{1}{\Delta y}\,\frac{1}{x\sinh^2 x}-\frac{x^2-\vec{x}\cdot\vec{y}}{\Delta^3 y}\,\frac{\cosh x}{x^2\sinh x}\,,\\
\tau_5^s&=\frac{y}{\Delta^3}\,\frac{\cosh x}{\sinh x}-\frac{x}{\sinh^2 x}\,\frac{1}{\Delta}\left(\frac{1}{y+\Delta-x}+\frac{1}{y+\Delta+x}\right).
\end{aligned}\tag{A10}$$

APPENDIX B: CALCULATION OF $V_{1\,\text{static}}$ FROM THE WILSON LOOP

I show in this Appendix that Eqs. (30f)–(34) for the order-g^2 static potential $V_{1\,\text{static}}$ are also obtained when the Wilson loop method[9] for computing the potential is applied to a static Euclidean background field. I write the Euclidean Wilson loop formula (neglecting spin terms) in the form

$$\left\{\int d[\vec{b}^\mu]\exp\left[-D\int d^4x\,\frac{1}{2g^2}\,(\vec{E}^j\cdot\vec{E}^j+\vec{B}^j\cdot\vec{B}^j)-iD\int d^4x\,\vec{b}^0\cdot\vec{J}^0(x)\right]\right\}$$

$$\times\left\{\int d[\vec{b}^\mu]\exp\left[-D\int d^4x\,\frac{1}{2g^2}\,(\vec{E}^j\cdot\vec{E}^j+\vec{B}^j\cdot\vec{B}^j)\right]\right\}^{-1}$$

$$\equiv\left\langle\exp\left[-iD\int d^4x\,\vec{b}^0(x)\cdot\vec{J}^0(x)\right]\right\rangle\underset{(\text{large }T)}{=}e^{-V_{\text{static}}T+i\Omega T}\,,\quad\int d^4x=\int_{-T/2}^{T/2}dx^0\int d^3x\,,\tag{B1}$$

with $\vec{J}^0(x)$ the static source current of Eq. (6). The exponent $-iD\int d^4x\vec{b}^0(x)\cdot\vec{J}^0(x)$ is just the natural generalization, for the case of static Yang-Mills point sources, of the Abelian Wilson loop integral $-i\oint dx^\mu b^\mu(x)$ evaluated over a quark-antiquark loop of temporal extent T. The only unconventional feature of Eq. (B1) is the fact that I have allowed for the presence of a phase term on the right-hand side, which I assume does not represent a true contribution to the potential energy.

To evaluate the functional integrals, I write $\vec{b}^\mu=\vec{b}_0^\mu+\vec{b}_q^\mu$, with $\vec{b}_0^\mu$ a static Euclidean background field solution. Shifting to $\vec{b}_q^\mu$ as the new variable in the functional integration, Eq. (B1) becomes

$$\left\{\int d[\vec{b}_q^\mu]\exp\left[-\int d^4x\,Q(\vec{b}_q^\mu)-iD\int d^4x\,\vec{b}_q^0(x)\cdot\vec{J}^0(x)\right]\right\}\left\{d[\vec{b}_q^\mu]\exp\left[-\int d^4x\,Q(\vec{b}_q^\mu)\right]\right\}^{-1}$$

$$=e^{-V_{\text{static}}T},\quad Q(\vec{b}_q^\mu)=\text{small fluctuation action quadratic in }\vec{b}_q^\mu, \tag{B2}$$

where the frequency in the phase term of Eq. (B1) has now been identified as

$$\Omega=-D\int d^3x\,\vec{b}_0^0(x)\cdot\vec{J}^0(x)\,. \tag{B3}$$

This is just the order-g^0 term which, we saw, was present in the free energy functional of Eq. (42) but not in the static potential or energy functional; I believe that the presence of the phase term is an indication that to do mean field theory properly in the presence of a background field, one should really not use Eq. (B1) but rather use the two-functional (free energy, energy) alternative developed in the text.

To evaluate the order-g^2 contribution to V_{static}, one develops the left-hand side of Eq. (B2) in a power series in the source current $\vec{J}^0$, through terms of order $(\vec{J}^0)^2$, and compares with the expansion $\exp(-V_{\text{static}}T)=1-V_{\text{static}}T+\cdots$ of the right-hand side, giving the formula

$$g^2V_{1\ \text{static}}T=\frac{D^2}{2}\int d^4x\,J^{a0}(x)\int d^4y\,J^{b0}(y)\left\{\int d[\vec{b}_q^\mu]\exp\left[-\int d^4x\,Q(\vec{b}_q^\mu)\right]b_q^{a0}(x)b_q^{b0}(y)\right\}$$
$$\times\left\{\int d[\vec{b}_q^\mu]\exp\left[-\int d^4x\,Q(\vec{b}_q^\mu)\right]\right\}^{-1}. \tag{B4}$$

The Gaussian integral on the right of Eq. (B4) may be readily evaluated[26] in the background gauge $D_0^\mu\vec{b}_q^\mu=0$ (with now $D_0^0=\partial/\partial x^0+\vec{b}_0^0\times$), giving the result

$$g^2V_{1\ \text{static}}T=g^2D\,\tfrac{1}{2}\int d^4x\int d^4y\,J^{a0}(x)J^{b0}(y)G^{a0,b0}(x,y\,|\,x^0-y^0)\,. \tag{B5}$$

Here $G^{a\mu,b\nu}(x,y\,|\,x^0-y^0)$ is just the time-dependent version of the propagator defined in Eq. (32),

$$[D_{0x}^\sigma D_{0x}^\sigma\vec{G}^{\mu,b\nu}(x,y\,|\,x^0-y^0)+2\vec{f}_0^{\mu\tau}(x)\times\vec{G}^{\tau,b\nu}(x,y\,|\,x^0-y^0)]^a=-Q^{a\mu,b\nu}(x,y)\delta(x^0-y^0)\,,$$
$$\int_{-\infty}^{\infty}d(x^0-y^0)G^{a\mu,b\nu}(x,y\,|\,x^0-y^0)=G^{a\mu,b\nu}(x,y)\,, \tag{B6}$$

and so Eq. (B5) gives

$$D^{-1}V_{1\ \text{static}}=\tfrac{1}{2}\int d^3x\,d^3y\,J^{a0}(x)G^{a0,b0}(x,y)J^{b0}(y)\,, \tag{B7}$$

in agreement with the result obtained in the text. [In order for Eq. (B7) to be convergent and unambiguous, it is of course still necessary to choose the source positions and orientations relative to the background so that $\vec{J}^\mu$ is orthogonal to the zero modes which satisfy Eq. (24b), and to subtract off the divergent Coulomb self-energies.] Thus, the order-g^2 confinement criterion stated in the discussion following Eq. (34) of the text agrees with the Wilson-loop criterion as applied to quarks in a static Euclidean background field.

APPENDIX C: EVALUATION OF THE SPIN-ORBIT POTENTIAL

Although the formalism developed in this paper is strictly applicable only to static quarks, it is easy to infer, by analogy with the Abelian case the rule for evaluating the spin-orbit interaction potential (which is the most important nonstatic correction). Equation (34) of the text, after averaging over self-dual and anti-self-dual background-field solutions, contains no charge-spin interaction term, and thus takes the form

$$V_{1\ \text{static}} = V_{1\ \text{static (charge-charge)}} + V_{1\ \text{static (spin-spin)}}\,. \quad \text{(C1)}$$

Keeping $x_1 \neq x_2$, so that the contact δ function does not contribute, the spin-spin potential can be written in the two equivalent forms

$$D^{-1}V_{1\ \text{static (spin-spin)}} = i\,\frac{\vec{Q}^{\text{eff}}_{(1)}\frac{1}{2}\sigma^j_{(1)}}{m_q}\cdot\vec{B}^j_{\sigma_{(2)}}(x_1) = i\,\frac{\vec{Q}^{\text{eff}}_{(2)}\frac{1}{2}\sigma^j_{(2)}}{m_q}\cdot\vec{B}^j_{\sigma_{(1)}}(x_2)\,, \quad \text{(C2)}$$

which identifies the color magnetic field $\vec{B}^j_{\sigma_{(2)}}(x_1)$ $[\vec{B}^j_{\sigma_{(1)}}(x_2)]$ induced at x_1 [at x_2] by the spin moment associated with $\sigma_{(2)}$ [with $\sigma_{(1)}$]. Similarly, let $\vec{E}^j_{Q_{(2)}}(x_1)$ $[\vec{E}^j_{Q_{(1)}}(x_2)]$ be the color electric fields induced at x_1 [at x_2] by the color source charge $\vec{Q}^{\text{eff}}_{(2)}$ $[\vec{Q}^{\text{eff}}_{(1)}]$, as obtained from the order-g^2 perturbation analysis. In terms of these quantities, the spin-orbit potential is given by

$$\begin{aligned} D^{-1}V_{1\ \text{spin-orbit}} = {} & \frac{-i}{2m_q}\{\vec{Q}^{\text{eff}}_{(1)}\cdot[L_1^j\vec{B}^j_{\sigma(2)}(x_1)+\vec{B}^j_{\sigma_{(2)}}(x_1)L_1^j]+\vec{Q}^{\text{eff}}_{(2)}\cdot[L_2^j\vec{B}^j_{\sigma_{(1)}}(x_2)+\vec{B}^j_{\sigma_{(1)}}(x_2)L_2^j]\} \\ & -\frac{1}{8m_q^{\,2}}\{\vec{Q}^{\text{eff}}_{(1)}\cdot[\vec{E}^l_{Q_{(2)}}(x_1)p_1^m+p_1^m\vec{E}^l_{Q_{(2)}}(x_1)]\sigma^n_{(1)}\epsilon^{lmn}+\vec{Q}^{\text{eff}}_{(2)}\cdot[\vec{E}^l_{Q_{(1)}}(x_2)p_2^m+p_2^m\vec{E}^l_{Q_{(1)}}(x_2)]\sigma^n_{(2)}\epsilon^{lmn}\}\,, \\ L_1^j = {} & \epsilon^{jlm}=(x_1-x_2)^l p_1^m\,,\quad L_2^j=\epsilon^{jlm}(x_2-x_1)^l p_2^m\,. \end{aligned} \quad \text{(C3)}$$

This expression is justified by the fact that it is the unique, gauge-invariant,[11] parity-even generalization of the Abelian spin-orbit interaction arising from the Breit equation.[39]

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[1]S. L. Adler, Phys. Rev. D 17, 3212 (1978).

[2]S. L. Adler, Phys. Rev. D 18, 411 (1978).

[3]A related (but not identical) approach to the problem of finding a semiclassical approximation to quantum chromodynamics has been proposed by R. Giles and L. McLerran, Phys. Lett. 79B, 447 (1978).

[4]The definition of P as anti-Hermitian, rather than Hermitian, is purely a convention. An investigation of generalized parametrized recipes for the color charge algebra by S-C. Lee (unpublished) has shown that there are parameter values for which the minimal algebra containing Q_q and $Q_{\bar{q}}$ is only two-dimensional, but that this two-dimensional algebra can nonetheless be embedded in a five-dimensional algebra with Jacobi and trace properties. The definitions and theorem of Sec. II A would have to be reformulated to apply to such degenerate cases, which may well be of physical interest.

[5]P. Cvitanović, R. J. Gonsalves, and D. E. Neville, Phys. Rev. D 18 3881 (1978); V. Rittenberg and D. Wyler, *ibid.* 18, 4806 (1978); I. Bars and C. M. Sommerfield (unpublished).

[6]The effective Lagrangian for each overlying algebra has the form of a classical Yang-Mills Lagrangian, multiplied by a scale factor D which depends on the underlying color group and the color state being studied. The same factor appears as a multiplier in the expression for the field energy.

[7]The notation follows that of Ref. 2, except that here I use the notation $\vec{b}^0$, rather than $\vec{\lambda}$, for the scalar potential.

[8]As explained in Ref. 2, the noncommutativity of the Pauli spin matrices $\sigma^m_{(n)}$ is to be ignored in all calculations. This approximation still permits the evaluation of spin interaction terms *bilinear* in the q and $\bar{q}$ spins. The spin currents $\vec{J}^k_{\text{spin}}$, $\vec{\mathfrak{J}}^m_{\text{spin}}$ are defined so as to satisfy

$$\int d^3x\,\delta\vec{B}^m\cdot\vec{\mathfrak{J}}^m_{\text{spin}} = \int d^3x\,\delta\vec{b}^k\cdot\vec{J}^k_{\text{spin}}\,;$$

the factor of i in $\vec{\mathfrak{J}}^m_{\text{spin}}$ (which was omitted in Ref. 2) is necessary to get the correct sign for the dipole-dipole interaction. Since terms linear in the spins are parity-odd and, as explained below, average to zero, all spin energies are real. The sign of the spin current $\vec{\mathfrak{J}}^m_{\text{spin}}$ has been taken from the Euclidean continuation analysis of F. Wilczek and A. Zee, Phys. Rev. Lett. 40, 83 (1978).

[9]For example, in evaluating the Wilson loop integral [K. G. Wilson, Phys. Rev. D 10, 2445 (1974)] to get the Coulomb force $1/(4\pi R)$ between test charges, one evaluates the singular Minkowski space integral $\int dt\,\langle T(A^0(R,t)A^0(0,0))\rangle$ by contour rotation to the absolutely convergent Euclidean form

$$\int_{-\infty}^{\infty} d\tau[4\pi^2(R^2+\tau^2)]^{-1} = 1/(4\pi R)\,.$$

The "$i\epsilon$" prescriptions which define a Minkowski field theory are all fixed by the continuation from the Euclidean region.

[10]I wish to thank C. M. Sommerfield for a helpful conversation about this.

[11]Specifically, the first term in Eq. (8a), divided by the total time interval T, is just the Minkowski-space field energy or "gluon energy" $E_{\text{gluon}} = \int d^3x\,T^{00}_{\text{gluon}}$, where $T^{\mu\nu}_{\text{gluon}}$ is the Minkowski-space gluon stress-energy tensor. Note that according to Eq. (8), $T^{-1}S_E$ is a Legendre transform of E_{gluon}, which is extremized

by the static equations with the quark source currents $\vec{J}^0, \vec{\mathfrak{J}}^m_{\rm spin}$ held fixed. Hence it is not surprising that S_E plays the role of a free energy in the stability analysis given in Sec. II D below. Both of the functionals $E_{\rm gluon}$ and S_E are manifestly gauge invariant under the general gauge transformation $\delta\vec{b}^j = D_j\vec{\phi}$, $\delta\vec{v} = \vec{v}\times\vec{\phi}$, $\vec{v} = \vec{b}^0$, $\vec{E}^j$, $\vec{B}^j$, $\vec{J}^0, \vec{\mathfrak{J}}^j_{\rm spin}$, which is an invariance of Eqs. (5) and (6). In my conventions, the Euclidean and Minkowski covariant components of dynamically independent fields are identical; the continuation from the Minkowski to the Euclidean region is accomplished by continuing the metric $g_{\mu\nu}$ from one of signature -1 to one of signature 1, being careful to include the appropriate factors $\sqrt{-g}$ (which continues as $1=\sqrt{-g}\to -i\sqrt{g} = -i$) in continuing the volume element d^4x and the Levi-Civita symbol $\epsilon_{\mu\nu\lambda\sigma}$, which are respectively a scalar and tensor density. The continuation of a convective current j_μ can be inferred from the continuation of the current $j_\mu = i\phi^*(\overleftrightarrow{\partial}/\partial x^\mu)\phi$ for a charged scalar field, giving the rule $j_\mu \to j_\mu$. Thus the Euclidean source current $\vec{J}^0$ is a Hermitian operator and is real for classical particle sources. However, since the spin vector of a particle transforms as a pseudovector density, the spin current $\vec{\mathfrak{J}}^m_{\rm spin}$ picks up as a factor of i on Euclidean continuation relative to a convective current, accounting for the factor of i in Eq. (6). See F. Wilczek and A. Zee, Phys. Rev. Lett. 40, 83 (1978).

[12]See Sec. 2 of S. Coleman, in the Proceedings of the 1977 International School of Physics "Ettore Majorana" (unpublished).

[13]See S. Coleman, Ref. 12, Sec. 1.

[14]A. M. Polyakov, Nucl. Phys. B120, 429 (1977); G. 't Hooft, Phys. Rev. D 14, 3432 (1976).

[15]See S. Coleman, Ref. 12, Sec. 3.2 for a discussion of the $S^3\to S^3$ homotopy classification. Moreover, the mapping $S^2\to S^3$ has only one homotopy class, and so does not define a topological quantum number.

[16]The argument which follows is a sharpening of the one given by M. A. Lohe, Phys. Lett. 70B, 325 (1977). Note, incidentally, that the entire asymptotic classification argument applies to the unexpanded potentials $\vec{b}^0$, $\vec{b}^j$, when the condition of finite $E_{-1\,\rm gluon}$ is replaced by the condition that $E_{\rm gluon}$ diverge only in the neighborhood of the source charges.

[17]This conclusion clearly still follows if the assumption $\vec{E}^j_0\sim x^{-2}$ is weakened to $\vec{E}^j_0\sim x^{-3/2-\epsilon}$, $\epsilon>0$ (which is the weakest power-law falloff consistent with finite energy).

[18]J. Arafune, P. G. O. Freund, and C. J. Goebel, J. Math. Phys. 16, 433 (1975).

[19]E. B. Bogomol'nyi, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. 24, 449 (1976)]; S. Coleman *et al.*, Phys. Rev. D 15, 544 (1977).

[20]It has been proved recently that self-dual solutions are the only finite-Euclidean-action solutions to the time-dependent Euclidean Yang-Mills equations that are strict action minima (rather than saddle points). Variational estimates of non-self-dual configurations, such as the monopole-antimonopole calculation of Magruder (Ref. 22 below), involve the imposition of constraints which modify the equations of motion from their pure Yang-Mills form.

[21]Note that there may be zero modes which satisfy Eq. (24b) but which are not normalizable in the norm defined in Eq. (27) below.

[22]This effect can be seen in the variational calculations of S. F. Magruder, Phys. Rev. D 17, 3257 (1978).

[23]The vanishing of $E_{0\,\rm gluon}$ follows immediately from letting δ be the $d/d\lambda$ operation defined just below; then

$$E_{0\ \rm gluon} = dE_{\rm gluon}/d\lambda\,|_{\lambda=0} = dV_{\rm static}/d\lambda\,|_{\lambda=0} = 0\,.$$

In Ref. 2, I reached the erroneous conclusion $V_{0\ \rm static}\neq 0$ by confusing $\int d^3x\,\delta\vec{b}^\mu_0\cdot\vec{J}^\mu$ with $\delta\int d^3x\,\vec{b}^\mu_0\cdot\vec{J}^\mu$. These two expressions of course differ by $\int d^3x\,\vec{b}^\mu_0\cdot\delta\vec{J}^\mu$, which is nonvanishing when δ includes a quark displacement. As stated earlier in the text, the part of Ref. 2 from Eq. (44) through Eq. (55) is incorrect, and should be replaced by the discussion of the present paper.

[24]The fact that it is only necessary to separate off an $|\vec{x}-\vec{y}|^{-1}$ term to get a finite answer is a consequence of a very general result of Hadamard, that logarithmic potentials are not needed in odd-dimensional spaces. See J. Hadamard, *Lectures on Cauchy's Problem in Nonlinear Partial Differential Equations* (Yale Univ. Press, New Haven, Conn., 1923).

[25]H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, New York, 1957), pp. 107 and 108.

[26]L. S. Brown, R. D. Carlitz, D. B. Creamer, and C. Lee, Phys. Rev. D 17, 1583 (1978).

[27]See Refs. 26 and 2, and also Sec. VII of N. Christ, E. J. Weinberg, and N. K. Stanton, Phys. Rev. D 18, 2013 (1978).

[28]The number of background fields allowed by the conditions of Eqs. (26) and (28) may well be finite. That is, for solutions with topological index $|n|$ larger than some n_0, it may not be possible to orthogonalize the quark source current to all the zero modes.

[29]R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976); C. Callan, R. Dashen, and D. Gross, Phys. Lett. 63B, 334 (1976); S. Coleman, Ref. 12, Sec. 3. A determination of the allowed values of ϕ will require an analysis of tunnelings of Euclidean static background solutions into themselves, analogous to the θ-vacuum analysis of the above references. The results of the above references suggest that the answer should be $\phi=\pm i\theta$, θ real, $\mathrm{Re}\phi=0$.

[30]The same negative sign appears in the Abelian case, where

$$T^{-1}S_E = \int d^3x\left[\frac{1}{2g^2}(E^jE^j+B^jB^j) - b^0J^0 - B^m\mathfrak{J}^m_{\rm spin}\right]$$

$$= \int d^3x\left[\frac{1}{2g^2}(E^jE^j+B^jB^j) - b^0J^0 - b^kJ^k_{\rm spin}\right].$$

Eliminating the source currents J^0 and $J^k_{\rm spin}$ by using the classical equations

$$\nabla^jE^j_c = g^2J^0\,,$$

$$\epsilon^{klm}\nabla^lB^m_c = g^2J^k_{\rm spin}$$

and integrating by parts, the above expression becomes *exactly*

$$T^{-1}S_E = \int d^3x\frac{1}{g^2}\left[\tfrac{1}{2}(E^jE^j+B^jB^j) - E^jE^j_c - B^jB^j_c\right]$$

$$= \int d^3x\frac{1}{g^2}\left[\tfrac{1}{2}(E^j_qE^j_q+B^j_qB^j_q) - \tfrac{1}{2}(E^j_cE^j_c+B^j_cB^j_c)\right],$$

with $E^j = E_c^j + E_q^j$, $B^j = B_c^j + B_q^j$. We see that the quantum fluctuation and classical contributions appear with opposite sign in $T^{-1}S_E$. The energy functional, on the other hand, is given by the positive-definite expression

$$E_{\text{gluon}} = \int d^3x T^{00}_{\text{gluon}}$$

$$= \int d^3x \frac{1}{g^2} \tfrac{1}{2}(E^j E^j + B^j B^j).$$

When the quantum fluctuations are zero, we have $T^{-1}S_{Ec} = -E_{\text{gluon}\,c}$. I wish to thank R. F. Dashen for a helpful conversation about the issues of sign reversals in the free energy and quantum stability.

[31]Note that classical stability, as defined below, is a very different concept from the quantum stability of equilibrium points referred to above.

[32]S. Coleman, Commun. Math. Phys. 55, 113 (1977); S. Deser, Phys. Lett. 64B, 463 (1976); R. Weder, Commun. Math. Phys. 57, 161 (1977).

[33]The topological quantum number is invariant under time evolution in the background gauge $D_{0\mu}\vec{b}_0^\mu = \partial_\mu \vec{b}_0^\mu = 0$, since then the leading asymptotic term $\vec{b}_0^0 \sim \kappa\hat{e}(\hat{r},t)$ must satisfy $(\partial/\partial t)\hat{e}(\hat{r},t) = 0$.

[34]Note, however, that there can be no classical Yang-Mills gluon bound states in algebraic chromodynamics, since non-Abelian classical overlying fields are present only in sectors of the theory where there are at least two quarks.

[35]In a recent paper [Phys. Lett. 79B, 242 (1978)], E. Mottola argues that the only normalizable zero modes are the three translations and a gauge mode. Mottola uses the procedures of L. S. Brown, R. D. Carlitz, and C. Lee, Phys. Rev. D 16, 417 (1977), to reduce the zero-mode problem in the self-dual case to the problem of an isospin-1 massless Dirac field scattering in a 't Hooft-Polyakov monopole background. According to this analysis, the number of normalizable zero modes of the Prasad-Sommerfield solution is $2k$, where k is the number of zero modes of the Dirac problem. Two sources of information on k are available. The Dirac problem has been studied by explicit partial-wave analysis by R. Jackiw and C. Rebbi, Phys. Rev. D 13, 3398 (1976), who find $k=2$. An alternative approach is to use generalized index theorem arguments for counting zero modes in three-dimensional open Euclidean space. The Dirac zero-eigenvalue problem separates into two independent two-component equations $L\psi_- = 0$, $L^\dagger\psi_+ = 0$, with respective zero-mode numbers k_-, k_+, and $k = k_- + k_+$. In a recent paper by C. J. Callias [Commun. Math. Phys. (to be published)], an index theorem is proved which in the isospin-1 case gives index $(L) = k_- - k_+ = 2n$, with n the topological quantum number of the 't Hooft-Polyakov background field. In the case of self-dual backgrounds and normalizable zero modes, the argument of Brown *et al.* can be used to show that $k_+ = 0$, giving $k = 2n$. Hence the number of normalizable isospin -1 Dirac modes in the Prasad-Sommerfield background is apparently $k=2$, in agreement with the conclusion reached by Jackiw and Rebbi. There are, however, loopholes in both of the above arguments for $k=2$, making them inapplicable to the case at hand. The argument of Jackiw and Rebbi in the isospin-1 case counts zero modes in the generic case of an arbitrary spherically symmetric background field, but does not search for possible additional zero modes arising from special properties of *particular* background solutions, and of course self-dual solutions are a very special subclass of the general case. The argument of Callias requires L to be Fredholm, which is true for nonzero fermion mass m, which Callias assumes. However, for massless fermions ($m=0$), as needed to analyze the self-dual zero-mode problem, L is non-Fredholm in the integer-isospin case, and the index theorem fails. The issue of whether the Prasad-Sommerfield solution has axial distortions is currently under study. Another open issue which deserves investigation is whether the restriction to real background solutions, tacitly assumed throughout this paper is necessary. It is possible that the mean-field approximation to the functional integral relevant to confinement is obtained, not from a real static Euclidean solution, but rather from a pair of complex-conjugate static Euclidean solutions; the Prasad-Sommerfield solution is known to have complex distrotions (see N. S. Manton, Ref. 37 below).

[36]An alternative construction of the scalar propagator has been given by P. Rossi [Pisa report (unpublished)].

[37]Discussions of the axially symmetric case, with inconclusive results, have been given by N. S. Manton, Nucl. Phys. B135, 319 (1978); M. A. Lohe, *ibid.* B142, 236 (1978); P. S. Jang, S. Y. Park, and K. C. Wali, Phys. Rev. D 17, 1641 (1978).

[38]If confining background solutions exist in the SU(2) case, I suspect that in the SU(j) case the classical solutions relevant for static confinement will involve only the SU(2) subgroups of SU(j), just as minimal SU(j) instantons are obtained by embedding the SU(2) instanton in an SU(2) subgroup of SU(j). [See, e.g., C. W. Bernard, N. H. Christ, A. H. Guth, and E. J. Weinberg, Phys. Rev. D 16, 2967 (1977).] If this is so, then the color charge algebra proposed in Ref. 1 already suffices for an approximate treatment of confinement, since the calculations of Ref. 5 for the qqq case show that the trace property does hold within each SU (2) subgroup of the SU (3) overlying algebra. Of course, it would be much better to find an algebra which does satisfy the conditions discussed in Sec. II A; a search for an improved ansatz, along the lines suggested in the added note to Ref. 2, is currently being pursued by S-C. Lee (unpublished).

[39]H. A. Bethe and E. E. Salpeter, Ref. 25, p. 181.